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THE RESTRAINED RAINBOW BONDAGE NUMBER OF A GRAPH

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Abstract. A restrained k-rainbow dominating function (RkRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1,2,\dots,k\}$ such that for any vertex $v \in V(G)$ with $f(v)=\emptyset$ the conditions $\bigcup_{u\in N(v)} f(u)=\{1,2,\dots,k\}$ and $|N(v)\cap \{u\in V \mid f(u)=\emptyset\}|\ge 1$ are fulfilled, where $N(v)$ is the open neighborhood of v. The weight of a restrained k-rainbow dominating function is the value $w(f)=\sum_{v\in V}|f(v)|$. The minimum weight of a restrained k-rainbow dominating function of G is called the restrained k-rainbow domination number of G, denoted by $\gamma_{rrk}(G)$. The restrained k-rainbow bondage number $b_{rrk}(G)$ of a graph G with maximum degree at least two is the minimum cardinality of all sets $F\subseteq E(G)$ for which $\gamma_{rrk}(G-F)>\gamma_{rrk}(G)$. In this paper, we initiate the study of the restrained k-rainbow bondage number in graphs and we present some sharp bounds for $b_{rr2}(G)$. In addition, we determine the restrained 2-rainbow bondage number of some classes of graphs.

1. Introduction

In this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v\in V$, the *open neighborhood* $N(v)$ is the set $\{u\in V(G)\mid uv\in E(G)\}$ and its *closed neighborhood* is $N[v]=N(v)\cup\{v\}$. Similarly, the *open neighborhood* of a set $S\subseteq V$ is the set $N(S)=\bigcup_{v\in S}N(v)$ and its *closed neighborhood* is $N[S]=N(S)\cup S$. The minimum and maximum degree in G are respectively denoted by $\delta(G)$ and $\Delta(G)$. A vertex of degree one is called a $leaf$, and its neighbor is called a *support vertex*. If v is a support vertex, then L_v will denote the set of all leaves adjacent to v. A support vertex is said to be an *end-stem* if all its neighbors except one of them are leaves. For $r,s\ge 1$, a double star $S(r,s)$ is a tree with exactly two vertices that are not leaves, with one adjacent to r leaves and the other to s leaves. For a vertex v in a rooted tree T, let $C(v)$ denote the set of children of v, $D(v)$ denote the set of descendants of v and $D[v]=D(v)\cup\{v\}$, and the depth of v, depth(v), is the largest distance from v to a vertex in $D(v)$. The *maximal subtree* at v is the subtree of T induced by $D(v)\cup\{v\}$, and is denoted by T_v. A subset S of vertices of G is a *dominating set* if $N[S]=V$.

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The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. For a more thorough treatment of domination parameters and for terminology not presented here see [17, 22].

A subset S of vertices of G is a *restrained dominating set* if $N[S]=V$ and the subgraph induced by $V-S$ has no isolated vertex. The *restrained domination number* $\gamma_r(G)$ is the minimum cardinality of a restrained dominating set of G. The restrained domination number was introduced by Domke et al. [14] and has been studied by several author (see for example [12, 13]). The *restrained bondage number* $b_r(G)$ of a nonempty graph G is the minimum cardinality among all sets of edges $F\subseteq E(G)$ for which $\gamma_r(G-F)>\gamma_r(G)$. The restrained bondage number has been investigated in [15, 18].

For a positive integer k, a k*-rainbow dominating function* (kRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1,2,\dots,k\}$ such that for any vertex $v\in V(G)$ with $f(v)=\emptyset$ the condition $\bigcup_{u\in N(v)} f(u)=\{1,2,\dots,k\}$ is fulfilled. The weight of a kRDF f is the value $w(f)=\sum_{v\in V}|f(v)|$. The k*-rainbow domination number* of a graph G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a kRDF of G. Note that $\gamma_{r1}(G)$ is the classical domination number $\gamma(G)$. The k-rainbow domination number was introduced by Brešar, Henning, and Rall [7] and has been studied by several authors (see for example [2, 3, 8, 9, 10, 19, 20, 21, 23, 24]).

A k-rainbow dominating function f is called a restrained k*-rainbow dominating function* (RkRDF) if the induced subgraph $G[\{v\in V\mid f(v)=\emptyset\}]$ has no isolated vertex. The *restrained* k*-rainbow domination number* of G, denoted by $\gamma_{rrk}(G)$, is the minimum weight of an RkRDF on G. A $\gamma_{rrk}(G)$*-function* is an RkRDF of G with $\omega(f)=\gamma_{rrk}(G)$. If f is a $\gamma_{rrk}(G)$-function, then since $V-\{v\in V\mid f(v)=\emptyset\}$ is a restrained dominating set, and since placing $\{1,2,\dots,k\}$ at the vertices of a restrained dominating set yields an RkRDF, we have

$$\max\{\gamma_{rk}(G),\gamma_r(G)\}\le\gamma_{rrk}(G)\le k\gamma_r(G).$$

The restrained k-rainbow domination number has been investigated in [1, 5].

Let G be a graph of order $n\ge k+1$ with $\gamma_{rrk}(G)<n$. The *restrained* k*-rainbow bondage number* $b_{rrk}(G)$ of G is the minimum cardinality of all sets $E'\subseteq E$ for which $\gamma_{rrk}(G-E')>\gamma_{rrk}(G)$. An edge set B with $\gamma_{rrk}(G-B)>\gamma_{rrk}(G)$ is called the *restrained* k*-rainbow bondage set*. A $b_{rrk}(G)$*-set* is a restrained k-rainbow bondage set of G of size $b_{rrk}(G)$. If B is a $b_{rrk}(G)$-set, then clearly $\gamma_{rrk}(G-B)=\gamma_{rrk}(G)+1$.

The k-rainbow bondage number $b_{rk}(G)$ for usual k-rainbow domination number was introduced by Dehgardi et al. in [11] and has been studied by several authors [4, 6].

One possible application of the concept of k-rainbow restrained domination is that of cities and emergency guards. Here, every vertex with a positive weight in a k-rainbow restrained dominating function, corresponds to a position of an emergency guard and each vertex not occupied by an emergency guard corresponds to a position of a city without any emergency guards, which is adjacent to at least one other deprived city. The k-rainbow restrained bondage number measures the vulnerability of the connection between situations under unpredictable events or attacks. The minimum k-rainbow restrained dominating function of cities plays an important role for dominating the whole situations with the minimum cost. So, we must consider whether its function remains safe under the unpredictable event or attack. Suppose that an unpredictable event happens. Then how many connection routes does it have to destroy so that the cost can not remains the same in order to k-rainbow restrained dominate the whole city? The minimum number of connection routes is just the k-rainbow restrained bondage number.

Our purpose in this paper is to initiate the study of the restrained k-rainbow bondage number in graphs. We first establish some sharp bounds for the restrained k-rainbow bondage number of a graph. In particular, we prove that for any tree T of order $n \geq 5$ with $\mathrm{diam}(T) \geq 3$ and different from P_5, P_6, $b_{rr2}(T) \leq (n-3)/2$. In addition, we determine the restrained 2-rainbow bondage number of some classes of graphs.

We make use of the following results in this paper.

Theorem A ([5])**.** *For $n \geq 4$, $\gamma_{rr2}(P_n) = \left\lceil \frac{2n+1}{3} \right\rceil + 1$ and $\gamma_{rr2}(P_n) = n$ otherwise.*

Corollary 1.1. *For $n \geq 7$, $b_{rr2}(P_n) = 1$.*

Proof. Let $P_n := v_1 v_2 \dots v_n$. It follows from Theorem A that

$$\gamma_{rr2}(P_n - v_3 v_4) = \gamma_{rr2}(P_{n-3}) + 3 = \left\lceil \frac{2(n-3)+1}{3} \right\rceil + 1 + 3 > \left\lceil \frac{2n+1}{3} \right\rceil + 1 = \gamma_{rr2}(P_n).$$

Hence $b_{rr2}(P_n) = 1$. □

Theorem B ([5])**.** *For $n \geq 6$, $\gamma_{rr2}(C_n) = 2\left\lceil \frac{n}{3} \right\rceil + 1$ when $n \equiv 2$ (mod 3) and $\gamma_{rr2}(C_n) = 2\left\lceil \frac{n}{3} \right\rceil$ otherwise.*

Corollary 1.2. *For $n \geq 6$,*

$$b_{rr2}(C_n) = \begin{cases} 1 & \text{if} \quad n \equiv 0 \pmod 3 \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Let $C_n := (v_1 v_2 \dots v_n)$. If $n = 3k$ $(k \geq 2)$, then it follows from Theorem A and B that

$$\gamma_{rr2}(C_n - v_1 v_n) = \gamma_{rr2}(P_n) = \left\lceil \frac{2n+1}{3} \right\rceil + 1 > 2\left\lceil \frac{n}{3} \right\rceil = \gamma_{rr2}(C_n).$$

Hence $b_{rr2}(C_n) = 1$ if $n \equiv 0 \pmod 3$. Now let $n \equiv 1 \pmod 3$. Then obviously $\left\lceil \frac{2n+1}{3} \right\rceil + 1 = 2\left\lceil \frac{n}{3} \right\rceil$ that implies $b_{rr2}(C_n) \geq 2$ in this case. On the other hand, Theorems A and B imply that

$$\gamma_{rr2}(C_n - \{v_1v_2, v_4v_5\}) = \gamma_{rr2}(P_{n-3}) + 3 = \left\lceil \frac{2(n-3)+1}{3} \right\rceil + 1 + 3 > 2\left\lceil \frac{n}{3} \right\rceil = \gamma_{rr2}(C_n).$$

Hence $b_{rr2}(C_n) = 2$ if $n \equiv 1 \pmod 3$. Finally let $n \equiv 2 \pmod 3$. It is easy to see that $\left\lceil \frac{2n+1}{3} \right\rceil + 1 = 2\left\lceil \frac{n}{3} \right\rceil + 1$ which implies that $b_{rr2}(C_n) \geq 2$ in this case. We deduce from Theorems A and B that

$$\gamma_{rr2}(C_n - \{v_1v_2, v_4v_5\}) = \gamma_{rr2}(P_{n-3}) + 3 = \left\lceil \frac{2(n-3)+1}{3} \right\rceil + 1 + 3 > 2\left\lceil \frac{n}{3} \right\rceil + 1 = \gamma_{rr2}(C_n)$$

and so $b_{rr2}(C_n) = 2$ in this case. This completes the proof. □

Theorem C ([5]). *Let G be a connected graph of order $n \geq 2$. Then $\gamma_{rr2}(G) = n$ if and only if $G \simeq K_{1,n-1}, C_4, C_5$ or $G = P_n$ for $n = 2,3,4,5,6$.*

Theorem D ([5]). *Let G be a graph of order $n \geq 2$. Then $\gamma_{rr2}(G) = 2$ if and only if $n = 2$ or $n \geq 3$ and $2 \leq \delta(G) \leq \Delta(G) = n-1$ or $3 \leq \delta(G) \leq \Delta(G) = n-2$ and there exist two distinct vertices u and v such that $V(G) - \{u, v\} \subseteq N(u) \cap N(v)$.*

Theorem E ([11]). *If $k \geq 2$ is an integer and $n \geq k+1$, then $b_{rk}(K_n) = \left\lceil \frac{kn}{k+1} \right\rceil$.*

Theorem F ([11]). *Let G be a graph of order $n \geq 3$. Then $\gamma_{r2}(G) = 2$ if and only if there exists a vertex set A with $|A| \leq 2$ such that every vertex of $V(G) - A$ is adjacent to every vertex of A.*

2. Bounds on the restrained rainbow bondage number

In this section we first establish a sharp upper bound on the restrained 2-rainbow bondage number of trees in terms of their order and then we present two sharp bounds on the restrained 2-rainbow bondage number of general graphs.

Observation 2.1. *If $T = S(r,s)$ is a double star of order $r+s+2 \geq 5$, then $b_{rr2}(T) = 1$.*

Theorem 2.2. *Let T be a tree of order $n \geq 5$. If $\mathrm{diam}(T) \geq 3$ and $T \notin \{P_5, P_6\}$, then*

$$b_{rr2}(T) \leq \frac{n-3}{2}.$$

Furthermore, this bound is sharp.

Proof. If $\operatorname{diam}(T) = 3$, then T is a double star of order at least 5 and it follows from Observation 2.1 that $b_{rr2}(T) = 1 \le \frac{n-3}{2}$. Assume that $\operatorname{diam}(T) \ge 4$. Let $P = v_1v_2\dots v_d$ be a diametral path in T such that $\deg(v_2)$ is as large as possible. Among all paths with this property we choose a path such that $|L_{v_3}|$ is as large as possible. Root T at v_d. We consider the following cases.

Case 1. $\deg(v_2) \ge 3$.
Suppose that $u \in L_{v_2} - \{v_1\}$. First let $\deg(v_3) \ge 3$. Assume that $L_{v_3} = \{x_1, x_2, \dots, x_k\}$ if $L_{v_3} \ne \emptyset$ and $N(v_3) \setminus (L_{v_3} \cup \{v_2, v_4\}) = \{y_1, y_2, \dots, y_t\}$ when $N(v_3) \setminus (L_{v_3} \cup \{v_2, v_4\}) \ne \emptyset$. We consider two subcases.

Subcase 1.1. $L_{v_3} \ne \emptyset$.
Let $B = \{v_3y_1, \dots, v_3y_t, v_3v_2, v_3v_4\}$. Clearly, $n \ge 2t+7$ and so

$$|B| = t+2 \le (n-7)/2+2 \le (n-3)/2.$$

Let $T_1, T_2, \dots, T_t, T_{t+1}, T_{t+2}$ be the components of $T - B$ containing $y_1, y_2, \dots, y_t, v_2, v_3$ respectively. Assume that f is a $\gamma_{rr2}(T-B)$-function. Then clearly $|f(x)| = 1$ for every vertex $x \in \bigcup_{i=1}^{t+2} V(T_i)$. Define $g : V(T) \to \mathscr{P}(\{1,2\})$ by $g(v_1) = \{1\}, g(u) = \{2\}, g(x_1) = \{1,2\}$, $g(v_2) = g(v_3) = \emptyset$, and $g(x) = f(x)$ otherwise. It is easy to see that g is a restrained 2-rainbow dominating function of T of weight less than $\omega(f)$ and hence $b_{rr2}(T) \le |B| \le (n-3)/2$ as desired.

Subcase 1.2. $L_{v_3} = \emptyset$.
Since $\deg(v_3) \ge 3$, we have $t \ge 1$. Assume that $B = \{v_3y_1, \dots, v_3y_t, v_3v_4\}$. Then clearly $n \ge 2t+6$ and so

$$|B| = t+1 \le (n-6)/2+1 < (n-3)/2.$$

Let $T_1, T_2, \dots, T_t, T_{t+1}$ be the components of $T - B$ containing $y_1, y_2, \dots, y_t, v_3$ respectively, and f be a $\gamma_{rr2}(T-B)$-function. Then $|f(x)| = 1$ for every vertex $x \in \bigcup_{i=1}^{t+1} V(T_i)$. Define $g : V(T) \to \mathscr{P}(\{1,2\})$ by $g(v_1) = \{1\}$, $g(u) = \{2\}, g(y_1) = \{1,2\}$, $g(v_2) = g(v_3) = \emptyset$ and $g(x) = f(x)$ otherwise. Obviously, g is a restrained 2-rainbow dominating function of T of weight less than $\omega(f)$ and hence $b_{rr2}(T) \le |B| < (n-3)/2$.

Now let $\deg(v_3) = 2$. By the choice of the diametral path, every vertex with depth 2 in $N(v_4) - \{v_5\}$, have degree 2. If $n = 6$ or 7, then clearly $b_{rr2}(T) = 1 < (n-3)/2$ and we are done. Suppose that $n \ge 8$. If $\deg(v_4) = 2$ or $N(v_4)\setminus\{v_3, v_5\} = L_{v_4}$, then let $B = \{v_5v_4, v_4v_3\}$, T_1, T_2 be the components of $T - B$ containing v_4, v_3, respectively, and f be a $\gamma_{rr2}(T-B)$-function. Clearly, $|f(x)| = 1$ for each $x \in V(T_1) \cup V(T_2)$. Then the function $g : V(T) \to \mathscr{P}(\{1,2\})$ defined by $g(v_4) = \{1,2\}, g(u) = \{1\}, g(v_1) = \{2\}$, $g(v_2) = g(v_3) = \emptyset$ and $g(x) = f(x)$ otherwise, is a restrained 2-rainbow dominating function of T of weight less than $\omega(f)$ and hence $b_{rr2}(T) \le 2 < (n-3)/2$ as desired. Henceforth, we assume that $\deg(v_4) \ge 3$ and v_4 has a neighbor of degree at least two other than v_3, v_5. Let $N(v_4) - \{v_3, v_5\} = L_{v_4} \cup \{y_1, y_2, \dots, y_t\}$. Clearly, $n \ge 2t+6$. If $n \ge 2t+7$

then let $B = \{v_4y_1,\dots,v_4y_t,v_3v_4,v_4v_5\}$, and if $n = 2t+6$ then let $B = \{v_4y_2,\dots,v_4y_t,v_3v_4,v_4v_5\}$. Then $|B| \le (n-3)/2$. Assume that f is a $\gamma_{rr2}(T-B)$-function. Clearly, $|f(x)| = 1$ for each $x \in V(T_{v_4})$. Define $g : V(T) \to \mathscr{P}(\{1,2\})$ by $g(v_4) = \{1,2\}, g(v_1) = \{1\}$, $g(u) = \{2\}, g(v_2) = g(v_3) = \emptyset$, and $g(x) = f(x)$ otherwise. Obviously, g is a restrained 2-rainbow dominating function of T of weight less than $\omega(f)$ and hence $b_{rr2}(T) \le |B| \le (n-3)/2$.

Case 2. $\deg(v_2) = 2$.
By the choice of the diametral path, we deduce that every end-steam on a diametral path has degree 2. In particular, any child of v_3 is a leaf or a support vertex of degree 2. Consider the following subcases.

Subcase 2.1. $\deg(v_3) \ge 3$.
Assume that $L_{v_3} = \{x_1,x_2,\dots,x_k\}$ if $L_{v_3} \ne \emptyset$ and $N(v_3)-\{v_4,v_2\} = L_{v_3} \cup \{y_1,y_2,\dots,y_t\}$ when $N(v_3)-\{v_4,v_2\} \ne L_{v_3}$. We distinguish the following.

- $N(v_3)-\{v_4,v_2\} = L_{v_3}$.
  If $n = 6$, then clearly $b_{rr2}(T) = 1 < \frac{n-3}{2}$. Hence, we assume that $n \ge 7$. Let $T' = T - \{v_4v_3, v_3v_2\}$ and f be a $\gamma_{rr2}(T')$-function. Clearly, $|f(x)| = 1$ for each vertex $x \in V(T_{v_3})$. If $f(v_4) = \emptyset$, then define the function $g : V(T) \to \mathscr{P}(\{1,2\})$ by $g(x_1) = \{1\}, g(v_3) = \emptyset, g(v_2) = \{2\}$ and $g(x) = f(x)$ otherwise. If $f(v_4) \ne \emptyset$, then let, without loss of generality, $1 \in f(v_4)$ and define the function $g : V(T) \to \mathscr{P}(\{1,2\})$ by $g(x_1) = \{2\}, g(v_1) = \{1,2\}, g(v_2) = g(v_3) = \emptyset$ and $g(x) = f(x)$ otherwise. It is easy to see that g is a restrained 2-rainbow dominating function of T of weight less than $\omega(f)$ and hence $b_{rr2}(T) \le 2 \le (n-3)/2$.

- $N(v_3)-\{v_4,v_2\} \ne L_{v_3}$.
  First let $|L_{v_3}| \ge 2$. Assume that $B = \{v_3y_1,\dots,v_3y_t,v_3v_2,v_3v_4\}$, f is a $\gamma_{rr2}(T-B)$-function and $T_1,T_2,\dots,T_t,T_{t+1},T_{t+2}$ are the components of $T-B$ containing $y_1,\dots,y_t,v_2,v_3$ respectively. Clearly, $|B| \le (n-3)/2$ and $|f(x)| = 1$ for each $x \in \bigcup_{i=1}^{t+2} V(T_i)$. Define the function $g : V(T) \to \mathscr{P}(\{1,2\})$ by $g(x_1) = \{1\}, g(x_2) = \{2\}, g(v_1) = \{1,2\}, g(v_2) = g(v_3) = \emptyset$ and $g(x) = f(x)$ otherwise. Obviously, g is a restrained 2-rainbow dominating function of T of weight less than $\omega(f)$ and hence $b_{rr2}(T) \le |B| \le (n-3)/2$.

  Now let $|L_{v_3}| \le 1$. Assume that $B = \{v_3y_1,\dots,v_3y_t,v_3v_4\}$ and f is a $\gamma_{rr2}(T-B)$-function. Obviously, $|B| \le (n-3)/2$ and $|f(x)| = 1$ for each $x \in V(T_{v_3})$. If $f(v_4) = \emptyset$, then the function $g : V(T) \to \mathscr{P}(\{1,2\})$ defined by $g(y_1) = \{1\}, g(v_2) = \{2\}, g(v_3) = \emptyset$ and $g(x) = f(x)$ otherwise, is a restrained 2-rainbow dominating function of T of weight less than $\omega(f)$, implying that $b_{rr2}(T) \le |B| \le (n-3)/2$. If $f(v_4) \ne \emptyset$, then let, without loss of generality, $1 \in f(v_4)$ and define $g : V(T) \to \mathscr{P}(\{1,2\})$ by $g(y_1) = \{2\}, g(v_1) = \{1,2\}, g(v_2) = g(v_3) = \emptyset$ and $g(x) = f(x)$ otherwise. Clearly, g is a restrained 2-rainbow dominating function of T of weight less than $\omega(f)$ and hence $b_{rr2}(T) \le (n-3)/2$.

Subcase 2.2. $\deg(v_3) = 2$.
By the choice of the diametral path, we deduce that every vertex with depth 2 on a diametral path has degree 2. Since $T \not\in \{P_5, P_6\}$, we must have $\mathrm{diam}(T) \geq 6$. Let $L_{v_4} = \{x_1, x_2, \ldots, x_k\}$ if $L_{v_4} \neq \emptyset$ and $N(v_4) - \{v_5, v_3\} = L_{v_4} \cup \{y_1, y_2, \ldots, y_t\}$ when $N(v_4) - \{v_5, v_3\} \neq L_{v_4}$. We distinguish the following.

(a) $\deg(v_4) \geq 3$ and $N(v_4) - \{v_5, v_3\} = L(v_4)$.
Then $k \geq 1$. If $k \geq 2$, then it is easy to see that $\gamma_{rr2}(T - \{v_5v_4, v_4v_3\}) > \gamma_{rr2}(T)$ that implies $b_{rr2}(T) \leq 2 \leq (n-3)/2$. Let $k = 1$, $T' = T - v_5v_4$ and let f be a $\gamma_{rr2}(T')$-function. If $f(v_5) = \emptyset$, then the function $g : V(T) \to \mathscr{P}(\{1,2\})$ defined by $g(v_3) = \{1\}, g(x_1) = \{2\}, g(v_4) = \emptyset$ and $g(x) = f(x)$ otherwise, is a restrained 2-rainbow dominating function of T of weight less than $\omega(f)$ and hence $b_{rr2}(T) = 1 < \frac{n-3}{2}$. If $f(v_5) \neq \emptyset$, then let, without loss of generality, $1 \in f(v_5)$ and define $g : V(T) \to \mathscr{P}(\{1,2\})$ by $g(v_2) = \{1,2\}, g(x_1) = \{2\}, g(v_3) = g(v_4) = \emptyset$ and $g(x) = f(x)$ otherwise. Clearly, g is a restrained 2-rainbow dominating function of T of weight less than $\omega(f)$ and so $b_{rr2}(T) = 1 < (n-3)/2$.

(b) $\deg(v_4) \geq 3$, $N(v_4) - \{v_5, v_3\} \neq L_{v_4}$ and $\deg(y_j) \geq 3$ for some j.
Let $j = 1$, $B = \{v_4y_1, \ldots, v_4y_t, v_3v_4, v_5v_4\}$ and f be a $\gamma_{rr2}(T - B)$-function. Clearly, $|B| < (n-3)/2$ and $|f(x)| = 1$ for each $x \in V(T_{v_4})$. Let $w_1, w_2 \in N(y_1) - \{v_4\}$ and define $g : V(T) \to \mathscr{P}(\{1,2\})$ by $g(w_1) = \{1\}, g(w_2) = \{2\}, g(v_3) = \{1,2\}, g(v_4) = g(y_1) = \emptyset$ and $g(x) = f(x)$ otherwise. It is easy to see that g is a restrained 2-rainbow dominating function of T of weight less than $\omega(f)$ and hence $b_{rr2}(T) < (n-3)/2$.

(c) $\deg(v_4) \geq 3$, $N(v_4) - \{v_5, v_3\} \neq L_{v_4}$ and $\deg(y_j) = 2$ for each j.
Let $B = \{v_4y_1, \ldots, v_4y_t, v_3v_4, v_5v_4\}$, f a $\gamma_{rr2}(T-B)$-function and $T_1, T_2, \ldots, T_t, T_{t+1}, T_{t+2}$ be the components of $T - B$ containing $y_1, \ldots, y_t, v_3, v_4$, respectively. Then $|f(x)| = 1$ for each $x \in \bigcup_{i=1}^{t+2} V(T_i)$ and $|B| \leq (n-3)/2$ because $n \geq 2t + 7$. If $f(v_5) = \emptyset$, then the function $g : V(T) \to \mathscr{P}(\{1,2\})$ defined by $g(v_3) = \{1\}, g(y_1) = \{2\}, g(v_4) = \emptyset$ and $g(x) = f(x)$ otherwise, is a restrained 2-rainbow dominating function of T of weight less than $\omega(f)$ and hence $b_{rr2}(T) \leq (n-3)/2$. If $f(v_5) \neq \emptyset$, then let, without loss of generality, $1 \in f(v_5)$ and define $g : V(T) \to \mathscr{P}(\{1,2\})$ by $g(v_2) = \{1,2\}, g(y_1) = \{2\}, g(v_3) = g(v_4) = \emptyset$ and $g(x) = f(x)$ otherwise. Clearly, g is a restrained 2-rainbow dominating function of T of weight less than $\omega(f)$ that implies $b_{rr2}(T) \leq (n-3)/2$.

(d) $\deg(v_4) = 2$.
Considering above, we may assume that any maximal subtree at each child of v_5 with depth 3, is the path P_4. Consider the following.

(i) $\deg(v_5) \geq 3$.
Assume that $N(v_5) - \{v_6, v_4\} = L_{v_5} \cup \{y_1, y_2, \ldots, y_t\}$ if $N(v_5) - \{v_6, v_4\} \neq L_{v_5}$ and $L_{v_5} = \{x_1, x_2, \ldots, x_k\}$ when $L_{v_5} \neq \emptyset$.

- $L_{v_5} \neq \emptyset$.
  If $N(v_5) - \{v_6, v_4\} = L_{v_5}$, then it is easy to see that $\gamma_{rr2}(T - \{v_5v_6, v_5v_4\}) > \gamma_{rr2}(T)$ that implies $b_{rr2}(T) \leq 2 < (n-3)/2$. Assume that $N(v_5) - \{v_6, v_4\} \neq L_{v_5}$ and $B = \{v_5y_1, \dots, v_5y_t, v_5v_4, v_5v_6\}$. As above $|B| < (n-3)/2$. Let f be a $\gamma_{rr2}(T-B)$-function. Then we may assume that $|f(v_i)| = |f(x_j)| = 1$ for $1 \leq i \leq 5$ and $1 \leq j \leq k$. Using an argument similar to that described in Case 3, we can see that $\gamma_{rr2}(T) < \gamma_{rr2}(T-B)$ implying that $b_{rr2} \leq (n-3)/2$.
- $L_{v_5} = \emptyset$ and T_{y_i} has a $\gamma_{rr2}(T_{y_i})$-function h such that $|h(y_i)| \geq 1$ for some i.
  Assume, without loss of generality, that $j = 1$ and $1 \in h(y_1)$. Let $B = \{v_5y_1, \dots, v_5y_t, v_5v_6\}$, f be a $\gamma_{rr2}(T-B)$-function and $T_1, T_2, \dots, T_t, T_{t+1}, T_{t+2}$ be the components of $T-B$ containing $y_1, \dots, y_t, v_6, v_5$ respectively. We may assume that $f|_{T_{y_1}} = h$ and $|f(v_i)| = 1$ for each $1 \leq i \leq 5$. If there exists a vertex $w \in \{v_6, y_2, \dots, y_t\}$ with $f(w) = \emptyset$, then define $g : V(T) \to \mathscr{P}(\{1,2\})$ by $g(v_4) = \{2\}, g(v_5) = \emptyset$ and $g(x) = f(x)$ otherwise. Clearly, g is a restrained 2-rainbow dominating function of T of weight less than $\omega(f)$ and so $b_{rr2}(T) \leq (n-3)/2$. Assume that $|f(w)| \geq 1$ for each $w \in \{v_6, y_1, \dots, y_t\}$. Suppose, without loss of generality, that $2 \in f(v_6)$. Then the function $g : V(T) \to \mathscr{P}(\{1,2\})$ defined by $g(v_5) = g(v_4) = \emptyset, g(v_3) = \{1,2\}$ and $g(x) = f(x)$ otherwise, is a restrained 2-rainbow dominating function of T of weight less than $\omega(f)$ and so $b_{rr2}(T) \leq |B| \leq (n-3)/2$.
- $L_{v_5} = \emptyset$ and For each i and each $\gamma_{rr2}(T_{y_i})$-function h, $h(y_i) = \emptyset$.
  Then $\deg_{T_{y_1}}(y_1) \geq 2$ and y_1 has a neighbor of degree at least two with exception of v_5. Let $N(y_1) \setminus (L_{y_1} \cup \{v_5\}) = \{z_1, z_2, \dots, z_s\}$. If the component of $T - v_6v_5$ containing v_6, has a γ_{rr2}-function f' with $|f'(v_6)| \geq 1$, then as above we can see that $b_{rr2}(T) \leq (n-3)/2$. Hence, we assume that every γ_{rr2}-function of the component of $T - v_6v_5$ containing v_6, assigns $\emptyset$ to v_6. Now let $B = \{v_5y_1, \dots, v_5y_t, v_5v_6, y_1z_1, \dots, y_1z_s\}$, f be a $\gamma_{rr2}(T-B)$-function and T_1 be the component of $T-B$ containing y_1. Obviously, $n \geq 2t + 2s + 6$ and so $|B| < (n-3)/2$. On the other hand, we may assume that $|f(y_1)| = 1$ and $|f(v_i)| = 1$ for $1 \leq i \leq 5$. Let h be a $\gamma_{rr2}(T_{y_1})$-function. By assumption we have $h(y_1) = \emptyset$ and $\sum_{x \in V(T_{y_1})} |f(x)| > \sum_{x \in V(T_{y_1})} |h(x)|$. Then the function $g : V(T) \to \mathscr{P}(\{1,2\})$ defined by $g(v_5) = g(v_3) = g(v_2) = \emptyset$, $g(v_4) = g(v_1) = \{1,2\}$, $g(u) = h(u)$ for $u \in V(T_{y_1})$ and $g(u) = f(u)$ otherwise, is a restrained 2-rainbow dominating function of T of weight less than $\omega(f)$ and so $b_{rr2}(T) \leq |B| \leq \frac{n-3}{2}$.

(ii) $\deg(v_5) = 2$.

Considering the above cases and subcases, we have $\deg(v_i) = 2$ for $2 \leq i \leq 5$. Similarly, we may assume that $\deg(v_i) = 2$ for $d-4 \leq i \leq d-1$. Using an argument similar to that described in the proof of (i), we can show that $b_{rr2}(T) \leq \frac{n-3}{2}$ when

$\deg(v_6) \geq 3$. Therefore, we suppose that $\deg(v_6) = 2$. If $\deg(v_7) = 2$, then it is easy to see that $\gamma_{rr2}(T) < \gamma_{rr2}(T - \{v_7v_8, v_7v_6\})$ that implies $b_{rr2}(T) \leq 2 \leq (n-3)/2$. Hence, we assume $\deg(v_7) \geq 3$. Let $N(v_7) - \{v_8, v_6\} = L_{v_7} \cup \{y_1, \ldots, y_t\}$ if $N(v_7) - \{v_8, v_6\} \neq L_{v_7}$ and $L_{v_7} = \{x_1, \ldots, x_k\}$ when $L_{v_7} \neq \emptyset$. Suppose that $B = \{v_7y_1, \ldots, v_7y_t, v_7v_6, v_7v_8\}$ and f is a $\gamma_{rr2}(T - B)$-function. Clearly, $|B| < (n-3)/2$ and we may assume that $|f(v_i)| = 1$ for $1 \leq i \leq 7$. Then the function $g : V(T) \to \mathscr{P}(\{1,2\})$ defined by $g(v_7) = g(v_4) = g(v_1) = \{1,2\}, g(v_6) = g(v_5) = g(v_3) = g(v_2) = \emptyset$ and $g(x) = f(x)$ otherwise, is a restrained 2-rainbow dominating function of T of weight less than $\omega(f)$ and so $b_{rr2}(T) \leq |B| \leq (n-3)/2$. All in all, we have $b_{rr2}(T) \leq (n-3)/2$ as desired.

To prove sharpness, let T be a wounded spider obtained from the star $K_{1,t}, (t \geq 3)$ by subdividing $t-2$ edges. It is easy to see that $n(T) = 2t-1$, $\gamma_{rr2}(T) = 2t-2$ and $b_{rr2}(T) = t-2 = (n-3)/2$. □

Theorem 2.3. *Let xyz be a path of length 2 in the graph G with $\delta(G) \geq 2$ and $\deg(y) \geq 3$. Then*

$$b_{rr2}(G) \leq \deg(x) + \deg(y) + \deg(z) - 4 - \pi(x,z)$$

where $\pi(x,z) = 1$ if $xz \in E(G)$ and $\pi(x,z) = 0$ otherwise.

Proof. Let $X \subseteq E(G)$ be the set consisting of all the edges incident with x, y or z with exception of the edges xy, yz. Clearly, $|X| = \deg(x)+\deg(y)+\deg(z)-4$ when $xz \notin E(G)$ and $|X| = \deg(x)+\deg(y)+\deg(z)-5$ when $xz \in E(G)$. Let $G_1 = G - X$ be the graph obtained from G by removing the edges of X. In G_1, the path xyz is a component. We show that $\gamma_{rr2}(G_1) > \gamma_{rr2}(G)$ implying that $b_{rr2}(G) \leq |X|$. Let f be a $\gamma_{rr2}(G_1)$-function. Since the path xyz is a component of G_1, we have $|f(x)| + |f(y)| + |f(z)| = 3$. If $xz \in E(G)$, then define $g : V(G) \to \mathscr{P}(\{1,2\})$ by $g(y) = \{1,2\}$, $g(x) = g(z) = \emptyset$ and $g(w) = f(w)$ for $w \in V(G) - \{x, y, z\}$. It is easy to see that g is a restrained 2-rainbow dominating function of G of weight less than $\omega(f)$ as desired.

Now let $xz \notin E(G)$. If there exists a vertex $u \in N(y) - \{x,z\}$ with $f(u) = \emptyset$, then the function $g : V(G) \to \mathscr{P}(\{1,2\})$ defined by $g(x) = \{1\}$, $g(z) = \{2\}$, $g(y) = \emptyset$ and $g(w) = f(w)$ for $w \in V(G) - \{x, y, z\}$, is a restrained 2-rainbow dominating function of G of weight less than $\omega(f)$ as desired. If there are two vertices $u \in N(x) - \{y\}$ and $v \in N(z) - \{y\}$ with $f(u) = f(v) = \emptyset$, then define $g : V(G) \to \mathscr{P}(\{1,2\})$ by $g(y) = \{1,2\}$, $g(x) = g(z) = \emptyset$ and $g(w) = f(w)$ for $w \in V(G) - \{x, y, z\}$. Clearly, g is a restrained 2-rainbow dominating function of G of weight less than $\omega(f)$ as desired. Since $\delta(G) \geq 2$, we thus can assume that $f(x_1) \neq \emptyset$ for a vertex $x_1 \in N(x)-\{y\}$ or $f(z_1) \neq \emptyset$ for a vertex $z_1 \in N(z)-\{y\}$, say $f(x_1) \neq \emptyset$ for a vertex $x_1 \in N(x)-\{y\}$. In addition, we may assume that $f(u) \neq \emptyset$ for each vertex $u \in N(y) - \{x,z\}$. Suppose that $u_1 \in N(y) - \{x,z\}$ (possibly $u_1 = x_1$). Assume, without loss of generality, that $1 \in f(u_1)$. Define $g : V(G) \to \mathscr{P}(\{1,2\})$ by $g(x) = g(y) = \emptyset, g(z) = \{2\}, g(x_1) = \{1,2\}$, and $g(w) = f(w)$ for $w \in V(G) - \{x, y, z, x_1\}$. It is easy to see that g is a restrained 2-rainbow dominating function of G with weight less than $\omega(f)$ as desired. This completes the proof. □

Corollary 2.4. *Let G be a connected graph of order $n \geq 6$ with $\delta(G) \geq 2$. Then*

$$b_{rr2}(G) \leq \delta(G) + 2\Delta(G) - 4.$$

The bound is sharp for cycles C_{3k+1} and C_{3k+2} where $k \geq 2$.

Proof. If $\Delta(G) = 2$, then the result is immediate by Corollary 1.2. Let $\Delta(G) \geq 3$. Assume that x is a vertex of minimum degree $\delta(G)$ such that x is adjacent to a vertex y of degree greater than $\delta(G)$. Since $\deg(y) \geq 3$, there is a path xyz in G satisfying the condition of Theorem 2.3 and the result follows by Theorem 2.3. □

Proposition 2.5. *Let xyz be a path of length 2 in the graph G such that $(N_G(x) \cap N_G(y)) \setminus \{z\} \neq \emptyset$. Then*

$$b_{rr2}(G) \leq \deg(x) + \deg(y) + \deg(z) - |N_G(x) \cap N_G(y)| - |N_G(x) \cap N_G(z)| - 2 - \pi(x,z)$$

where $\pi(x,z) = 1$ if $xz \in E(G)$ and $\pi(x,z) = 0$ otherwise.

Proof. Let $X \subseteq E(G)$ be the set consisting of all the edges incident with x, y or z with exception of all edges between y and $N_G(x) - \{z\}$ and all edges between z and $N_G(x) - \{y\}$. Clearly, $|X| = \deg(x) + \deg(y) + \deg(z) - |N_G(x) \cap N_G(y)| - |N_G(x) \cap N_G(z)| - 3$ if $xz \in E(G)$ and $|X| = \deg(x) + \deg(y) + \deg(z) - |N_G(x) \cap N_G(y)| - |N_G(x) \cap N_G(z)| - 2$ when $xz \notin E(G)$. Let $G_1 = G - X$ be the graph obtained from G by removing the edges of X. In G_1, the vertex x is isolated and the neighbors of y or z lie in $N_G(x)$. We show that $\gamma_{rr2}(G_1) > \gamma_{rr2}(G)$ that implies $b_{rr2}(G) \leq |X|$ as desired. Let f be a $\gamma_{rr2}(G_1)$-function. Since x is isolated in G_1, we may assume that $f(x) = \{1\}$. If $f(y) = \emptyset$, then the function $g : V(G) \to \mathscr{P}(\{1,2\})$ defined by $g(x) = \emptyset$ and $g(w) = f(w)$ for $w \in V(G) - \{x\}$, is a restrained 2-rainbow dominating function of G of weight less than $\omega(f)$ as desired. Assume that $|f(y)| \geq 1$. If $f(y) = \{1,2\}$, then the function $g : V(G) \to \mathscr{P}(\{1,2\})$ defined by $g(y) = \{2\}$ and $g(w) = f(w)$ for $w \in V(G) - \{y\}$, is a restrained 2-rainbow dominating function of G with weight less than $\omega(f)$ as desired.

Let $|f(y)| = 1$. Assume, without loss of generality, that $f(y) = \{1\}$. If $f(z) = \emptyset$, then the function $g : V(G) \to \mathscr{P}(\{1,2\})$ given by $g(x) = \emptyset$ and $g(w) = f(w)$ for $w \in V(G) - \{x\}$ is a restrained 2-rainbow dominating function of G with weight less than $\omega(f)$ as desired. Suppose $|f(z)| \geq 1$. If there exists a vertex $w_1 \in (N_G(x) \cap N_G(y))$ with $f(w_1) = \emptyset$, then the function g defined by $g(x) = \{1\}, g(z) = \{2\}$, $g(y) = \emptyset$ and $g(w) = f(w)$ for $w \in V(G) - \{x, y, z\}$ is a restrained 2-rainbow dominating function of G with weight less than $\omega(f)$ as desired. Hence, we assume $f(w) \neq \emptyset$ for each $w \in (N_G(x) \cap N_G(y)) - \{z\}$. Assume that $w \in (N_G(x) \cap N_G(y)) - \{z\}$. Define the function g by $g(w) = \{1,2\}, g(x) = g(y) = \emptyset$ and $g(a) = f(a)$ for $a \in V(G) - \{w, x, y\}$. Clearly, g is a restrained 2-rainbow dominating function of G with weight less than $\omega(f)$ as desired.

Thus $\gamma_{rr2}(G_1) > \gamma_{rr2}(G)$ implying that $b_{rr2}(G) \le |X|$ and the proof is complete. □

3. Complete graphs and complete bipartite graphs

In this section, we determine the restrained 2-rainbow bondage number of complete graphs and complete bipartite graphs.

Observation 3.1. *For every graph G with $\gamma_{rr2}(G) = \gamma_{r2}(G) < n$, $b_{r2}(G) \ge b_{rr}(G)$.*

Proof. For any $b_{r2}(G)$-set B, we have $\gamma_{rr2}(G-B) \ge \gamma_{r2}(G-B) > \gamma_{r2}(G) = \gamma_{rr2}(G)$ that implies $b_{r2}(G) \ge b_{rr2}(G)$. □

Proposition 3.2. *For $n \ge 9$,*

$$b_{rr2}(K_n) = \left\lceil \frac{2n}{3} \right\rceil.$$

Proof. By Observation 3.1 and Theorem E, we have $b_{rr2}(K_n) \le \left\lceil \frac{2n}{3} \right\rceil$. Now we show that $b_{rr2}(K_n) \ge \lceil \frac{2n}{3} \rceil$. Let $V(K_n) = \{x_1, x_2, \ldots, x_n\}$ be the vertex set of K_n and let B be a $b_{rr2}(K_n)$-set. Assume, to the contrary, that $|B| < \lceil \frac{2n}{3} \rceil$. It follows from Theorem E that $\gamma_{r2}(K_n - B) = 2$. By Theorem F, $\Delta(K_n - B) = n-1$ or there are two vertices, say x_1, x_2, such that $\{x_3, \ldots, x_n\} \subseteq N(x_1) \cap N(x_2)$. First let $\Delta(K_n - B) = n-1$ and $\deg_{K_n - B}(x_1) = n-1$. If $\delta(K_n - B) \ge 2$, then clearly $\gamma_{rr2}(K_n - B) = 2$ which is a contradiction. Hence, $\deg_{K_n-B}(x_i) = 1$ for some $2 \le i \le n$ that implies $|B| \ge n-2 \ge \lceil \frac{2n}{3} \rceil$, a contradiction. Now let $\Delta(K_n - B) \le n-2$. Then $\{x_3, \ldots, x_n\} \subseteq N(x_1) \cap N(x_2)$. If $\delta(K_n - B) \ge 3$, then clearly $\gamma_{rr2}(K_n - B) = 2$ which is a contradiction. Assume that $\deg_{K_n-B}(x_i) = 2$ for some $3 \le i \le n$, say $i = 3$. It follows that $\{x_3x_4, \ldots, x_3x_n\} \subseteq B$. Since $\Delta(K_n - B) \le n-2$, we deduce that $x_1x_2 \in B$ and so $|B| \ge n-2 \ge \lceil \frac{2n}{3} \rceil$, a contradiction. Thus $b_{rr2}(K_n) = \lceil \frac{2n}{3} \rceil$ and the proof is complete. □

Proposition 3.3. *For $3 \le n \le 8$,*

$$b_{rr2}(K_n) = n-2.$$

Proof. If $n = 3$, then we have $b_{rr2}(K_3) = 1$. Let $n \ge 4$ and $V(K_n) = \{x_1, x_2, \ldots, x_n\}$ be the vertex set of K_n. It follows from Theorem D that $\gamma_{rr2}(K_n - \{x_1x_2, \ldots, x_1x_{n-1}\}) \ge 3$. Also the function f defined by $f(x_1) = \{1\}$, $f(x_2) = \{1,2\}$ and $f(x) = \emptyset$ otherwise, is an R2RDF of $K_n - B$ of weight 3 that implies $\gamma_{rr2}(K_n - \{x_1x_2, \ldots, x_1x_{n-1}\}) = 3 > \gamma_{rr2}(K_n)$. Hence $b_{rr2}(K_n) \le n-2$. If $n = 4$, then it follows from Theorem D that $\gamma_{rr2}(K_4 - e) = 2$ for each $e \in E(K_4)$ and so $b_{rr2}(K_4) = 2$. If $n = 5$, then clearly for every two edges $e_1, e_2 \in E(K_5)$, we have $\Delta(K_5 - \{e_1, e_2\}) = 4$ and $\delta(K_5 - \{e_1, e_2\}) \ge 2$. Hence, $\gamma_{rr2}(K_5 - \{e_1, e_2\}) = 2$ by Theorem D that implies $b_{rr2}(K_5) = 3$. Let $6 \le n \le 8$. Since $n-2 = \lceil \frac{2n}{3} \rceil$, using an argument similar to that described in the proof of Proposition 3.2 leads to $b_{rr2}(K_n) = n-2$. □

Proposition 3.4. *For integers* $2 \le p \le q$ *and* $q \ne 2$,

$$b_{rr2}(K_{p,q}) = \begin{cases} 2 & \text{if } p = 2 \\ 3 & \text{if } p = 3 \\ p-1 & \text{otherwise.} \end{cases}$$

Proof. Let $X = \{x_1, x_2, \dots, x_p\}$ and $Y = \{y_1, y_2, \dots, y_q\}$ be the partite sets of $K_{p,q}$, and let B be a $b_{rr2}(K_{p,q})$-set. We note that $\gamma_{rr2}(K_{p,q}) = 4$ for $q \ge p \ge 2$.

First let $p = 2$. For any edge $x_i y_j \in E(K_{2,q})$, say $i = 1, j = 1$, the function $f : V(K_{2,q}) \to \mathscr{P}(\{1,2\})$ defined by $f(y_1) = \{1\}, f(y_2) = \{2\}, f(x_1) = \{1,2\}$ and $f(u) = \emptyset$ otherwise, is a $\gamma_{rr2}(K_{2,q} - x_1 y_1)$-function and so $\gamma_{rr}(K_{p,q} - x_1 y_1) = 4$. It follows that $b_{rr2}(K_{2,q}) \ge 2$. On the other hand, it is clear that $\gamma_{rr2}(K_{2,q} - \{x_1 y_1, x_2 y_1\}) = 5$ that implies $b_{rr2}(K_{2,q}) = 2$.

Now let $p = 3$. Let e_1, e_2 be two arbitrary distinct edges of $K_{3,q}$. If e_1 and e_2 have a common endpoint, then we may assume, without loss of generality, that $e_1 = x_1 y_1, e_2 = x_2 y_1$ or $e_1 = x_1 y_1, e_2 = x_1 y_2$, and if e_1 and e_2 have no common endpoint, then we may assume that $e_1 = x_1 y_1$ and $e_2 = x_2 y_2$. If $e_1 = x_1 y_1, e_2 = x_2 y_1$ or $e_1 = x_1 y_1, e_2 = x_1 y_2$ and $q = 3$ or $e_1 = x_1 y_1, e_2 = x_2 y_2$ then define $f : V(K_{3,q}) \to \mathscr{P}(\{1,2\})$ by $f(x_1) = f(y_1) = \{1\}, f(x_2) = f(y_2) = \{2\}$ and $f(u) = \emptyset$ otherwise, and if $e_1 = x_1 y_1, e_2 = x_1 y_2$ and $q \ge 4$ then define $f : V(K_{3,q}) \to \mathscr{P}(\{1,2\})$ by $f(x_3) = f(y_q) = \{1,2\}$ and $f(u) = \emptyset$ otherwise. It is easy to see that f is a $\gamma_{rr2}(K_{3,q} - \{e_1, e_2\})$-function of weight 4 and hence $b_{rr2}(K_{3,q}) \ge 3$. It is easy to see that $\gamma_{rr2}(K_{3,q} - \{x_1 y_1, x_2 y_1, x_3 y_1\}) = 5$ that implies $b_{rr2}(K_{3,q}) = 3$.

Finally let $p \ge 4$. Considering $B = \{x_2 y_1, x_3 y_1, \dots, x_p y_1\}$, one can see that $\deg_{K_{p,q} - B}(y_1) = 1$ and $\gamma_{rr2}(K_{p,q} - B) = 5$. It follows that $b_{rr2}(K_{p,q}) \le p - 1$. Now we show that $b_{rr2}(K_{p,q}) \ge p - 1$. Let F be an arbitrary set of edges with $|F| \le p - 2$. It is clear that $\delta(K_{p,q} - F) \ge 2$ and $\deg_{K_{p,q}-F}(x_{i_1}) = \deg_{K_{p,q}-F}(x_{i_2}) = q, \deg_{K_{p,q}-F}(y_{j_1}) = \deg_{K_{p,q}-F}(y_{j_2}) = p$ for some $1 \le i_1, i_2 \le p$ and $1 \le j_1, j_2 \le q$. Define $f : V(K_{p,q}) \to \mathscr{P}(\{1,2\})$ by $f(x_{i_1}) = f(y_{j_1}) = \{1,2\}$ and $f(u) = \emptyset$ otherwise. It is easy to see that f is a $\gamma_{rr2}(K_{p,q} - B)$-function of weight 4 and hence $b_{rr2}(K_{p,q}) \ge p - 1$. Thus $b_{rr2}(K_{p,q}) = p - 1$ when $p \ge 4$ and the proof is complete. □

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