TAMKANG JOURNAL OF MATHEMATICS
Volume 49, Number 1, 25-34, March 2018
doi:10.5556/j.tkjm.49.2018.2369

This paper is available online at http://journals.math.tku.edu.tw/index.php/TKJM/pages/view/onlinefirst

SOME APPLICATIONS OF CERTAIN INTEGRAL OPERATORS INVOLVING FUNCTIONS WITH BOUNDED RADIUS ROTATIONS

KHALIDA INAYAT NOOR, BUSHRA MALIK AND SYED ZAKAR HUSSAIN BUKHARI

Abstract. Integral transforms map equations from their original domains into others where manipulations and solutions may be much easier than in original domains. To get back in the original environment, we use the idea of inverse of the integral transform. A function analytic and locally univalent in a given simply connected domain is said to be of bounded boundary rotation if its range has bounded boundary rotation which is defined as the total variation of the direction angle of the tangent to the boundary curve under a complete circuit.

The main objective of the present article is to study some applications of certain integral operators to functions of bounded radius rotation involving Janowski functions. We discuss some inclusion results under certain assumption on parameters involve in operators as well as in related subclasses of analytic functions. Most of these results are best possible. We also relate our findings with the existing literature of the subjects.

1. Introduction

Let $\mathcal{H}(\mathbb{U})$ represent the class of all analytic functions f defined in the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ and for a positive integer n and $a \in \mathbb{C}$, let

$$\mathcal{H}[a,n] := \left\{ f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots \right\}.$$

Also, the subclass $\mathcal{A}$ of the class $\mathcal{H}[a,n]$ is defined as:

$$\mathcal{A} := \left\{ f \in \mathcal{H}[0,1] : f'(0) = 1 \right\}. \tag{1.1}$$

The class of univalent functions is represented by $\mathcal{S}$ and it is a subclass of the class $\mathcal{A}$, whereas, $\mathcal{S}^*, \mathcal{C}, \mathcal{K}$ and $\mathcal{Q}$ are the well-known classes of starlike, convex, close-to-convex and quasi-convex functions respectively. The Hadamard product or convolution for f and g is given by

$$\left(f * g\right)(z) = z + a_2 b_2 z^2 + a_3 b_3 z^3 + \dots \qquad (z \in \mathbb{U}),$$

Received December 3, 2016, accepted September 20, 2017.
2010 *Mathematics Subject Classification*. Primary 30C45, Secondary 30C80.
Key words and phrases. Hypergeometric functions, integral operator, differential subordination.
Corresponding author: Syed Zakar Hussain Bukhari.

where f is defined by (1.1) and $g(z) = z + a_2z^2 + a_3z^3 + ...$ For $f, g \in \mathcal{H}(\mathbb{U})$, we say that f *is subordinate to* g and write as $f(z) \prec g(z)$, if there exists a *Schwarz function* w, that is, $w \in \mathcal{H}(\mathbb{U})$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z)), z \in \mathbb{U}$. Let $\mathcal{P}$ denote the well-known class of *Carathéodory functions* p such that $p \in \mathcal{H}(\mathbb{U})$, with $p(0) = 1$ and $\operatorname{Re} p(z) > 0, z \in \mathbb{U}$. Also $\mathcal{P}(\varrho)$ represents the class of *Carathéodory functions* p such that $p \in \mathcal{H}(\mathbb{U})$ with $p(0) = 1$ and $\operatorname{Re} p(z) > \varrho, (0 \le \varrho < 1, z \in \mathbb{U})$. For details, see [5]. Using subordination defined above, Janowski [6] introduced the class $\mathcal{P}[A, B]$ for $-1 \le B < A \le 1$. A function p analytic in $\mathbb{U}$ such that $p(0) = 1$ belongs to the class $\mathcal{P}[A, B]$, if $p(z) = \frac{1+Aw(z)}{1+Bw(z)}$, where w is a Schwarz function. Geometrically, the image $p(\mathbb{U})$ of $p \in \mathcal{P}[A, B]$ lies inside the open unit disk centered on the real axis with diameter ends at $p(-1)$ and $p(1)$. Clearly $\mathcal{P}[A, B] \subset \mathcal{P}\left(\frac{1-A}{1-B}\right)$. For a general bilinear transformation $\frac{1+A_1z}{1+B_1z}$, $A_1 \in \mathbb{C}$, $B_1 \in [-1, 0]$ and $A_1 \ne B_1$, the class $\mathcal{P}[A_1, B_1]$ consists of functions p with $p(0) = 1$ and $p(z) \prec \frac{1+A_1z}{1+B_1z}$, see [16].

A functions p analytic in $\mathbb{U}$ such that $p(0) = 1$ belongs to the class $\mathcal{P}_k[A, B]$, if and only if

$$p(z) = \frac{1}{2}\int_{-\pi}^{\pi} \frac{1 + Aze^{-it}}{1 + Bze^{-it}} d\varphi(t) \qquad (-1 \le B < A \le 1, k \ge 2, z \in \mathbb{U}),$$

where φ is a real valued function of bounded variation on $[-\pi, \pi]$ that satisfies the conditions $\int_{-\pi}^{\pi} d\varphi(t) = 2$ and $\int_{-\pi}^{\pi} |d\varphi(t)| \le 2$ or equivalently, $p \in \mathcal{P}_k[A, B]$ if and only if there exist p_1, $p_2 \in \mathcal{P}[A, B]$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z) \qquad (k \ge 2, z \in \mathbb{U}). \tag{1.2}$$

From (1.2), we note that $\mathcal{P}_k[1 - 2\varrho, -1] = \mathcal{P}_k(\varrho)$, see [20] and $\mathcal{P}_k[1, -1] = \mathcal{P}_k$, for reference, see [23]. The classes $V_k[A, B]$ and $R_k[A, B]$ are related to the class $\mathcal{P}_k[A, B]$ can be defined as:

$$f \in V_k[A, B] \iff \frac{(zf'(z))'}{f'(z)} \in \mathcal{P}_k[A, B] \text{ and } f \in R_k[A, B] \iff \frac{zf'(z)}{f(z)} \in \mathcal{P}_k[A, B].$$

For various related classes, we refer [1, 2, 3, 8, 9, 12, 13, 14, 15, 24]. For $f \in \mathcal{A}$, we consider the following one parameter families of integral operators

$$I_\eta^\mu f(z) = \frac{(\eta + 1)^\mu}{\Gamma(\mu) z^\eta} \int_0^z t^{(\eta-1)} \left(\log \frac{z}{t}\right)^{(\mu-1)} f(t)dt,$$
$$\Im_\eta^\mu f(z) = \binom{\mu + \eta}{\eta} \frac{\mu}{z^\eta} \int_0^z t^{(\eta-1)} \left(1 - \frac{t}{z}\right)^{(\mu-1)} f(t)dt,$$

and $J_\eta f(z) = \frac{\eta+1}{z^\eta} \int_0^z t^{(\eta-1)} f(t)dt$, where $\mu \ge 0$, $\eta > -1, z \in \mathbb{U}$, Γ denotes the familiar gamma function and $\binom{\mu}{\eta} = \frac{\Gamma(\mu-1)}{\Gamma(\mu-\eta+1)\Gamma(\eta+1)}$. The operator I_η^μ was studied by Patel et al. [22] and the operators I_η^μ, J_η were introduced by Jung et al. [7]. Later Owa [21] investigated certain properties of the operators I^μ and $\Im_\eta^\mu$. For $\eta > -1$, the operator J_η is the generalized Bernardi-Libera-Livingston integral operator studied by Owa and Srivasrava [19] and Srivastava and Owa [25].

For $\eta = 1, 2, \ldots$, the operator J_η was studied by Bernardi [25] and J_1 is the Libera integral operator [10]. Using the integral representation of gamma and beta functions, it can be shown that for $f \in \mathscr{A}$:

$$I_\eta^\mu f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\eta+1}{\eta+n}\right)^{\alpha} a_n z^n, \tag{1.3}$$

$$\Im_\eta^\mu f(z) = \binom{\mu+\eta}{\eta} z\, {}_2F_1(1, \eta, \mu+\eta; z) * f(z), \tag{1.4}$$

and

$$J_\eta f(z) = z + \sum_{\nu=2}^{\infty} \left(\frac{\eta+1}{\eta+\nu}\right) a_\nu z^\nu, \tag{1.5}$$

where $\mu \geq 0$, $\eta > -1, z \in \mathbb{U}$. By virtue of (1.3), (1.4), (1.5) and $\mu \geq 0$, $\eta > -1$, we have the following identities:

$$z\left(I_\eta^\mu f(z)\right)' = (\eta+1) I_\eta^{\mu-1} f(z) - \eta I_\eta^\mu f(z), z \in \mathbb{U} \tag{1.6}$$

and

$$z\left(\Im_\eta^\mu f(z)\right)' = (\mu+\eta) \Im_\eta^{\mu-1} f(z) - (\mu+\eta-1) \Im_\eta^\mu f(z). \tag{1.7}$$

2. Preliminaries

To establish our main results, we will use the following lemmas.

Lemma 1. *Let* δ, γ *with* $\mathrm{Re}(\delta+\gamma) > 0$ *and let* $A_1 \in \mathbb{C}$, $B_1 \in [-1, 0]$ *satisfy either*

$$\mathrm{Re}\left[\delta(1 + A_1 B_1) + \gamma(1 + B_1^2)\right] \geq \left|\delta A_1 + \overline{\delta} B_1 + B_1(\gamma + \overline{\gamma})\right|,$$

when $B_1 \in (-1, 0]$ *or* $\delta(1 + A_1) > 0$ *and* $\mathrm{Re}\left[\delta(1 - A_1) + 2\gamma\right] \geq 0$, *when* $B_1 = -1$. *If* $p(0) = 1$ *satisfies*

$$p(z) + \frac{zp'(z)}{\delta p(z) + \gamma} \prec \frac{1 + A_1 z}{1 + B_1 z} \qquad (z \in \mathbb{U}), \tag{2.1}$$

then $p(z) \prec q(z) \prec \frac{1+A_1 z}{1+B_1 z}, z \in \mathbb{U}$, *where* $q(z) + \frac{zq'(z)}{\delta q(z)+\gamma} = \frac{1+A_1 z}{1+B_1 z}$ $\quad (z \in \mathbb{U})$ *has univalent solution in* $\mathbb{U}$ *given by*

$$q(z) = \begin{cases} \left(\delta \int_0^1 \left[\frac{1+B_1 t z}{1+B_1 z}\right]^{\delta\left(\frac{A_1}{B_1}-1\right)} t^{(\delta+\gamma-1)} dt\right)^{-1} - \frac{\gamma}{\delta}, & B_1 \neq 0 \\ \left(\delta \int_0^1 e^{\delta A_1 (t-1) z} t^{(\delta+\gamma-1)} dt\right)^{-1} - \frac{\gamma}{\delta}, & B_1 = 0. \end{cases} \tag{2.2}$$

In addition q *is the best dominant of* (2.1).

Lemma 2. *Let* α *be a positive measure on the unit interval* $I = [0, 1]$. *Let* $g(t, z)$ *be a function analytic in* $\mathbb{U}$ *for each* $t \in I$ *and integrable in* t *for each* $z \in \mathbb{U}$ *and for almost all* $t \in I$. *Suppose that* $\mathrm{Re}\{g(t, z)\} > 0$ *in* $\mathbb{U}$, $g(t, -r)$ *is real for* r *and* $\mathrm{Re}\left(\frac{1}{g(t,z)}\right) \geq \frac{1}{g(t,-r)}$ *for* $|z| \leq r < 1$ *and* $t \in I$. *If* $g(z) = \int_I g(t, z) d\alpha(t)$, *then* $\mathrm{Re}\left(\frac{1}{g(z)}\right) \geq \frac{1}{g(-r)}, |z| \leq r$.

Lemma 3. *For real or complex numbers a, b, c $(c \neq 0,-1,-2,\dots)$ and* $\operatorname{Re} c > \operatorname{Re} b > 0$, *we have*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b,c;z) \qquad (z \in \mathbb{U}), \tag{2.3}$$

where

$$ {}_2F_1(a,b,c;z) = (1-z)^{-a}\,{}_2F_1\left(a, c-b, c; \frac{z}{1-z}\right) \tag{2.4}$$

and ${}_2F_1(a,b,c;z) = {}_2F_1(b,a,c;z)$.

For the proof of the above lemmas, we refer [11].

Lemma 4. *Let $p \in \mathscr{P}(\varrho)$ with $p(0)=1$, $0 \le \varrho < 1$. Then*

$$\operatorname{Re} p(z) > 2\varrho - 1 + \frac{2(1-\varrho)}{1+|z|} \qquad (z \in \mathbb{U}).$$

Lemma 5. *Let $p_i \in \mathscr{P}(\varrho_i) : p_i(0)=1$, $0 \le \varrho_i < 1$, $i = 1,2$. Then $p_1 * p_2 \in \mathscr{P}(\varrho_3)$ for $\varrho_3 = 1-2(1-\varrho_1)(1-\varrho_2)$. This result is best possible.*

For reference of Lemma 4 and Lemma 5, see [21] and [26] respectively.

3. Main results

Theorem 1. *Let $f \in \mathscr{A}$ be such that*

$$\frac{\Im_\eta^{\mu-2} f(z)}{\Im_\eta^{\mu-1} f(z)} \in \mathscr{P}_k[A_1,B_1] \qquad (z \in \mathbb{U}), \tag{3.1}$$

for $-1 \le B_1 \le 0$, $A_1 \neq B_1$, $k \ge 2$, $\mu \ge 2$, $\eta > -1$ and $\mu+\eta-1 > 0$. Then

$$\frac{\Im_\eta^{\mu-1} f(z)}{\Im_\eta^{\mu} f(z)} \in \mathscr{P}_k[A_1^*,B_1],\; A_1^* = \frac{(\mu+\eta-1)A_1 + B_1}{\mu+\eta} \qquad (z \in \mathbb{U}).$$

(i) *Further, if $f \in \mathscr{A}$ satisfies (3.1), then we have*

$$\frac{\Im_\eta^{\mu-1} f(z)}{\Im_\eta^{\mu} f(z)} \prec \frac{1}{(\mu+\eta)Q(z)} = \widetilde{q}(z),$$

where

$$Q(z) = \begin{cases} \displaystyle\int_0^1 \left[\frac{1+B_1 tz}{1+B_1 z}\right]^{(\mu+\eta-1)\left(\frac{A_1}{B_1}-1\right)} t^{(\mu+\eta-1)}\,dt, & B_1 \neq 0 \\ \displaystyle\int_0^1 t^{(\mu+\eta-1)} e^{(\mu+\eta-1)A_1(t-1)z}\,dt, & B_1 = 0 \end{cases} \tag{3.2}$$

(ii) *Moreover, if* $-1 \le B_1 < 0$, *and* $B_1 < \operatorname{Re} A_1 \le \min\left\{\frac{-2B_1}{\mu+\eta-1}, \frac{\mu+\eta-B_1}{\mu+\eta-1}\right\}$, *then for* $f \in \mathscr{A}$ *satisfying* (3.1), *we have* $\frac{\Im_\eta^{\mu-1} f(z)}{\Im_\eta^{\mu} f(z)} \in \mathscr{P}_k(\varrho_1)$, *where*

$$\varrho_1 = \left[{}_2F_1\left(1, \frac{(\mu+\eta-1)(B_1-A_1)}{B_1}, \mu+\eta+1; \frac{B_1}{B_1-1}\right)\right]^{-1}. \tag{3.3}$$

This result is best possible.

Proof. Let

$$\frac{\Im_\eta^{\mu-1} f(z)}{\Im_\eta^{\mu} f(z)} = p(z) = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z) \qquad (z \in \mathbb{U}), \tag{3.4}$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. On logarithmic differentiation of (3.4) and using (1.7), we obtain

$$\frac{\Im_\eta^{\mu-2} f(z)}{\Im_\eta^{\mu-1} f(z)} = \frac{(\mu+\eta)p(z)-1}{(\mu+\eta-1)} + \frac{zp'(z)}{(\mu+\eta-1)p(z)} \qquad (z \in \mathbb{U}). \tag{3.5}$$

From (3.4), we can write

$$\frac{(\mu+\eta)p(z)-1}{(\mu+\eta-1)} = \left(\frac{k}{4}+\frac{1}{2}\right)\frac{(\mu+\eta)p_1(z)-1}{(\mu+\eta-1)} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{(\mu+\eta)p_2(z)-1}{(\mu+\eta-1)}.$$

Let

$$\frac{(\mu+\eta)p(z)-1}{(\mu+\eta-1)} = \psi(z) = \left(\frac{k}{4}+\frac{1}{2}\right)\psi_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)\psi_2(z) \qquad (z \in \mathbb{U}),$$

where $\psi_i(z) = \frac{(\mu+\eta)p_i(z)-1}{(\mu+\eta-1)}$, $i = 1,2, z \in \mathbb{U}$. Thus (3.5) becomes

$$\frac{\Im_\eta^{\mu-2} f(z)}{\Im_\eta^{\mu-1} f(z)} = \psi(z) + \frac{z\psi'(z)}{(\mu+\eta-1)\psi(z)+1} \qquad (z \in \mathbb{U}). \tag{3.6}$$

On simplification as in [12] and [17], we have

$$\psi(z) + \frac{z\psi'(z)}{D\psi(z)+1} = \frac{k+2}{4}\left(\psi_1(z) + \frac{z\psi_1'(z)}{D\psi_1(z)+1}\right) - \frac{k-2}{4}\left(\psi_2(z) + \frac{z\psi_2'(z)}{D\psi_2(z)+1}\right),$$

where $D = \mu+\eta-1$. From (3.6), it is clear that

$$\psi_i(z) + \frac{z\psi_i'(z)}{(\mu+\eta-1)\psi_i(z)+1} \in \mathscr{P}[A_1, B_1] \text{ for } i = 1,2.$$

Now using Lemma 1 for $\delta = \mu+\eta-1$, $\gamma = 1 : \delta+\gamma = \mu+\eta > 0$, we have

$$\psi_i(z) \prec q(z) \prec \frac{1+A_1 z}{1+B_1 z} \qquad (z \in \mathbb{U}), \tag{3.7}$$

where q is the best dominant and is given by (2.2) for $\delta = \mu+\eta-1$ and $\gamma = 1$. Again from (3.7), we have

$$p_i(z) \prec \frac{1}{(\mu+\eta)Q(z)} \qquad (z \in \mathbb{U}),$$

where Q is given by (3.2). Next we show that

$$\operatorname*{Inf}_{|z|<1}\{\operatorname{Re} q(z)\} = q(-1). \tag{3.8}$$

Setting $a = (\mu+\eta-1)\frac{(B_1-A_1)}{B_1}$, $b = \mu+\eta$ and $c = \mu+\eta+1$ so that $\mu+\eta+1 > \mu+\eta > 0$ in (3.2) and by using (2.3), (3.6) and (3.7), we see that for $B_1 \neq 0$

$$Q(z) = (1+B_1 z)^a \int_0^1 t^{b-1}(1+B_1 tz)^{-a} dt = \frac{\Gamma(b)}{\Gamma(c)}\, {}_2F_1\left(1, a, c; \frac{B_1 z}{B_1 z+1}\right). \tag{3.9}$$

To prove (3.8), we have to show that

$$\operatorname{Re}\left(\frac{1}{Q(z)}\right) \geq \frac{1}{Q(-1)} \qquad (z \in \mathbb{U}).$$

Again from (3.2) and (3.9) for $-1 \leq B_1 < 0$ such that

$$B_1 < \operatorname{Re} A_1 \leq \min\left\{\frac{-2B_1}{\mu+\eta-1}, \frac{\mu+\eta-B_1}{\mu+\eta-1}\right\},$$

we have

$$Q(z) = \int_0^1 g(t,z)\, d\alpha(t) \qquad (z \in \mathbb{U}),$$

where $g(t,z) = \frac{1+B_1 z}{1+(1-t)B_1 z}$ and

$$d\alpha(t) = \frac{\Gamma(b)}{\Gamma(c)\Gamma(c-b)} t^{a-1}(1-t)^{c-a-1} dt$$

is positive measure on $[0,1]$. For $-1 \leq B_1 < 0$, we find that $\operatorname{Re} g(t,z) > 0$, where $g(t,-r)$ is real for $|z| \leq r < 1$, $t \in [0,1]$ and

$$\operatorname{Re}\left(\frac{1}{g(t,z)}\right) = \operatorname{Re}\left(\frac{1+(1-t)B_1 z}{1+B_1 z}\right) \geq \frac{1+(1-t)B_1 r}{1-B_1 r} = \frac{1}{g(t,-r)}.$$

Thus by making use of Lemma 2 and letting $r \to 1^-$, we prove that $\operatorname{Re}\frac{1}{Q(z)} \geq \operatorname{Re}\frac{1}{Q(-1)}$, $z \in \mathbb{U}$. In the case $A_1 = \varrho_0 = \min\left\{\frac{-2B_1}{\mu+\eta-1}, \frac{\mu+\eta-B_1}{\mu+\eta-1}\right\}$, we have the required result by taking $A_1 \to \varrho_0^+$. The result is best possible because of best dominant property of q. □

For special cases, we refer, [15, 19].

Theorem 2. *Let $\mu \geq 2$, $\eta > -1$ and $\mu+\eta > 1$. If $f \in \mathscr{A}$ satisfies*

$$\frac{\Im_\eta^{\mu-2} f(z)}{\Im_\eta^{\mu-1} f(z)} \in \mathscr{P}_k[A_1, B_1] \qquad (z \in \mathbb{U}),$$

for $A_1 \in \mathbb{C}$, $-1 \leq B_1 < 0$, $A_1 \neq B_1$ and $B_1 < \operatorname{Re} A_1 \leq \min\left\{\frac{-2B_1}{\mu+\eta-1}, \frac{\mu+\eta-B_1}{\mu+\eta-1}\right\}$, then $\Im_\eta^\mu f \in R_k(\varrho_0)$, where $\varrho_0 = (\mu+\eta)(1-\varrho_1)$ and ϱ_1 is defined by (3.3).

Proof. From (1.7), we have

$$1+\frac{1}{\mu+\eta}\left\{\frac{z\left(\Im_\eta^\mu f(z)\right)'}{\Im_\eta^\mu f(z)}-1\right\}=\frac{\Im_\eta^{\mu-1} f(z)}{\Im_\eta^\mu f(z)} \qquad (z\in\mathbb{U}).$$

By applying Theorem 1, we have

$$1+\frac{1}{\mu+\eta}\left\{\frac{z\left(\Im_\eta^\mu f(z)\right)'}{\Im_\eta^\mu f(z)}-1\right\}\in\mathscr{P}_k(\varrho_1),$$

where ϱ_1 is defined by (3.3). This yields the required result. □

Theorem 3. *Let $f\in\mathscr{A}$ satisfies*

$$\frac{I_\eta^{\mu-2} f(z)}{I_\eta^{\mu-1} f(z)}\in\mathscr{P}_k[A_1,B_1],\quad (k\geq 2,\mu\geq 2,\eta>-1,z\in\mathbb{U}), \tag{3.10}$$

for $A_1\in\mathbb{C}$, $B_1\in[-1,0]$, $A_1\neq B_1$ and $\mu+\eta-1>0$, then

$$\frac{I_\eta^{\mu-1} f(z)}{I_\eta^{\mu} f(z)}\prec\frac{1}{(\eta+1)Q(z)}=\widetilde{q(z)}\prec\mathscr{P}_k[A_1,B_1] \qquad (z\in\mathbb{U}), \tag{3.11}$$

where $Q(z)=\int_0^1\left[\frac{1+B_1tz}{1+B_1z}\right]^{(\eta+1)\left(\frac{A_1}{B_1}-1\right)}t^\eta dt$ for $B_1\neq 0$ and $Q(z)=\int_0^1 t^\eta e^{(\eta+1)A_1(t-1)z}dt$ when $B_1=0$. Also q is the best dominant of (3.2)*. Furthermore, if $-1\leq \operatorname{Re}A_1\leq 1$, then*

$$\frac{I_\eta^{\mu-1} f(z)}{I_\eta^{\mu} f(z)}\in\mathscr{P}_k(\varrho) \qquad (z\in\mathbb{U}),$$

where $\varrho=\frac{1}{{}_2F_1\left(1,\frac{(\eta+1)(B_1-A_1)}{B_1},\eta+2;\frac{B_1}{B_1-1}\right)}$. The result is best possible.

Proof. Set

$$\frac{I_\eta^{\mu-1} f(z)}{I_\eta^{\mu} f(z)}=p(z)=\left(\frac{k}{4}+\frac{1}{2}\right)p_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)p_2(z) \qquad (z\in\mathbb{U}), \tag{3.12}$$

where p is analytic in $\mathbb{U}$ with $p(0)=1$. One logarithmic differentiation on both sides of (3.12) and using the identity (1.6), we have

$$\frac{I_\eta^{\mu-2} f(z)}{I_\eta^{\mu-1} f(z)}=p(z)+\frac{zp'(z)}{(\eta+1)p(z)} \qquad (z\in\mathbb{U}).$$

From (3.10), we have

$$p(z)+\frac{zp'(z)}{(\eta+1)p(z)}\in\mathscr{P}_k[A_1,B_1].$$

Using (3.12) and the same convolution technique used in Theorem 1, we have

$$p_i(z)+\frac{zp_i'(z)}{(\eta+1)p_i(z)}\in\mathscr{P}[A_1,B_1],\ i=1,2 \qquad (z\in\mathbb{U}). \tag{3.13}$$

Now using Lemma 1 for $\delta=\eta+1$, $\gamma=0$ and $-1\le \operatorname{Re} A_1\le 1$,

$$p_i(z)\prec q(z)\prec\frac{1+A_1z}{1+B_1z} \qquad (z\in\mathbb{U}),$$

where q is best dominant of (3.13) and is given by (2.2), this proves (3.11). Proceeding as in Theorem 1, the remaining part of Theorem 3 follows. □

For $\eta=1$, $A_1=1-2\varrho$, $B_1=-1$, $0\le\varrho<1$ and $k=2$, if $f\in\mathscr{A}$ satisfies

$$\frac{I^{\mu-2}f(z)}{I^{\mu-1}f(z)}\in\mathscr{P}(\varrho),\ \ 0\le\varrho<1 \qquad (z\in\mathbb{U}),$$

then

$$\frac{I^{\mu-1}f(z)}{I^{\mu}f(z)}\in\mathscr{P}(\varrho_1),\text{ where }\ \varrho_1=\frac{1}{{}_2F_1(1,4(1-\varrho),3;\frac{1}{2})}.$$

This improves the result of Owa [18].

Theorem 4. *Let $\mu>1$, $\eta>-1$, $\lambda<1$ and $-1\le B_i<A_i\le 1$ for $i=1,2$. If*

$$(1-\lambda)\frac{\Im_\eta^{\mu-1}f_i(z)}{z}+\lambda\frac{\Im_\eta^{\mu}f_i(z)}{z}\in\mathscr{P}_k[A_i,B_i],\ i=1,2,\ k\ge 2 \qquad (z\in\mathbb{U}), \tag{3.14}$$

then

$$(1-\lambda)\frac{\Im_\eta^{\mu-1}F(z)}{z}+\lambda\frac{\Im_\eta^{\mu}F(z)}{z}\in\mathscr{P}_k(1-2\varrho,-1) \qquad (z\in\mathbb{U}),$$

where $F=f_1*f_2$ and $\varrho=1-\frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left(1-\chi\int_0^1\frac{u^{\chi-1}}{1+u}du\right)$ for $\chi=\frac{\mu+\eta}{1-\lambda}$. The result is sharp when $B_1=B_2=-1$.

Proof. Set $H_i(z)=(1-\lambda)\frac{\Im_\eta^{\mu-1}f_i(z)}{z}+\lambda\frac{\Im_\eta^{\mu}f_i(z)}{z}$, $i=1,2$, $z\in\mathbb{U}$. From (3.14), we see that $H_i\in\mathscr{P}_k[A_i,B_i]\subset\mathscr{P}_k\left(\frac{1-A_i}{1-B_i}\right)$, $i=1,2,z\in\mathbb{U}$. Let

$$H_i(z)=\left(\frac{k}{4}+\frac{1}{2}\right)h_i(z)-\left(\frac{k}{4}-\frac{1}{2}\right)p_i(z),\ i=1,2 \qquad (z\in\mathbb{U}),$$

where $h_i,\ p_i\in\mathscr{P}(\varrho_i)$, $i=1,2$. By making use of (1.7) and (3.4), we obtain

$$\Im_\eta^{\mu}f_i(z)=\chi z^{1-\chi}\int_0^z t^{\chi-1}H_i(t)dt,\ i=1,2,\chi=\frac{\mu+\eta}{1-\lambda},z\in\mathbb{U}. \tag{3.15}$$

Using above equation (3.15), a simple computation shows that

$$\Im_\eta^{\mu}F(z)=\chi z^{1-\chi}\int_0^z t^{\chi-1}H(t)dt,\chi=\frac{\mu+\eta}{1-\lambda},z\in\mathbb{U}, \tag{3.16}$$

where $H(z) = \chi z^{1-\chi} \int_0^z t^{\chi-1}(H_1 * H_2)(t)dt$. For $\chi = \frac{\mu+\eta}{1-\lambda}$, we can also write

$$H(z) = \frac{k+2}{4}\chi z^{1-\chi}\int_0^z t^{\chi-1}(h_1 * h_2)(t)dt - \frac{k-2}{4}\chi z^{1-\chi}\int_0^z t^{\chi-1}(p_1 * p_2)(t)dt. \tag{3.17}$$

From Lemma 5, we have $h_1 * h_1 \in \mathscr{P}(\varrho_3)$ and $p_1 * p_1 \in \mathscr{P}(\varrho_3)$, where $\varrho_3 = 1-2(1-\varrho_1)(1-\varrho_2)$. Now using (3.16), (3.17) and Lemma 4, we have

$$\operatorname{Re}\chi z^{1-\chi}\int_0^z t^{\chi-1}(h_1 * h_2)(t)dt > 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left(1-\chi\int_0^1 \frac{u^{\chi-1}}{1+u}du\right),$$

where $\chi = \frac{\mu+\eta}{1-\lambda}$. Similarly, for $\chi = \frac{\mu+\eta}{1-\lambda}$, we can prove

$$\operatorname{Re}\left\{\chi z^{1-\chi}\int_0^z t^{\chi-1}(p_1 * p_2)(t)dt\right\} > 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left(1-\chi\int_0^1 \frac{u^{\chi-1}}{1+u}du\right).$$

When $B_1 = B_2 = -1$, we consider the functions $f_i \in \mathscr{A}, i = 1,2$, which satisfy the condition (3.14) and is given by

$$\Im_\eta^\mu f_i(z) = \chi z^{1-\chi}\int_0^z t^{\chi-1}\left\{\frac{k+2}{4}\frac{1+A_i t}{1-t} - \frac{k-2}{4}\frac{1+A_i t}{1-t}\right\}dt,$$

where $\chi = \frac{\mu+\eta}{1-\lambda}$. It follows from (3.16) that for $\chi = \frac{\mu+\eta}{1-\lambda}$, we have

$$H(z) = \frac{k+2}{4}\chi\int_0^1 u^{\chi-1}\left(1-A_{1,2}+\frac{A_{1,2}}{1-uz}\right)du - \frac{k-2}{4}\chi\int_0^1 u^{\chi-1}\left(1-A_{1,2}+\frac{A_{1,2}}{1-uz}\right)du.$$

where $A_{1,2} = (1+A_1)(1+A_2)$. Therefore $H(z) \to 1-(1+A_1)(1+A_2)\left(1-\chi\int_0^1 \frac{u^{\chi-1}}{1+u}du\right)$ for $z \to -1^+$ and $\chi = \frac{\mu+\eta}{1-\lambda}$. Thus the proof is completed. □

Acknowledgement

The authors are thankful to the Rector CIIT Islamabad and Worthy Vice Chancellor MUST for promotion of research conducive environment in their relevant Institutions.

References

[1] F. S. M. Al Sarari and S. Latha, *On symmetrical functions with bounded boundary rotation*, J. Math. Comput. Sci., **4(3)**(2014), 494–502

[2] S. Z. H. Bukhari and K. I. Noor, *Some subclasses of analytic functions with bounded Mocanu variation*, Acta Math. Hungarica, **126(3)**(2010), 199–208.

[3] A. Cetinkaya, Y. Kahramaner and Y. Polatoğlu , *Harmonic mappings related to the bounded boundary rotation*, Int. J. Math. Anal., **8(57)**(2014), 2837–2843.

[4] S. D. Bernardi, *The radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc., **24**(1970), 312–318.

[5] A. W. Goodman, *Univalent functions*, Vol. I,II, Mariner Publ Comp. Tampa, Florida, 1983.

[6] W. Janowski, *Some extremal problems for certain families of analytic functions I*, Ann. Polon. Math., **28**(1973), 297–326.

[7] I. B. Jung, Y. C. Kim and H. M. Srivastava, *The hardy space of analytic functions associated with certain one-parameter family of integral operator*, J. Math. Anal. Appl., **176**(1993), 138–147.

[8] S. Kanas and J. Kowalczyk, *A note on Briot-Bouquet-Bernoulli differential subordination*, Comment. Math. Univ. Carolin., **46(2)**(2005), 339–347.

[9] S. Kanas and D. K.-Smęt, *Harmonic mappings related to functions with bounded boundary rotation and norm of the pre-Schwarzian derivative*, Bull. Korean Math. Soc., **51(3)**(2014), 803–812.

[10] R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc., **16**(1965), 755–758.

[11] S. S. Miller and P. T. Mocanu, *Differential subordination theory and applications*, M. Dekker Inc., N.Y., 2000.

[12] K. I. Noor, *On a generalization of $\alpha-$convexity*, J. Inequal. Pure Appl. Math., **8(16)** (2007), 4 pp.

[13] K. I. Noor and S. Z. H. Bukhari, *On analytic functions related with generalized Robertson functions*, Appl. Math. Comput., **215**(2009), 2965-2970.

[14] K. I. Noor and N. E. Cho, *Some convolution properties of certain classes of analytic functions*, Appl. Math. Lett., **21(11)**(2008), 1155–1160.

[15] K. I. Noor and A. Muhammad, *On analytic functions with generalized bounded Mocanu variation*, Appl. Math. Comput., **196**(2008), 802–811.

[16] K. I. Noor, *Applications of certain operators to the classes related with generalized Janowski functions*, Integral Transforms Spec. Funct., **21(8)**(2010), 557–567.

[17] K. I. Noor and S. Hussain, *On certain analytic functions associated with Ruscheweyh derivatives and bounded Mocanu variation*, J. Math. Anal. Appl., **340**(2008), 1145–1152.

[18] S. Owa, *Properties of certain integral operators*, G. Math. J., **2(5)**(1995), 535–545.

[19] S. Owa and H. M. Srivastava, *Some applications of the generalized Libera operator*, Proc. Japan Acad., **62**(1986), 125–128.

[20] K. Padmanabhan and R. Parvatham, *Properties of a class of functions with bounded boundary rotation*, Ann. Polon. Math., **31**(1975), 311–323.

[21] D. Z. Pashkouleva, *The starlikeness and spiral-convexity of certain subclasses of analytic functions in Current Topics in Analytic Function Theory*, 266–273, World Sci., Singapore, 1992.

[22] J. Patel and P. Sahoo, *Some applications of differential subordination to certain one parameter families of integral operators*, Indian J. Pure Appl. Math., **35(10)**(2004), 1167–1177.

[23] B. Pinchuk, *Functions with bounded boundary rotation*, Israel J. Math., **10**(1971), 7–16.

[24] Y. Polatoğlu, M. Bolcal, A. Şen and E. Yavuz, *A study on the generalization of Janowski functions in the unit disk*, Acta. Math. Acad. Paedag Nyíregyháziensis., **22**(2006), 27–31.

[25] H. M. Srivastava and S. Owa, *A certain one-parameter additive family of operators defined on analytic functions*, J. Math. Anal. Appl., **118**(1986), 80–87.

[26] J. Stankiewics and Z. Stankiewics, *Some applications of the Hadamard convolution in the theory of functions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A, **40**(1986), 251–265.

Department of Mathematics, COMSATS Institute of Information Technology, Islamabad, Pakistan.

E-mail: khalidanoor@hotmail.com

Department of Mathematics, COMSATS Institute of Information Technology, Islamabad, Pakistan.

E-mail: bushramalik@comsats.edu.pk

Department of Mathematics, Mirpur University of Science and Technology(MUST) Mirpur-10250(AJK), Pakistan.

E-mail: fatmi@must.edu.pk