A CLASS OF SPACES AND THEIR ANTI SPACES

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Abstract. The present paper is a continuation of the study of $S$-closed and $s$-closed topological spaces as introduced by Thompson [11] and Maio and Noiri [7] respectively. Although there is no relation between compactness with $S$-closedness or $s$-closedness, this paper yields some new characterizations of these concepts in terms of compactness.

1. Introduction

Since the introduction of semi-open sets by N. Levine [6], many mathematicians have introduced many new topological properties, using semi-open sets. Maio and Noiri [7] initiated the study of a class of topological spaces under the terminology “$s$-closed spaces”, which is properly contained in the class of $S$-closed spaces as introduced by Thompson [11] and subsequently studied extensively by many mathematicians. Ganster and Reilly [4] have shown a remarkable result towards the distinction between these concepts that every infinite topological space can be represented as a closed subspace of a connected $S$-closed space which is not $s$-closed. The aim of this paper is to study these topological properties viz. $S$-closedness and $s$-closedness via compactness which reflect the distinction between the concepts of compactness and $S$-closedness or $s$-closedness. This, however, leads us to establish in a straightforward manner certain important characterization theorems of $S$-closed spaces and $s$-closed spaces which are already well-known. In the last section, we introduce and characterize the class of anti-$S$-closed and anti-$s$-closed spaces.

By $(X, T)$ or simply by $X$ we shall denote a topological space, and for a subset $A$ of $X$, the closure of $A$ and the interior of $A$ will be denoted by $\text{cl } A$ and $\text{int } A$ respectively. A subset $A$ of $X$ is said to be semi-open [6] if there exists an open set $U$ of $X$ such that $U \subset A \subset \text{cl } U$. Biswas [2] used semi-open sets to define semi-closed sets and semi-closure of a set. A subset $A$ of $X$ is semi-closed iff $X - A$ is semi-open and the semi-closure of $A$, denoted by $\text{scl } A$, is the intersection of all semi-closed sets containing $A$ [2]. A set which is semi-open as well as semi-closed is said to be a semi-regular set [7], Maio and Noiri [7] characterized semi-regular sets in terms of regular open sets as follows: a set

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A is semi-regular iff there exists a regular open set $U$ of $X$ such that $U \subset A \subset \text{cl } U$. The family of all semi-open (resp. semi-regular, regular-open, regular closed) sets of $X$ will be denoted by $\text{SO}(X)$ (resp. $\text{SR}(X)$, $\text{RO}(X)$, $\text{RC}(X)$). While the collection of all members of $\text{SO}(X)$ (resp. $\text{SR}(X)$, $\text{RO}(X)$, $\text{RC}(X)$) each containing a point $x$ of $X$ will be denoted by $\text{SO}(x)$ (resp. $\text{SR}(x)$, $\text{RO}(x)$, $\text{RC}(x)$). A subset $A$ of $X$ is said to be $S$-closed [9] (resp. $s$-closed [7]) relative to $X$ or simply an $S$-set ($s$-set) iff every cover of $A$ by sets of $\text{SO}(X)$ admits a finite subfamily whose closures (resp. semi-closures) cover $A$. In case $A = X$ and $A$ is an $S$-set ($s$-set), then $X$ is called an $S$-closed [11] (resp. $s$-closed [7]) space.

2. S–Closed and s–Closed Spaces

Analogous to a well known theorem on compactness, Asha Mathur [8] and Maio and Noiri [7] respectively proved that a topological space $X$ is $S$-closed (resp. $s$-closed) iff every regular closed (resp. semi-regular) cover of $X$ has a finite subcover. Although it is well known that compactness and $S$-closedness (resp. $s$-closedness) are independent notions, it is our intention in this section to study such spaces with the help of compactness. An important and useful consequence of such study is to achieve a new approach which not only simplifies (in a straightforward way) the proofs of some well-known characterization theorems of $S$-closed and $s$-closed spaces but also improves some characterization theorem of such spaces. Joseph and Kwack [5] and Ganguly and Basu [3] initiated respectively $(\theta, s)$-continuous function and $\gamma$-continuous function to study $S$-closed (resp. $s$-closed) spaces. Using those functions, we derive that a topological space $X$ is $S$-closed (resp. $s$-closed) iff it is a $(\theta, s)$-continuous (resp. $\gamma$-continuous) image of a compact space. For these purposes we require some definitions and results.

**Definition 2.1.** A filter base $\mathcal{F}$ on $X$ is said to $s$-accumulate [11] (resp. SR-accumulate [7]) to $x \in X$ iff for each $V \in \text{RC}(x)$ (resp. $V \in \text{SR}(x)$) there is an $F \in \mathcal{F}$ satisfying $F \subset V$.

Joseph and Kwack [5] and Maio and Noiri [7] respectively established that $\text{RC}(x) = \{\text{cl } V : V \in \text{SO}(x)\}$ and $\text{SR}(x) = \{\text{scl } V : V \in \text{SO}(x)\}$. Therefore an equivalent formulation of the above definition is that a filter base $\mathcal{F}$ on $X$ is said to have an $s$-accumulation (resp. SR-accumulation) point $x$ iff for each $F \in \mathcal{F}$ and for each $V \in \text{RC}(x)$ (resp. $V \in \text{SR}(x)$), $F \cap V \neq \phi$.

**Definition 2.2.** A filter base $\mathcal{F}$ on $X$ is said to $s$-converge [11] (resp. SR-converge [7]) to $x \in X$ if for each $V \in \text{RC}(x)$ (resp. $V \in \text{SR}(x)$) there is an $F \in \mathcal{F}$ satisfying $F \cap V \neq \phi$.

The corresponding definitions for nets are obvious.

**Definition 2.3.** Let $(X, T)$ be a topological space. We define $T_{\text{RC}}$-topology (resp. $T_{\text{SR}}$-topology) on $X$ as the topology on $X$ which has $\text{RC}(X)$ (resp. $\text{SR}(X)$) as a subbase. It is to be noted that intersection of even two regular closed (resp. semi-regular) sets may
fail to be regular closed (resp. semi-regular). Therefore these collections do not form a base for topology.

**Definition 2.4.** A filter base \( \mathcal{I} \) in \((X, T)\) is said to be \( T_{RC} \)-convergent (resp. \( T_{SR} \)-convergent) to \( x \) if \( \mathcal{I} \) converges to \( x \) in \((X, T_{RC})\) (resp. in \((X, T_{SR})\)).

**Proposition 2.5.** A filterbase \( \mathcal{I} \) in \((X, T)\) \( s \)-converges (resp. SR-converges) to \( x \) iff \( \mathcal{I} \) \( T_{RC} \)-converges (resp. \( T_{SR} \)-converges) to \( x \).

**Proof.** Straightforward.

The corresponding proposition using nets is also obvious.

**Definition 2.6.** A filter base \( \mathcal{I} \) on \((X, T)\) is said to have \( x \) as a \( T_{RC} \)-accumulation (resp. \( T_{SR} \)-accumulation) point if \( x \) is an accumulation point of \( \mathcal{I} \) in \((X, T_{RC})\) (resp. in \((X, T_{SR})\)).

Similarly, \( T_{RC} \) (resp. \( T_{SR} \))-accumulation point of a net can be defined.

**Remark 2.7.** Every \( T_{RC} \)-accumulation (resp. \( T_{SR} \)-accumulation) point of a filter or a net is also an \( s \)-accumulation (resp. SR-accumulation) point. But the converse is not necessarily true follows from the following example.

**Example 2.8.** Let \( X = \mathbb{R} \), be the set of reals with the usual topology then \((X, T_{RC})\) (resp. \((X, T_{SR})\)) is clearly the discrete topology. Let \( x_n = (-1)^n \frac{1}{n} \) for each positive integer \( n \), then the net \( \{x_n\}_{n \in \mathbb{N}} \) and the filter \( \mathcal{I} \) based on the net \( \{x_n\}_{n \in \mathbb{N}} \) both have 0 as the \( s \)-accumulation (resp. SR-accumulation) point. But 0 is not a \( T_{RC} \)-accumulation (resp. \( T_{SR} \)-accumulation) point of \( \{x_n\}_{n \in \mathbb{N}} \) or \( \mathcal{I} \).

**Theorem 2.9.** A topological space \((X, T)\) is \( S \)-closed iff \((X, T_{RC})\) is compact.

**Proof.** Let \((X, T)\) be \( S \)-closed. Then every regular closed cover of \( X \) has a finite subcover. But the collection of all regular closed sets of \((X, T)\) is a subbase for \( T_{RC} \). Therefore every subbasic open cover of \((X, T_{RC})\) has a finite subcover. By Alexander subbase theorem, \((X, T_{RC})\) is compact.

Conversely, let \((X, T_{RC})\) be compact. Since \( RC(X) \subset T_{RC} \), every regular closed cover of \((X, T)\) has a finite subcover. So \((X, T)\) is \( S \)-closed by [Theorem 1 of Asha Mathur [8]].

**Theorem 2.10.** A topological space \((X, T)\) is \( s \)-closed iff \((X, T_{SR})\) is compact.

**Proof.** It is similar to Theorem 2.9 and is thus omitted.

The following theorem for \( S \)-closed spaces improves Theroem 1 of Asha Mathur [8], Theorem 1.3 of T. Noiri [9] and Theorem 2 of Thompson [11]; and the theorem for \( s \)-closed spaces improves proposition 3.1 of Maio and Noiri [7].

**Theorem 2.11.** Let \((X, T)\) be a topological space. Then the following are equivalent.
i) \((X, T)\) is \(S\)-closed (resp. \(s\)-closed)

ii) every proper regular open (resp. Semi-regular) set is an \(S\)-set (resp. \(s\)-set) in \((X, T)\).

iii) every closed set of \((X, T_{RC})\) [resp. \((X, T_{SR})\)] is an \(S\)-set (resp. \(s\)-set) in \((X, T)\).

iv) every family of regular open (resp. Semi-regular) subsets of \((X, T)\) with the finite intersection property (f.i.p. for short) has non-void intersection.

v) every family of closed subsets of \((X, T_{RC})\) [resp. \((X, T_{SR})\)] with the f.i.p. has non-void intersection.

vi) every filter base in \((X, T)\) has an \(s\)-accumulation (resp. \(SR\)-accumulation) point.

vii) every net in \((X, T)\) has an \(s\)-accumulation (resp. \(SR\)-accumulation) point.

viii) every filter base in \((X, T)\) has a \(T_{RC}\)-accumulation (resp. \(T_{SR}\)-accumulation) point.

ix) every net in \((X, T)\) has a \(T_{RC}\)-accumulation (resp. \(T_{SR}\)-accumulation) point.

x) every net in \((X, T)\) has a \(T_{RC}\)-convergent (resp. \(T_{RS}\) convergent) subnet.

xi) every filter \(\zeta\) in \((X, T)\) has a sub-ordinate filter \(\zeta_0\) of \(\zeta\) which is \(T_{RC}\)-convergent (resp. \(T_{SR}\)-convergent).

xii) every universal net in \((X, T)\) is \(T_{RC}\)-convergent (resp. \(T_{SR}\)-convergent).

xiii) every ultrafilter in \((X, T)\) is \(T_{RC}\)-convergent (resp. \(T_{SR}\)-convergent).

Proof. The facts discussed above prove the theorem immediately.

Definition 2.12. A function \(f : (X, T) \rightarrow (Y, T')\) is said to be \((\theta, s)\)-continuous [5] (resp. \(\gamma\)-continuous [3]) if for each \(x \in X\) and each \(W \in SO(f(x))\), there is an open set \(V\) containing \(x\) such that \(f(V) \subseteq cl W\) (resp. \(f(V) \subseteq scl W\)).

Since \(RC(x) = \{cl W : W \in SO(x)\}\) [5] (resp. \(SR(x) = \{scl W : W \in SO(x)\}\) [7]), the above definition can equivalently be stated as: a function \(f : (X, T) \rightarrow (Y, T')\) is \((\theta, s)\)-continuous (resp. \(\gamma\)-continuous) iff \(f^{-1}(W)\) is open in \(X\), for every \(W \in RC(Y)\) (resp. \(W \in SR(Y)\)).

Theorem 2.13. A topological space \((X, T)\) is \(S\)-closed iff it is a \((\theta, s)\)-continuous image of a compact space.

Proof. Let \((X, T)\) be \(S\)-closed. Then by Theorem 2.9, \((X, T_{RC})\) is compact. Let \(i : (X, T_{RC}) \rightarrow (X, T)\) be the identity function, which is obviously \((\theta, s)\)-continuous. Therefore there exist a compact space and a \((\theta, s)\)-continuous function such that the \(S\)-closed space \((X, T)\) is the \((\theta, s)\)-continuous image of a compact space.

Conversely, let \(f : (Y, T^*) \rightarrow (X, T)\) be \((\theta, s)\)-continuous surjection and \((Y, T^*)\) be compact. Let \(\{V_\alpha : \alpha \in I\}\) be a cover of \((X, T)\) be regular closed sets of \((X, T)\). Then \(\{f^{-1}(V_\alpha) : \alpha \in I\}\) is a cover of the compact space \((Y, T^*)\) by open sets of \((Y, T^*)\). Therefore there exists a finite subset \(I_0\) of \(I\) such that \(\{f^{-1}(V_\alpha) : \alpha \in I_0\}\) covers \(Y\) and hence \(\{V_\alpha : \alpha \in I_0\}\) covers \(X\). Therefore \((X, T)\) is \(S\)-closed.

Theorem 2.14. A topological space \((X, T)\) is \(s\)-closed iff it is a \(\gamma\)-continuous image of a compact space.

Proof. The proof is similar to Theorem 2.13 and thus omitted.
Maio and Norir [7] initiated quasi-irresolute function and established that such functions preserve \( s \)-sets. Here we introduce a weaker form of quasi-irresolute function which also has the same property.

**Definition 2.15.** ([7]) A function \( f : X \to Y \) is said to be quasi-irresolute if for each \( x \in X \) and each \( V \in \text{SO}(f(x)) \) there exists a \( U \in \text{SO}(x) \) such that \( f(U) \subset \text{scl} \ V \).

**Definition 2.16.** A function \( f : (X, T) \to (Y, T') \) is said to be weakly quasi-irresolute if \( f : (X, T_{\text{SR}}) \to (Y, T'_{\text{SR}}) \) is continuous.

**Theorem 2.17.** Every quasi-irresolute function is weakly quasi-irresolute.

**Proof.** Let \( f : (X, T) \to (Y, T') \) be quasi-irresolute. Then \( f^{-1}(V) \) is semi-regular set in \( (X, T) \) for every \( V \in \text{SR}(Y) \). But the collection of all semi-regular sets of \( (Y, T'_{\text{SR}}) \) is a subbase for \( (Y, T'_{\text{SR}}) \). Hence \( f : (X, T_{\text{SR}}) \to (Y, T'_{\text{SR}}) \) is continuous. Therefore \( f : (X, T) \to (Y, T') \) is weakly quasi-irresolute.

**Remark 2.18.** That the converse of the above theorem is not necessarily true follows from the following example.

**Example 2.19.** Let \( f : R \to R \) where \( R \) is the set of reals with the usual topology \( T \) be a function defined by

\[
f(x) = \begin{cases} 
  r_1 & \text{if } x \text{ is rational,} \\
  r_2 & \text{if } x \text{ is irrational, with } r_2 > r_1 > 0.
\end{cases}
\]

Clearly \( T_{\text{SR}} \) is the discrete topology; therefore \( f : (R, T_{\text{SR}}) \to (R, T_{\text{SR}}) \) is continuous and hence \( f \) is weakly quasi-irresolute function. But \( f \) is not a quasi-irresolute function. In fact if \( \delta \) be such that \( 0 < \delta < \frac{|r_2 - r_1|}{2} \), then the open interval \( J = (r_2 - \delta, r_2 + \delta) \) is a semi-regular set in \( (R, T) \); but \( f^{-1}(J) \) is the set of all irrationals of \( R \). Hence \( f^{-1}(J) \) is not even a semi-open set in \( R \).

Hence we get an improved result of Corollary 5.1 of Maio and Norir [7].

**Theorem 2.20.** If \( f : (X, T) \to (Y, T') \) is weakly quasi-irresolute and \( K \) is an \( s \)-set of \( (X, T) \), then \( f(K) \) is an \( s \)-set in \( Y \).

**Proof.** Let \( \{U_\alpha : \alpha \in I\} \) be a cover of \( f(K) \) by semi-regular sets of \( (Y, T') \). Since \( f : (X, T) \to (Y, T') \) is weakly quasi-irresolute, \( \{f^{-1}(U_\alpha) : \alpha \in I\} \) is a cover of \( K \) by open sets of \( (X, T_{\text{SR}}) \). By Theorem 2.10, \( K \) is compact in \( (X, T_{\text{SR}}) \). Therefore, there exists a finite subset \( I_0 \) of \( I \) such that \( K \subset \bigcup_{\alpha \in I_0} f^{-1}(U_\alpha) \). Which implies \( f(K) \subset \bigcup_{\alpha \in I_0} U_\alpha \). Therefore \( f(K) \) is an \( s \)-set in \( Y \).

**Corollary 2.21.** If \( f : (X, T) \to (Y, T') \) is weakly quasi-irresolute surjection and \( (X, T) \) is s-closed then \( (Y, T') \) is also s-closed.

**Definition 2.22.** ([7]) A space \( (X, T) \) is said to be weakly Hausdorff if every point of \( X \) is the intersection of regular closed sets of \( X \).
The following theorem improves Corollary 5.2 of Maio and Noiri [7].

**Theorem 2.23.** Let $f : (X, T) \rightarrow (Y, T')$ be weakly quasi-irresolute, $(X, T)$ is $s$-closed and $(Y, T')$ be weakly Hausdorff. Then the image of each semi-$\theta$-closed set [7] in $X$ is semi-$\theta$-closed in $Y$.

**Proof.** Let $K$ be a semi-$\theta$-closed set in $(X, T)$. Then by Proposition 4.2 of Maio and Noiri [7], $K$ is an $s$-set in $X$. By Theorem 2.20, $f(K)$ is an $s$-set in $Y$. Therefore by Proposition 4.3 of Maio and Noiri [7], $f(K)$ is semi-$\theta$-closed set in $(Y, T)$.

3. Anti-$S$-Closed and Anti-$s$-Closed Spaces

P. Bankston [1] studied topological anti-properties. Reilly & Vamanamurthy [10] extended these concepts to semi-compact spaces. In a similar fashion [10], here we introduce and characterize two new topological anti-properties under the terminology ‘anti-$S$-closedness’ and ‘anti-$s$-closedness’ along with their mutual relationships.

**Definition 3.1.** A topological space $(X, T)$ is said to be anti-$S$-closed (resp. anti-$s$-closed) if only the finite subsets of $(X, T)$ are $S$-sets (resp. $s$-sets) of $(X, T)$.

An infinite subset $A$ of $(X, T)$ is said to be anti-$S$-closed (resp. anti-$s$-closed) relative to $X$ if only the finite subsets of $A$ are $S$-sets (resp. $s$-sets) in $(X, T)$.

**Theorem 3.2.** A topological space $(X, T)$ is anti-$S$-closed iff for every infinite set $N$ of $X$ and each point $x$ of $X$, there exists a regular closed set $R$ containing $x$ such that $N \setminus R$ is not an $S$-set in $(X, T)$.

**Proof.** Let the given condition hold. We have to show that $(X, T)$ is anti-$S$-closed. Let $N$ be any infinite set and let $x \in X$. Then by hypothesis, there exists a $R \in RC(x)$ such that $N \setminus R$ is not an $S$-set. Therefore there exists a cover $A$ of $N \setminus R$ by regular closed sets of $X$ which has no finite subcover. So $N$ is not an $S$-set in $(X, T)$. Therefore $(X, T)$ is anti-$S$-closed.

Conversely, let $(X, T)$ be anti-$S$-closed space. Let $N$ be any infinite subset of $X$ and let $x$ be any point of $X$. Then by definition of anti-$S$-closed space, $N$ and hence $N \cup \{x\}$ is not an $S$-set. Therefore there exists a cover $A$ of $N \cup \{x\}$ by regular closed sets which has no finite subcover. Hence there exists a member $R \in A$ such that $x \in R$. So $N \setminus R$ is not an $S$-set in $(X, T)$.

**Theorem 3.3.** A topological space $(X, T)$ is anti-$s$-closed iff for every infinite set $N$ of $X$ and each point $x$ of $X$, there exists a $V \in SR(x)$ such that $N \setminus V$ is not an $s$-set in $(X, T)$.

**Proof.** The proof is similar to that of the above theorem.

**Theorem 3.4.** If $(X, T)$ is anti-$S$-closed then it is anti-$s$-closed.
Proof. The proof immediately follows because of the fact that every s-set is an S-set.

Remark 3.5. That the converse of the above theorem is not necessarily true follows from the following example.

Example 3.6. Let \( X \) be set of all integers with the topology \( T \) having the base \( \{ X, \{0\}, \{-1\}, \{-2\}, \ldots \} \). Then no infinite set of \( X \) is an s-set; if we have \( Z^+ \), the set of positive integers, then \( \{\{0,1\},\{0,2\},\{0,3\},\ldots\} \) is a semi-open cover of \( Z^+ \) and \( \text{scl} \{0,n\} = \{0,n\} \). Then \( Z^+ \) is not an s-set. But \( Z^- \) is an S-set; if we consider \( Z - Z^+ \) then \( \{\{0\},\{-1\},\{-2\},\ldots\} \) is a semi-open cover of \( Z - Z^+ \) and \( \text{scl}\{-n\} = \{-n\} \). Therefore it has no finite subcover. So \( Z - Z^+ \) is not an s-set. If \( L \subset X \) be such that it is infinite and contains finitely many points from \( Z^+ \) then again this can be shown to be a non s-set; if it contains infinitely many elements from \( Z^+ \), the same thing happens. Thus \( X \) is anti-s-closed but not anti-S-closed.

Theorem 3.7. Any topological space \((X,T)\) which is not S-closed (resp. not s-closed) has a proper infinite subset which is anti-S-closed (resp. anti-s-closed) relative to \( X \).

Proof. Since \((X,T)\) is not S-closed (resp. not s-closed), there exists, in particular, a countable cover \( A \) of \( X \) by regular closed (resp. semi-regular) sets which has no finite subcover. We pick up the points \( x_{m+1} \in X - \bigcup_{i=1}^{\infty} V_i \) (where \( V_i \in A \)). Then the set \( \{x_m, m \in N\} \), where \( N \) is the set of naturals, is a proper infinite subset of \( X \) which is not S-set (resp. s-set). Therefore every infinite subset of \( \{x_m : m \in N\} \) is not an S-set (resp. s-set) in \((X,T)\). Hence the infinite subset \( \{x_m : m \in N\} \) is anti-S-closed (resp. anti-s-closed) relative to \( X \).

Definition 3.8. A topological space \((X,T)\) is said to be hereditarily S-closed (resp. hereditarily s-closed) if each of its subsets is S-set (resp. s-set) in \((X,T)\).

Theorem 3.9. A topological space \((X,T)\) is hereditarily S-closed (resp. hereditarily s-closed) iff \((X,T)\) is anti-(anti-S-closed) [resp. anti-(anti-s-closed)].

Proof. Let \((X,T)\) be anti-(anti-S-closed) [resp. anti-(anti-s-closed)]. If possible let \((X,T)\) be not hereditarily S-closed (resp. hereditarily s-closed). Then there exists a subset \( B \) of \( X \) such that \( B \) is not an S-set (resp. s-set) and hence \( B \) must be infinite. Therefore by Theorem 3.7, \( B \) has an infinite subset \( M \) which is anti-S-closed (resp. anti-s-closed) relative to \( X \) a contradiction to the definition of anti-(anti-S-closed) [resp. anti-(anti-s-closed)].

Conversely, let \((X,T)\) be hereditarily S-closed (resp. hereditarily s-closed). If possible, let \((X,T)\) be not anti-(anti-S-closed) [resp. anti-(anti-s-closed)]. Then by definition there exists an infinite subset \( V \) of \( X \) which is anti-S-closed (resp. anti-s-closed) relative to \( X \). Therefore \( V \) is not an S-set (resp. s-set) in \((X,T)\)-a contradiction.
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References


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