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A CLASS OF SPACES AND THEIR ANTI SPACES

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Abstract. The present paper is a continuation of the study of S-closed and s-closed topological spaces as introduced by Thompson [11] and Maio and Noiri [7] respectively. Although there is no relation between compactness with S-closedness or s-closedness, this paper yields some new characterizations of these concepts in terms of compactness.

1. Introudction

Since the introduction of semi-open sets by N. Levine [6], many mathematicians have introduced many new topological properties, using semi-open sets. Maio and Noiri [7] initiated the study of a class of topological spaces under the terminology "s-closed spaces", which is properly contained in the class of S-closed spaces as introduced by Thompson [11] and subsequently studied extensively by many mathematicians. Ganster and Reilly [4] have shown a remarkable result towards the distinction between these concepts that every infinite topological space can be represented as a closed subspace of a connected S-closed space which is not s-closed. The aim of this paper is to study these topological properties viz. S-closedness and s-closedness via compactness which reflect the distinction between the concepts of compactness and S-closeness or s-closedness. This, however, leads us to establish in a straight forward manner certain important characterization theorems of S-closed spaces and s-closed spaces which are already well-known. In the last section, we introduce and characterize the class of anti-S-closed and anti-s-closed spaces.

By (X, T) or simply by X we shall denote a topological space, and for a subset A of X, the closure of A and the interior of A will be denoted by cl A and int A respectively. A subset A of X is said to be *semi-open* [6] if there exists an open set U of X such that $U \subset A \subset \text{cl } U$. Biswas [2] used semi-open sets to define semi-closed sets and semi-closure of a set. A subset A of X is *semi-closed* iff $X - A$ is semi-open and the *semi-closure of* A, denoted by scl A, is the intersection of all semi-closed sets containing A [2]. A set which is semi-open as well as semi-closed is said to be a *semi-regular* set [7]. Maio and Noiri [7] characterized semi regular sets in terms of regular open sets as follows: a set

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A is semi-regular iff there exists a regular open set U of X such that $U \subset A \subset \text{cl}\, U$. The family of all semi-open (resp. semi-regular, regular-open, regular closed) sets of X will be denoted by $\text{SO}(X)$ (resp. $\text{SR}(X)$, $\text{RO}(X)$, $\text{RC}(X)$). While the collection of all members of $\text{SO}(X)$ (resp. $\text{SR}(X)$, $\text{RO}(X)$, $\text{RC}(X)$) each containing a point x of X will be denoted by $\text{SO}(x)$ (resp. $\text{SR}(x)$, $\text{RO}(x)$, $\text{RC}(x)$). A subset A of X is said to be S-closed [9] (resp. s-closed [7]) relative to X or simply an *S-set* (*s-set*) iff every cover of A by sets of $\text{SO}(X)$ admits a finite subfamily whose closures (resp. semi-closures) cover A. In case $A = X$ and A is an S-set (s-set), then X is called an *S-closed* [11] (resp. *s-closed* [7]) space.

2. S−Closed and s−Closed Spaces

Analogous to a well known theorem on compactness, Asha Mathur [8] and Maio and Noiri [7] respectively proved that a topological space X is S-closed (resp. s-closed) iff every regular closed (resp. semi-regular) cover of X has a finite subcover. Although it is well known that compactness and S-closedness (resp. s-closedness) are independent notions, it is our intention in this section to study such spaces with the help of compactness. An important and useful consequence of such study is to achieve a new approach which not only simplifies (in a straightforward way) the proofs of some well-known characterization theorems of S-closed and s-closed spaces but also improves some characterization theorem of such spaces. Joseph and Kwack [5] and Ganguly and Basu [3] initiated respectively (θ, s)-continuous function and γ-continuous function to study S-closed (resp. s-closed) spaces. Using those functions, we derive that a topological space X is S-closed (resp. s-closed) iff it is a (θ, s)-continuous (resp. γ-continuous) image of a compact space. For these purposes we require some definitions and results.

Definition 2.1. A filter base $\Im$ on X is said to s-accumulate [11] (resp. SR-accumulate [7]) to $x \in X$ iff for each $V \in \text{SO}(x)$ and each $F \in \Im$ satisfy $F \cap \text{cl}\, V \neq \phi$ (resp. $F \cap \text{scl}\, V \neq \phi$).

Joseph and Kwack [5] and Maio and Noiri [7] respectively established that $\text{RC}(x) = \{\text{cl}\, V : V \in \text{SO}(x)\}$ and $\text{SR}(x) = \{\text{scl}\, V : V \in \text{SO}(x)\}$. Therefore an equivalent formulation of the above definition is that a filter base $\Im$ on X is said to have an s-accumulation (resp. SR-accumulation) point x iff for each $F \in \Im$ and for each $V \in \text{RC}(x)$ (resp. $V \in \text{SR}(x)$), $F \cap V \neq \phi$.

Definition 2.2. A filter base $\Im$ on X is said to s-converge [11] (resp. SR-converge [7]) to x iff for each $V \in \text{RC}(x)$ (resp. $V \in \text{SR}(x)$) there is an $F \in \Im$ satisfying $F \subset V$.

The corresponding definitions for nets are obvious.

Definition 2.3. Let (X, T) be a topological space. We define T_{RC}-topology (resp. T_{SR}-topology) on X as the topology on X which has $\text{RC}(X)$ (resp. $\text{SR}(X)$) as a subbase. It is to be noted that intersection of even two regular closed (resp. semi-regular) sets may fail to be regular closed (resp. semi-regular). Therefore these collections do not form a base for topology.

Definition 2.4. A filter base $\Im$ in (X, T) is said to be T_{RC}-convergent (resp. T_{SR}-convergent) to x if $\Im$ converges to x in (X, T_{RC}) (resp. in (X, T_{SR})).

Proposition 2.5. *A filterbase $\Im$ in (X, T) s-converges (resp. SR-converges) to x iff $\Im$ T_{RC}-converges (resp. T_{SR}-converges) to x.*

Proof. Straightforward.

The corresponding proposition using nets is also obvious.

Definition 2.6. A filter base $\Im$ on (X, T) is said to have x as a T_{RC}-accumulation (resp. T_{SR}-accumulation) point if x is an accumulation point of $\Im$ in (X, T_{RC}) (resp. in (X, T_{SR})).

Similarly, T_{RC} (resp. T_{SR})-accumulation point of a net can be defined.

Remark 2.7. Every T_{RC}-accumulation (resp. T_{SR}-accumulation) point of a filter or a net is also an s-accumulation (resp. SR-accumulation) point. But the converse is not necessarily true follows from the following example.

Example 2.8. Let $X = R$, be the set of reals with the usual topology then (X, T_{RC}) (resp. (X, T_{SR})) is clearly the discrete topology. Let $x_n = (-1)^n.1/n$ for each positive integer n, then the net $\{x_n\}_{n\in N}$ and the filter $\Im$ based on the net $\{x_n\}_{n\in N}$ both have 0 as the s-accumulation (resp. SR-accumulation) point. But 0 is not a T_{RC}-accumulation (resp. T_{SR}-accumulation) point of $\{x_n\}_{n\in N}$ or $\Im$.

Theorem 2.9. *A topological space (X, T) is S-closed iff (X, T_{RC}) is compact.*

Proof. Let (X, T) be S-closed. Then every regular closed cover of X has a finite subcover. But the collection of all regular closed sets of (X, T) is a subbase for T_{RC}. Therefore every subbasic open cover of (X, T_{RC}) has a finite subcover. By Alexander subbase theorem, (X, T_{RC}) is compact.

Conversely, let (X, T_{RC}) be compact. Since $\text{RC}(X) \subset T_{\text{RC}}$, every regular closed cover of (X, T) has a finite subcover. So (X, T) is S-closed by [Theorem 1 of Asha Mathur [8]].

Theorem 2.10. *A topological space (X, T) is s-closed iff (X, T_{SR}) is compact.*

Proof. It is similar to Theorem 2.9 and is thus omitted.

The following theorem for S-closed spaces improves Theroem 1 of Asha Mathur [8], Theorem 1.3 of T. Noiri [9] and Theorem 2 of Thompson [11]; and the theorem for s-closed spaces improves proposition 3.1 of Maio and Noiri [7].

Theorem 2.11. *Let (X, T) be a topological space. Then the following are equivalent.*

i) (X, T) *is* S*-closed* (*resp.* s*-closed*)
ii) *every proper regular open* (*resp. Semi-regular*) *set is an* S*-set* (*resp.* s*-set*) *in* (X, T).
iii) *every closed set of* (X, T_{RC}) [*resp.* (X, T_{SR})] *is an* S*-set* (*resp.* s*-set*) *in* (X, T).
iv) *every family of regular open* (*resp. Semi-regular*) *subsets of* (X, T) *with the finite intersection property* (*f.i.p. for short*) *has non-void intersection.*
v) *every family of closed subsets of* (X, T_{RC}) [*resp.* (X, T_{SR})] *with the f.i.p. has non-void intersection.*
vi) *every filter base in* (X, T) *has an* s*-accumulation* (*resp. SR-accumulation*) *point.*
vii) *every net in* (X, T) *has an* s*-accumulation* (*resp. SR-accumulation*) *point.*
viii) *every filter base in* (X, T) *has a* T_{RC}*-accumulation* (*resp.* T_{SR}*-accumulation*) *point.*
ix) *every net in* (X, T) *has a* T_{RC}*-accumulation* (*resp.* T_{SR}*-accumulation*) *point.*
x) *every net in* (X, T) *has a* T_{RC}*-convergent* (*resp.* T_{RS} *convergent*) *subnet.*
xi) *every filter* $\Im$ *in* (X, T) *has a sub-ordinate filter* $\Im_0$ *of* $\Im$ *which is* T_{RC}*-convergent* (*resp.* T_{SR}*-convergent*).
xii) *every universal net in* (X, T) *is* T_{RC}*-convergent* (*resp.* T_{SR}*-convergent*).
xiii) *every ultrafilter in* (X, T) *is* T_{RC}*-convergent* (*resp.* T_{SR}*-convergent*).

Proof. The facts discussed above prove the theorem immediately.

Definition 2.12. A function $f : (X, T) \to (Y, T')$ is said to be (θ, s)-continuous [5] (resp. γ-continuous [3]) if for each $x \in X$ and each $W \in \mathrm{SO}(f(x))$, there is an open set V containing x such that $f(V) \subset \mathrm{cl}\ W$ (resp. $f(V) \subset \mathrm{scl}\ W$).

Since $\mathrm{RC}(x) = \{\mathrm{cl}\ W : W \in \mathrm{SO}(x)\}$ [5] (resp. $\mathrm{SR}(x) = \{\mathrm{scl}\ W : W \in \mathrm{SO}(x)\}$ [7]), the above definition can equivalently be stated as: a function $f : (X, T) \to (Y, T')$ is (θ, s)-continuous (resp. γ-continuous) iff $f^{-1}(W)$ is open in X, for every $W \in \mathrm{RC}(Y)$ (resp. $W \in \mathrm{SR}(Y)$).

Theorem 2.13. *A topological space* (X, T) *is* S*-closed iff it is a* (θ, s)*-continuous image of a compact space.*

Proof. Let (X, T) be S-closed. Then by Theorem 2.9, (X, T_{RC}) is compact. Let $i : (X, T_{\mathrm{RC}}) \to (X, T)$ be the identity function, which is obviously (θ, s)-continuous. Therefore there exist a compact space and a (θ, s)-continuous function such that the S-closed space (X, T) is the (θ, s)-continuous image of a compact space.

Conversely, let $f : (Y, T^*) \to (X, T)$ be (θ, s)-continuous surjection and (Y, T^*) be compact. Let $\{V_\alpha : \alpha \in I\}$ be a cover of (X, T) be regular closed sets of (X, T). Then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a cover of the compact space (Y, T^*) by open sets of (Y, T^*). Therefore there exists a finite subset I_0 of I such that $\{f^{-1}(V_\alpha) : \alpha \in I_0\}$ covers Y and hence $\{V_\alpha : \alpha \in I_0\}$ covers X. Therefore (X, T) is S-closed.

Theorem 2.14. *A topological space* (X, T) *is* s*-closed iff it is a* γ*-continuous image of a compact space.*

Proof. The proof is similarl to Theorem 2.13 and thus omitted.

Maio and Norir [7] initiated quasi-irresolute function and established that such functions preserve s-sets. Here we introduce a weaker form of quasi-irresolute function which also has the same property.

Definition 2.15. ([7]) A function $f : X \to Y$ is said to be quasi-irresolute if for each $x \in X$ and each $V \in \mathrm{SO}(f(x))$ there exists a $U \in \mathrm{SO}(x)$ such that $f(U) \subset \mathrm{scl}\, V$.

Definition 2.16. A function $f : (X, T) \to (Y, T')$ is said to be weakly quasi-irresolute if $f : (X, T_{\mathrm{SR}}) \to (Y, T'_{\mathrm{SR}})$ is continuous.

Theorem 2.17. *Every quasi-irresolute function is weakly quasi-irresolute.*

Proof. Let $f : (X, T) \to (Y, T')$ be quasi-irresolute. Then $f^{-1}(V)$ is semi-regular set in (X, T) for every $V \in \mathrm{SR}(Y)$. But the collection of all semi-regular sets of (Y, T') is a subbase for (Y, T'_{SR}). Hence $f : (X, T_{\mathrm{SR}}) \to (Y, T'_{\mathrm{SR}})$ is continuous. Therefore $f : (X, T) \to (Y, T')$ is weakly quasi-irresolute.

Remark 2.18. That the coverse of the above theorem is not necessarily true follows from the following example.

Example 2.19. Let $f : R \to R$ where R is the set of reals with the usual topology T be a function defined by

$$f(x) = \begin{cases} r_1 & \text{if } x \text{ is rational,} \\ r_2 & \text{if } x \text{ is irrational, with } r_2 > r_1 > 0. \end{cases}$$

Clearly T_{SR} is the discrete topology; therefore $f : (R, T_{\mathrm{SR}}) \to (R, T_{\mathrm{SR}})$ is continuous and hence f is weakly quasi-irresolute function. But f is not a quasi-irresolute function. In fact if δ be such that $0 < \delta < |(r_2 - r_1)/2|$, then the open interval $J = (r_2 - \delta, r_2 + \delta)$ is a semi-regular set in (R, T); but $f^{-1}(J)$ is the set of all irrationals of R. Hence $f^{-1}(J)$ is not even a semi-open set in R.

Hence we get an improved result of Corollary 5.1 of Maio and Noiri [7].

Theorem 2.20. *If $f : (X, T) \to (Y, T')$ is weakly quasi-irresolute and K is an s-set of (X, T), then $f(K)$ is an s-set in Y.*

Proof. Let $\{U_\alpha : \alpha \in I\}$ be a cover of $f(K)$ by semi-regular sets of (Y, T'). Since $f : (X, T) \to (Y, T')$ is weakly quasi-irresolute, $\{f^{-1}(U_\alpha) : \alpha \in I\}$ is a cover of K by open sets of (X, T_{SR}). By Theorem 2.10, K is compact in (X, T_{SR}). Therefore, there exists a finite subset I_0 of I such that $K \subset \cup\{f^{-1}(U_\alpha) : \alpha \in I_0\}$. Which implies $f(K) \subset \cup_{\alpha \in I_0} U_\alpha$. Therefore $f(K)$ is an s-set in Y.

Corollary 2.21. *If $f : (X, T) \to (Y, T')$ is weakly quasi-irresolute surjection and (X, T) is s-closed then (Y, T') is also s-closed.*

Definition 2.22. ([7]) A space (X, T) is said to be weakly Hausdorff if every point of X is the intersection of regular closed sets of X.

The following theorem improves Corollary 5.2 of Maio and Noiri [7].

Theorem 2.23. *Let $f : (X,T) \to (Y,T')$ be weakly quasi-irresolute, (X,T) is s-closed and (Y,T') be weakly Hausdorff. Then the image of each semi-θ-closed set* [7] *in X is semi-θ-closed in Y.*

Proof. Let K be a semi-θ-closed set in (X,T). Then by Proposition 4.2 of Maio and Noiri [7], K is an s-set in X. By Theorem 2.20, $f(K)$ is an s-set in Y. Therefore by Proposition 4.3 of Maio and Noiri [7], $f(K)$ is semi-θ-closed set in (Y,T).

3. Anti−S−Closed and Anti−s−Closed Spaces

P. Bankston [1] studied topological anti-properties. Reilly & Vamanamurthy [10] extended these concepts to semi-compact spaces. In a similar fashion [10], here we introduce and characterize two new topological anti-properties under the terminology 'anti-S-closedness' and 'anti-s-closedness' along with their mutual relationships.

Definition 3.1. A topological space (X,T) is said to be anti-S-closed (resp. anti-s -closed) if only the finite subsets of (X,T) are S-sets (resp. s-sets) of (X,T).

An infinite subset A of (X,T) is said to be anti-S-closed (resp. anti-s-closed) relative to X if only the finite subsets of A are S-sets (resp. s-sets) in (X,T).

Theorem 3.2. *A topological space (X,T) is anti-S-closed iff for every infinite set N of X and each point x of X, there exists a regular closed set R containing x such that $N \setminus R$ is not an S-set in (X,T).*

Proof. Let the given condition hold. We have to show that (X,T) is anti-S-closed. Let N be any infinite set and let $x \in X$. Then by hypothesis, there exists a $R \in RC(x)$ such that $N \setminus R$ is not an S-set. Therefore there exists a cover $\mathcal{A}$ of $N \setminus R$ by regular closed sets of X which has no finite subcover. So N is not an S-set in (X,T). Therefore (X,T) is anti-S-closed.

Conversely, let (X,T) be anti-S-closed space. Let N be any infinite subset of X and let x be any point of X. Then by definition of anti-S-closed space, N and hence $N \cup \{x\}$ is not an S-set. Therefore there exists a cover $\mathcal{A}$ of $N \cup \{x\}$ by regular closed sets which has no finite subcover. Hence there exists a member $R \in \mathcal{A}$ such that $x \in R$. So $N \setminus R$ is not an S-set in (X,T).

Theorem 3.3. *A topological space (X,T) is anti-s-closed iff for every infinite set N of X and each point x of X, there exists a $V \in SR(x)$ such that $N \setminus V$ is not an s-set in (X,T).*

Proof. The proof is similar to that of the above theorem.

Theorem 3.4. *If (X,T) is anti-S-closed then it is anti-s-closed.*

Proof. The proof immediately follows because of the fact that every s-set is an S-set.

Remark 3.5. That the converse of the above theorem is not necessarily true follows from the following example.

Example 3.6. Let X be set of all integers with the topology T having the base $\{X, \{0\}, \{-1\}, \{-2\}, \ldots\}$. Then no infinite set of X is an s-set; if we have Z^+, the set of positive integers, then $\{\{0,1\}, \{0,2\}, \{0,3\}, \ldots\}$ is a semi-open cover of Z^+ and scl $\{0,n\} = \{0,n\}$. Then Z^+ is not an s-set. But Z^+ is an S-set; if we consider $Z - Z^+$ then $\{\{0\}, \{-1\}, \{-2\}, \ldots\}$ is a semi-open cover of $Z - Z^+$ and scl$\{-n\} = \{-n\}$. Therefore it has no finite subcover. So $Z - Z^+$ is not an s-set. If $L \subset X$ be such that it is infinite and contains finitely many points from Z^+ then again this can be shown to be a non s-set; if it contains infinitely many elements from Z^+, the same thing happens. Thus X is anti-s-closed but not anti-S-closed.

Theorem 3.7. *Any topological space* (X, T) *which is not* S*-closed* (*resp. not* s*-closed*) *has a proper infinite subset which is anti-*S*-closed* (*resp. anti-*s*-closed*) *relative to* X.

Proof. Since (X, T) is not S-closed (resp. not s-closed), there exists, in particular, a countable cover $\mathcal{A}$ of X by regular closed (resp. semi-regular) sets which has no finite subcover. We pick up the points $x_{m+1} \in X - \cup_{i=1}^{n} V_i$ (where $V_i \in \mathcal{A}$). Then the set $\{x_m, m \in N\}$, where N is the set of naturals, is a proper infinite subset of X which is not S-set (resp. s-set). Therefore every infinite subset of $\{x_m : m \in N\}$ is not an S-set (resp. s-set) in (X, T). Hence the infinite subset $\{x_m : m \in N\}$ is anti-S-closed (resp. anti-s-closed) relative to X.

Definition 3.8. A topological space (X, T) is said to be hereditarily S-closed (resp. hereditarily s-closed) if each of its subsets is S-set (resp. s-set) in (X, T).

Theorem 3.9. *A topological space* (X, T) *is hereditarily* S*-closed* (*resp. hereditarily* s*-closed*) *iff* (X, T) *is anti-*(*anti-*S*-closed*) [*resp. anti-*(*anti-*s*-closed*)].

Proof. Let (X, T) be anti-(anti-S-closed) [resp. anti-(anti-s-closed)]. If possible let (X, T) be not hereditarily S-closed (resp. hereditarily s-closed). Then there exists a subset B of X such that B is not an S-set (resp. s-set) and hence B must be infinite. Therefore by Theorem 3.7, B has an infinite subset M which is anti-S-closed (resp. anti-s-closed) relative to X-a contradiction to the definition of anti-(anti-S- closed) [resp. anti-(anti-s-closed)].

Coversely, let (X, T) be hereditarily S-closed (resp. hereditarily s-closed). If possible, let (X, T) be not anti-(anti-S-closed) [resp. anti-(anti-s-closed)]. Then by definition there exists an infinite subset V of X which is anti-S-closed (resp. anti-s-closed) relative to X. Therefore V is not an S-set (resp. s-set) in (X, T)-a contradiction.

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