TAMKANG JOURNAL OF MATHEMATICS
Volume 49, Number 1, 35-48, March 2018
doi:10.5556/j.tkjm.49.2018.2383
This paper is available online at http://journals.math.tku.edu.tw/index.php/TKJM/pages/view/onlinefirst

FRAME FOR OPERATORS IN FINITE DIMENSIONAL HILBERT SPACE

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Abstract. In this paper, we study frames for operators (K-frames) in finite dimensional Hilbert spaces and express the dual of K-frames. Some properties of K-dual frames are investigated. Furthermore, the notion of their oblique K-dual and some properties are presented.

1. Introduction

Frames were first introduced by Duffin and Schaeffer in the study of nonharmonic Fourier series in 1952, [8], and were widely studied from 1986 since the great work by Daubechies et al, [7]. Now, frames play an important role not only in the theoretic but also in many kinds of applications, for example, signal processing [10], filter bank theory [3] and many other fields [9, 13, 15].
The notion of K-frames was considered for the first time in [11], in connection with atomic decompositions for operators in Hilbert spaces. Basic properties and examples of K-frames are given in [11] and [12].

Let $K \in B(\mathscr{H})$, the space of all bounded linear operators on a Hilbert space $\mathscr{H}$. A sequence $\{\varphi_j\}_{j\in\mathbb{J}}$ is said to be a K-frame for $\mathscr{H}$ if there exist constants $A, B > 0$ such that

$$A\|K^* x\|^2 \leq \sum_{j\in\mathbb{J}} |\langle x, \varphi_j\rangle|^2 \leq B\|x\|^2, \quad (x \in \mathscr{H}). \tag{1.1}$$

We call A, B the lower and the upper K-frame bounds for $\{\varphi_j\}_{j\in\mathbb{J}}$, respectively. If $K = I_{\mathscr{H}}$, then $\{\varphi_j\}_{j\in\mathbb{J}}$ is the ordinary frame. If only the right inequality holds, then $\{\varphi_j\}_{j\in\mathbb{J}}$ is called a Bessel sequence. Suppose that $\Phi = \{\varphi_j\}_{j\in\mathbb{J}}$ is a K-frame for $\mathscr{H}$. The operator $T_\Phi : \mathscr{H} \to \ell^2(\mathbb{J})$ defined by $T_\Phi(x) = \{\langle x, \varphi_j\rangle\}_{j\in\mathbb{J}}$ is called the analysis operator. T_Φ is bounded and $T_\Phi^* : \ell^2(\mathbb{J}) \to \mathscr{H}$ is given by $T_\Phi^*(\{c_j\}_{j\in\mathbb{J}}) = \sum_{j\in\mathbb{J}} c_j\varphi_j$. T_Φ^* is called the pre-frame or synthesis operator. The

Received December 20, 2016, accepted November 17, 2017.
2010 *Mathematics Subject Classification*. 42C15, 46B15.
Key words and phrases. K-frame, K-dual, oblique K-dual.
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operator $S_\Phi : \mathcal{H} \to \mathcal{H}$ defined by $S_\Phi(x) = T_\Phi^* T_\Phi(x) = \sum_{j\in\mathbb{J}} \langle x, \varphi_j\rangle \varphi_j$ is called the frame operator of Φ. Note that, frame operator of a K-frame is not invertible on $\mathcal{H}$ in general, but it is invertible on the subspace $R(K) \subset \mathcal{H}$, that $R(K)$ is the range of K, and for all $x \in S_\Phi(R(K))$ we have

$$B^{-1}\|x\| \le \|S_\Phi^{-1} x\| \le A^{-1}\|K^\dagger\|^2\|x\|, \tag{1.2}$$

where $K^\dagger$ is the pseudo-inverse of K. If K is invertible then S_Φ is invertible. Also, in this case, we can see that any K-frame for $\mathcal{H}^N$ is a frame for $\mathcal{H}^N$. Since we can write

$$\frac{A}{\|K^{-1}\|^2}\|x\|^2 \le A\|K^* x\|^2 \le \sum_{j=1}^{M} |\langle x, \varphi_j\rangle|^2,$$

and for all $x \in \mathcal{H}^N$ we have

$$A\|x\|^2 = A\|K^*(K^*)^{-1}x\|^2 \le \sum_{j=1}^{M} |\langle (K^*)^{-1}x, \varphi_j\rangle|^2 = \sum_{j=1}^{M} |\langle x, K^{-1}\varphi_j\rangle|^2.$$

Hence the sequence $\{K^{-1}\varphi_j\}_{j=1}^M$ is a frame.

Given a positive integer N. Throughout this paper, we suppose that $\mathcal{H}^N$ is a real or complex N-dimensional Hilbert space. By $\langle\cdot,\cdot\rangle$ and $\|.\|$ we denote the inner product on $\mathcal{H}^N$ and its corresponding norm, respectively. I_H is the identity operator on $\mathcal{H}^N$. For two Hilbert spaces $\mathcal{H}_1$ and H_2 we denote by $B(\mathcal{H}_1, \mathcal{H}_2)$ the collection of all bounded linear operators between $\mathcal{H}_1$ and $\mathcal{H}_2$, and we abbreviate $B(\mathcal{H}, \mathcal{H})$ by $B(\mathcal{H})$. In particular, $B(\mathcal{H}^N) = \mathcal{M}_{N\times N}(\mathbb{C})$. Denote by P_W the orthogonal projection of $\mathcal{H}$ onto a closed subspace $W \subseteq \mathcal{H}$.

Finite frames and its properties were proposed by several researchers. In particular, the theory of finite frames in $\mathcal{H}^N$ was developed by P. G. Casazza et al [4]. Also, the concept of oblique dual frames and their properties in finite dimensional Hilbert space were presented by X. C. Xiao, Y. C. Zhu and X. M. Zeng [18].

The paper is organized in the following manner. In Section 2, we study the notion of a finite K-frames and prove some properties in finite dimensional Hilbert space. In particular, we give a simple way to construct new K-frames from given ones. Also, we extend Theorem 1.1 in [10] to the setting of K-dual frame pairs. In Section 3, we introduce the concept of K-dual of K-frames in $\mathcal{H}^N$ and its properties are discussed. Also, in the last part of Section 3, the oblique K-dual is investigated.

2. Finite K-frame

Frames in finite dimensional spaces, i.e., finite frames, are a very important class of frames due to their significant relevance in applications. In this section, we present K-frame theory in finite-dimensional Hilbert spaces. Let $K \in B(\mathcal{H}^N)$ and $\Phi = \{\varphi_j\}_{j=1}^M$ be a family of vectors in $\mathcal{H}^N$. If $A\|K^* x\|^2 = \sum_{j=1}^M |\langle x, \varphi_j\rangle|^2$, then Φ is called an A-tight K-frame and if $\|K^* x\|^2 = \sum_{j=1}^M |\langle x, \varphi_j\rangle|^2$, then Φ is called a tight K-frame or Parseval K-frame. We can see that, Φ is an A-tight K-frame if and only if $S_\Phi = AKK^*$. If $\|\varphi_j\| = 1$ for all $j = 1, 2, \ldots, M$, this is an unit norm K-frame. Also, if a K-frame Φ is independent in $\mathcal{H}^N$ then it is call minimal K-frame.

Note that, if $K \in B(\mathcal{H}^N)$ and $\Phi = \{\varphi_j\}_{j=1}^M$ is a frame for H^N, then Φ is a K-frame. Since for any $K \neq 0$ we can write

$$\frac{A}{\|K\|^2}\|K^* x\|^2 \le A\|x\|^2 \le \sum_{j=1}^{M} |\langle x, \varphi_j\rangle|^2, \; (x \in \mathcal{H}^N).$$

Remark 2.1.

(i) If $span\{\varphi_j : j = 1, 2, \ldots, M\} = R(K)$, then $\{\varphi_j\}_{j=1}^M$ is a frame for $R(K)$. Thus $\{\varphi_j\}_{j=1}^M$ is a K-frame for $R(K)$.

(ii) Any K-frame is not necessary a frame, in general. For example, let $\mathcal{H}^3 = \mathbb{R}^3$ and $K = P_{\mathbb{R}^2}$. Then $\{e_1, e_2\}$ is a K-frame which is not a frame since

$$span\{e_1, e_2\} \neq \mathbb{R}^3.$$

Note that, if Φ is a K-frame for $\mathcal{H}^N$, then by Proposition 3.1 in [17], there exists a sequence $\Psi = \{\psi_j\}_{j=1}^M \subseteq \mathcal{H}^N$ such that

$$Kx = \sum_{j=1}^{M} \langle x, \psi_j\rangle \varphi_j, \; (x \in \mathcal{H}^N)$$

and this means that $R(K) \subseteq span\{\varphi_j : j = 1, 2, \ldots, M\}$. Furthermore, we have the following proposition.

Proposition 2.2 ([16])**.** *A sequence $\Phi = \{\varphi_j\}_{j=1}^M$ is a K-frame for $\mathcal{H}^N$ if and only if $R(K) \subseteq span\{\varphi_j : j = 1, 2, \ldots, M\}$.*

Now, for an arbitrary K-frame, we obtain the optimal lower and upper K-frame bounds by eigenvalues of its frame operator.

Proposition 2.3.

(i) *Let $0 \neq K \in B(\mathcal{H}^N)$. Let $\Phi = \{\varphi_j\}_{j=1}^M$ be a K-frame for $R(K)$ with K-frame operator S_Φ with eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N > 0$. Then λ_1 is the optimal upper K-frame bound and if $\lambda_N \neq 0$ then $\frac{\lambda_N}{\|K\|^2}$ is the optimal lower K-frame bound.*

(ii) *Suppose that* $\Phi = \{\varphi_j\}_{j=1}^M$ *is a* K*-frame for* $R(K)$ *and* $\{\lambda_j\}_{j=1}^N$ *denotes the eigenvalues for* S_Φ *and each eigenvalue appears in the list corresponding to its algebraic multiplicity. Then*

$$\sum_{j=1}^{N} \lambda_j = \sum_{j=1}^{M} \|\varphi_j\|^2.$$

Proof. For the proof of (i), suppose that $\{e_j\}_{j=1}^N$ is an orthonormal eigen basis of the frame operator S_Φ with associated eigenvalues $\{\lambda_j\}_{j=1}^N$ given in decreasing order. Hence we can write $x = \sum_{j=1}^N \langle x, e_j\rangle e_j$, for all $x \in \mathcal{H}^N$. Also, we have

$$S_\Phi x = \sum_{j=1}^{N} \langle x, e_j\rangle S_\Phi e_j = \sum_{j=1}^{N} \lambda_j \langle x, e_j\rangle e_j,$$

and thus

$$\begin{aligned}\sum_{j=1}^{M} |\langle x, \varphi_j\rangle|^2 = \langle S_\Phi x, x\rangle &= \sum_{j=1}^{N} \lambda_j |\langle x, e_j\rangle|^2 \\ &\le \lambda_1 \sum_{j=1}^{N} |\langle x, e_j\rangle|^2 = \lambda_1 \|x\|^2.\end{aligned}$$

For the lower bound we have

$$\begin{aligned}\frac{\lambda_N}{\|K\|^2} \|K^* x\|^2 \le \lambda_N \|x\|^2 = \lambda_N &\sum_{j=1}^{M} |\langle x, e_j\rangle|^2 \\ \le &\sum_{j=1}^{N} \lambda_j |\langle x, e_j\rangle|^2 = \langle S_\Phi x, x\rangle.\end{aligned}$$

The proof of secondly part is similar to the proof of Theorem 1.1.12 [6]. ☐

Now, we introduce a constructive method to extend a given frame to a tight K-frame.

Theorem 2.4. *Let* $K \in B(\mathcal{H}^N)$*. Let* $\Phi = \{\varphi_j\}_{j=1}^M$ *be a frame for* $\mathcal{H}^N$*. Assume that the frame operator* S_Φ *has the eigenvalues* $\{\lambda_j\}_{j=1}^N$*, ordered as* $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N > 0$*. Let* $\{e_j\}_{j=1}^N$ *be a corresponding eigenbasis. Then the collection* $\{K\varphi_j\}_{j=1}^M \cup \{\sqrt{\lambda_1 - \lambda_j}\, K e_j\}_{j=2}^N$ *is a* λ_1*-tight* K*-frame for* $\mathcal{H}^N$*.*

Proof. Set $\{\psi_j\}_{j=2}^N = \{\sqrt{\lambda_1 - \lambda_j}\, K e_j\}_{j=2}^N$. Suppose that $\tilde{S}$ is the frame operator for $\{K\varphi_j\}_{j=1}^M \cup \{\psi_j\}_{j=2}^N$. Now, for any $x \in \mathcal{H}^N$ we can write

$$\begin{aligned}\tilde{S}x &= \sum_{j=1}^{M} \langle x, K\varphi_j\rangle K\varphi_j + \sum_{j=2}^{M} \langle x, \psi_j\rangle \psi_j \\ &= K \sum_{j=1}^{M} \langle K^* x, \varphi_j\rangle \varphi_j + \sum_{j=2}^{M} \langle x, \psi_j\rangle \psi_j\end{aligned}$$

$$= K\sum_{j=1}^{N}\lambda_j\langle K^*x,e_j\rangle e_j + K\sum_{j=2}^{N}\sqrt{\lambda_1-\lambda_j}\langle K^*x,e_j\rangle\sqrt{\lambda_1-\lambda_j}e_j$$
$$= K\sum_{j=1}^{N}\lambda_j\langle K^*x,e_j\rangle e_j + K\sum_{j=2}^{N}(\lambda_1-\lambda_j)\langle K^*x,e_j\rangle e_j$$
$$= \lambda_1 KK^*x.$$

Therefore $\{K\varphi_j\}_{j=1}^{M}\cup\{\sqrt{\lambda_1-\lambda_j}\,Ke_j\}_{j=2}^{N}$ is a λ_1-tight K-frame for $\mathcal{H}^N$. □

Example 2.5. Let $K\in B(\mathcal{H}^6)$. Consider the frame $\Phi=\{\varphi_j\}_{j=1}^{5}:=\{3e_1\}\cup\{e_j\}_{j=2}^{5}$ for the subspace $W=span\{e_j, j=1,2,\dots,5\}$ of $\mathcal{H}^6=\mathbb{C}^6$ where $\{e_j\}_{j=1}^{6}$ is the orthonormal basis for $\mathcal{H}^6$. We have

$$S_\Phi x=\sum_{j=1}^{5}\lambda_j\langle x,e_j\rangle e_j=\sum_{j=1}^{5}\langle x,\varphi_j\rangle\varphi_j\,,\ (x\in W).$$

Thus we can see that the eigenvalues for the frame operator S_Φ are $\lambda_1=9$, $\lambda_j=1$, $j=2,\dots,5$. Hence by Theorem 2.1 there exist 4 vectors $\{\psi_j\}_{j=1}^{4}=\{\sqrt{9-\lambda_j}\,Ke_j\}_{j=2}^{5}$ such that $\{K\varphi_j\}_{j=1}^{5}\cup\left\{\sqrt{9-\lambda_j}\,Ke_j\right\}_{j=2}^{5}$ is a 9-tight K-frame for W.

In the following proposition, we express two inequality of A-tight K-frames.

Proposition 2.6.

(i) *If* $\Phi=\{\varphi_j\}_{j=1}^{M}$ *is an* A*-tight* K*-frame for* $\mathcal{H}^N$*, then*

$$\max_{j=1,2,\dots,M}\|\varphi_j\|^2\le A\|K\|^2.$$

(ii) *If* $\Phi=\{\varphi_j\}_{j=1}^{M}$ *is an unit norm* A*-tight* K*-frame for* $\mathcal{H}^N$*, then*

$$A\|K\|^2N\ge M.$$

Proof.

(i) Note that for any $j=1,\dots,M$, we have

$$\|\varphi_j\|^4=|\langle\varphi_j,\varphi_j\rangle|^2\le\sum_{i=1}^{M}|\langle\varphi_j,\varphi_i\rangle|^2=A\|K^*\varphi_j\|^2\le A\|K^*\|^2\|\varphi_j\|^2.$$

Thus, $\max_{j=1,2,\dots,M}\|\varphi_j\|^2\le A\|K\|^2$.

(ii) Let $\{e_i\}_{i=1}^{N}$ be an orthonormal basis for $\mathcal{H}^N$. Then we can write

$$M=\sum_{j=1}^{M}\|\varphi_j\|^2=\sum_{j=1}^{M}\sum_{i=1}^{N}|\langle e_i,\varphi_j\rangle|^2=\sum_{i=1}^{N}\sum_{j=1}^{M}|\langle e_i,\varphi_j\rangle|^2$$
$$=\sum_{i=1}^{N}A\|K^*e_i\|^2\le A\|K\|^2\sum_{i=1}^{N}\|e_i\|^2=A\|K\|^2N.$$
□

Remark 2.7. In the frame setting, if $\Phi = \{\varphi_j\}_{j=1}^M$ is a finite sequence which is a frame for $\mathscr{H}$, then $\mathscr{H}$ is finite-dimensional [2]. But in the K-frame setting, it is not true. For example, we define $K : \ell^2 \longrightarrow \ell^2$ with $Kx = \sum_{j=1}^M \langle x, e_j\rangle e_j$. Clearly $\{Ke_j\}_{j=1}^\infty = \{e_j\}_{j=1}^M$ is a K-frame but ℓ^2 is not finite-dimensional.

If we have information on the lower K-frame bound of an unit norm K-frame, we can provide a criterion for how many elements we can remove so that the rest of the elements forms a K-frame.

Proposition 2.8. *Suppose that* $0 \neq K \in B(\mathscr{H}^N)$ *and* $\Phi = \{\varphi_j\}_{j=1}^M$ *is an unit norm* K*-frame for* $\mathscr{H}^N$ *with the lower* K*-frame bound* A*. Then for any index set* $I \subseteq \{1,\dots,M\}$ *such that* $\mid I \mid \|K^\dagger \mid_{R(K)} \|^2 < A$*, the family* $\{\varphi_j\}_{j\notin I}$ *is a* K*-frame for* $R(K)$*.*

Proof. Let $\Phi = \{\varphi_j\}_{j=1}^N$ be an unit norm K-frame for $\mathscr{H}^N$, then we can write

$$\sum_{j\in I} \mid \langle x, \varphi_j\rangle \mid^2 \leq \sum_{j\in I} \|\varphi_j\|^2 \|x\|^2 = \mid I \mid \|x\|^2, \; (x \in R(K)).$$

Now, since for any $x \in R(K)$

$$\frac{A}{\|K^\dagger \mid_{R(K)} \|^2}\|x\|^2 \leq A\|K^* x\|^2 \leq \sum_{j\in I} \mid \langle x, \varphi_j\rangle \mid^2 + \sum_{j\notin I} \mid \langle x, \varphi_j\rangle \mid^2 .$$

So we have

$$\left(\frac{A}{\|K^\dagger \mid_{R(K)} \|^2} - \mid I \mid\right)\|x\|^2 \leq \sum_{j\notin I} \mid \langle x, \varphi_j\rangle \mid^2, \; (x \in R(K)).$$

Then Φ is a frame and hence is a K-frame for $R(K)$. □

In the last part of this section, we study conditions under which a linear combination of two K-frames is K-frame too.

Definition 2.9. Let $K \in B(\mathscr{H}^N)$ and $\Phi = \{\varphi_j\}_{j=1}^M$ and $\Psi = \{\psi_j\}_{j=1}^M$ be K-frames for $\mathscr{H}^N$. Φ and Ψ are called strongly disjoint if $R(T_\Phi) \perp R(T_\Psi)$, where T_Φ and T_Ψ are the analysis operators of the sequences Φ and Ψ, respectively.

Theorem 2.10. *Suppose that* $K \in B(\mathscr{H}^N)$ *and* $\Phi = \{\varphi_j\}_{j=1}^M$ *and* $\Psi = \{\psi_j\}_{j=1}^M$ *are strongly disjoint tight* K*-frames for* $\mathscr{H}^N$*. Also, assume that* $A, B \in B(\mathscr{H}^N)$ *are operators such that* $AKK^*A^* + BKK^*B^* = I_{N\times N}$*, then* $\{A\Phi + B\Psi\}$ *is a* K*-frame for* $\mathscr{H}^N$*. In particular, if* $KK^* = \frac{1}{2(|\alpha|^2+|\beta|^2)} I_{N\times N}$*, then* $\{\alpha\Phi + \beta\Psi\}$ *is a* K*-frame for* $\mathscr{H}^N$*.*

Proof. For any $x \in \mathscr{H}^N$

$$\sum_{j=1}^M \mid \langle x, A\varphi_j + B\psi_j\rangle \mid^2 = \|\{\langle x, A\varphi_j + B\psi_j\rangle\}_{j=1}^M\|^2$$

$$
\begin{aligned}
&= \|\{\langle A^* x, \varphi_j\rangle + \langle B^* x, \psi_j\rangle\}_{j=1}^M\|^2 \\
&= \|\{\langle A^* x, \varphi_j\rangle\}_{j=1}^M\|^2 + \|\{\langle B^* x, \psi_j\rangle\}_{j=1}^M\|^2 \\
&+ \langle\{A^* x, \varphi_j\rangle\}_{j=1}^M, \{B^* x, \psi_j\rangle\}_{j=1}^M\rangle_{\mathcal{H}^N} \\
&= \sum_{j=1}^M |\langle A^* x, \varphi_j\rangle|^2 + \sum_{j=1}^M |\langle B^* x, \psi_j\rangle|^2 \\
&= \|K^* A^* x\|^2 + \|K^* B^* x\|^2 = \|x\|^2.
\end{aligned}
$$

Therefore $\{A\varphi_j + B\psi_j\}_{j=1}^M$ is a tight frame for $\mathcal{H}^N$ and so is a K-frame for $\mathcal{H}^N$. □

Corollary 2.11. *Assume that* $(\Phi_1, \Phi_2, ..., \Phi_k)$ *is strongly disjoint k-tuple of tight K-frames on* $\mathcal{H}^N$ *and* $A_i \in B(\mathcal{H}^N)$ *such that* $\sum_{i=1}^k A_i K K^* A_i^* = I_{N\times N}$. *Then* $\{\sum_{i=1}^k A_i \Phi_i\}$ *is a K-frame for* $\mathcal{H}^N$.

Proposition 2.12. *With assumption of Theorem 2.2, if* $AKK^*A^* + BKK^*B^* = KK^* I_{N\times N}$, *then* $\{A\Phi + B\Psi\}$ *is a tight K-frame for* $\mathcal{H}^N$.

Remark 2.13. Theorem 2.2, Corollary 2.1 and Proposition 2.5 actually hold in infinite-dimensional Hilbert spaces.

3. Dual of K-frame

Dual frames are important to reconstruct of vectors (or signals) in terms of the frame elements. In the other words, two frames $\Phi = \{\varphi_j\}_{j\in\mathbb{J}}$ and $\Psi = \{\psi_j\}_{j\in\mathbb{J}}$ are dual frames for $\mathcal{H}$ if for all $x \in \mathcal{H}$,

$$x = \sum_{j\in\mathbb{J}} \langle x, \psi_j\rangle \varphi_j = \sum_{j\in\mathbb{J}} \langle x, \varphi_j\rangle \psi_j. \tag{3.1}$$

Also, if $\{\varphi_j\}_{j\in\mathbb{J}}$ is a K-frame, a Bessel sequence $\Psi = \{\psi_j\}_{j\in\mathbb{J}}$ is called a K-dual of $\{\varphi_j\}_{j\in\mathbb{J}}$ (see [1]) if

$$Kx = \sum_{j\in\mathbb{J}} \langle x, \psi_j\rangle \varphi_j, \quad (x \in \mathcal{H}). \tag{3.2}$$

We can see that, for every K-frame of $\mathcal{H}$ there exists at least a Bessel sequence $\{\psi_j\}_{j\in\mathbb{J}}$ which satisfies in K-dual equality and the sequences $\{\varphi_j\}_{j\in\mathbb{J}}$ and $\{\psi_j\}_{j\in\mathbb{J}}$ in (3.2) are not interchangeable in general [17]. Now, we study this notion in finite dimensional Hilbert spaces.

Definition 3.1. If $\Phi = \{\varphi_j\}_{j=1}^M$ is a K-frame for $\mathcal{H}^N$, a sequence $\Psi = \{\psi_j\}_{j=1}^M$ is called a K-dual frame for Φ if

$$Kx = \sum_{j=1}^M \langle x, \psi_j\rangle \varphi_j, \ (x \in \mathcal{H}^N). \tag{3.3}$$

The systems $\Phi = \{\varphi_j\}_{j=1}^M$ and $\Psi = \{\psi_j\}_{j=1}^M$ are referred to as a K-dual frame pair.

If T_Φ^* and T_Ψ^* are the $N\times M$ matrices whose j-th columns are $\Phi=\{\varphi_j\}_{j=1}^M$ and $\Psi=\{\psi_j\}_{j=1}^M$, respectively, then (3.3) is equivalent to $K=T_\Phi^* T_\Psi$.

Note that $\Phi=\{\varphi_j\}_{j=1}^M$ and $\Psi=\{\psi_j\}_{j=1}^M$ are not interchangeable in general. Indeed, $\Phi=\{\varphi_j\}_{j=1}^M$ and $\Psi=\{\psi_j\}_{j=1}^M$ in (3.3) are interchangeable if and only if K is self adjoint. Also, always there is at least a K-dual of any arbitrary K-frame.

Example 3.2. Consider the standard orthonormal basis $\Psi=\{\psi_j\}_{j=1}^3=\{e_j\}_{j=1}^3$ of Hilbert space $\mathbb{C}^3$. Define $K\in B(\mathbb{C}^3)$ as follows $Ke_1=e_1, Ke_2=e_1, Ke_3=e_2$. Set $\Phi=\{\phi_j\}_{j=1}^3=\{Ke_j\}_{j=1}^3$, that is, $\Phi=\{\phi_j\}_{j=1}^3$ is a K-frame. Since $x=\sum_{j=1}^3\langle x,e_j\rangle e_j$, $x\in\mathbb{C}^3$, then $Kx=\sum_{j=1}^3\langle x,\psi_j\rangle\varphi_j$. Hence Ψ is a K-dual of Φ which is not interchangeable. The frame operator of Φ is $S=\begin{bmatrix}2&0&0\\0&1&0\\0&0&0\end{bmatrix}$.

In the following proposition, trace formula for a tight K-frames is stated that is associated to its K-dual.

Proposition 3.3. *Let $\Phi=\{\varphi_j\}_{j=1}^M$ be a tight K-frame for $\mathcal{H}^N$. Then*

$$Tr(K)=\sum_{j=1}^M\langle\varphi_j,\psi_j\rangle, \tag{3.4}$$

where $\Psi=\{\psi_j\}_{j=1}^M$ is a K-dual of $\Phi=\{\varphi_j\}_{j=1}^M$.

Proof. We know that if $\{e_i\}_{i=1}^N$ is an arbitrary orthonormal basis for $\mathcal{H}^N$, then by definition, $Tr(K)=\sum_{i=1}^N\langle Ke_i,e_i\rangle$. Now, we can write

$$\begin{aligned}Tr(K)&=\sum_{i=1}^N\langle Ke_i,e_i\rangle=\sum_{i=1}^N\langle\sum_{j=1}^M\langle e_i,\psi_j\rangle\varphi_j,e_i\rangle=\sum_{j=1}^M\sum_{i=1}^N\langle e_i,\psi_j\rangle\langle\varphi_j,e_i\rangle\\&=\sum_{j=1}^M\sum_{i=1}^N\langle\varphi_j,e_i\rangle\langle e_i,\psi_j\rangle=\sum_{j=1}^M\langle\sum_{i=1}^N\langle\varphi_j,e_i\rangle e_i,\psi_j\rangle=\sum_{j=1}^M\langle\varphi_j,\psi_j\rangle.\end{aligned}$$

□

In the following theorem, we characterize the scalar sequences $\nu=\{\nu_j\}_{j=1}^M$ for which there exists a K-dual pair of frames $\{\varphi_j\}_{j=1}^M$ and $\{\psi_j\}_{j=1}^M$ such that $\nu_j=\langle\varphi_j,\psi_j\rangle$ for all $j=1,2,M$.

Theorem 3.4. *Let $K\in B(\mathcal{H}^N)$ and $\nu=\{\nu_j\}_{j=1}^M\subset\mathbb{C}$ with $M>dim(R(K))=rank(K)$ be given. Suppose that there exist K-dual frame pairs $\{\varphi_j\}_{j=1}^M$ and $\{\psi_j\}_{j=1}^M$ for $\mathcal{H}^N$ such that $\nu_j=\langle\varphi_j,\psi_j\rangle$ for all $j=1,2,M$. Then there exists a tight K^*-frame $\{\theta_j\}_{j=1}^M$ and a corresponding dual frame $\Gamma=\{\gamma_j\}_{j=1}^M$ for $\mathcal{H}^N$ such that $\nu_j=\langle\theta_j,\gamma_j\rangle$ for all $j=1,2,M$. Furthermore $Tr(K)=\sum_{j=1}^M\nu_j$.*

Proof. Fix $\nu = \{\nu_j\}_{j=1}^M \subset \mathbb{C}$ such that there exists a K-dual frame pairs $\{\varphi_j\}_{j=1}^M$ and $\{\psi_j\}_{j=1}^M$ such that satisfying $\nu_j = \langle \varphi_j, \psi_j \rangle$ for all $j = 1, 2, M$. Hence we can write $T_\Phi^* T_\Psi = K$ and $\nu = \text{diag}(T_\Phi^* T_\Psi)$, where diag($\cdot$) denotes the column vector of entries on the main diagonal of a matrix. Set $\theta_j = K^* S_\Phi^{-\frac{1}{2}} P_{S_\Phi^{\frac{1}{2}}(R(K))} \varphi_j$ and $\gamma_j = S_\Phi^{\frac{1}{2}} (K^\dagger |_{R(K)})^* \psi_j$. Thus $\Theta = \{\theta_j\}_{j=1}^M$ is a tight K^*-frame, since

$$\begin{aligned}
S_\Theta x &= \sum_{j=1}^M \langle x, \theta_j \rangle \theta_j \\
&= \sum_{j=1}^M \langle x, K^* S_\Phi^{-\frac{1}{2}} P_{S_\Phi^{\frac{1}{2}}(R(K))} \varphi_j \rangle K^* S_\Phi^{-\frac{1}{2}} P_{S_\Phi^{\frac{1}{2}}(R(K))} \varphi_j \\
&= K^* S_\Phi^{-\frac{1}{2}} P_{S_\Phi^{\frac{1}{2}}(R(K))} \sum_{j=1}^M \langle x, K^* S_\Phi^{-\frac{1}{2}} P_{S_\Phi^{\frac{1}{2}}(R(K))} \varphi_j \rangle \varphi_j \\
&= K^* S_\Phi^{-\frac{1}{2}} P_{S_\Phi^{\frac{1}{2}}(R(K))} \sum_{j=1}^M \langle (S_\Phi^{-\frac{1}{2}})^* K x, \varphi_j \rangle \varphi_j \\
&= K^* S_\Phi^{-\frac{1}{2}} P_{S_\Phi^{\frac{1}{2}}(R(K))} S_\Phi (S_\Phi^{-\frac{1}{2}})^* K x \\
&= K^* S_\Phi^{-\frac{1}{2}} P_{S_\Phi^{\frac{1}{2}}(R(K))} S_\Phi^{\frac{1}{2}} S_\Phi^{\frac{1}{2}} (S_\Phi^{-\frac{1}{2}})^* K x \\
&= K^* S_\Phi^{\frac{1}{2}} (S_\Phi^{-\frac{1}{2}})^* K x \\
&= K^* (S_\Phi^{-\frac{1}{2}} S_\Phi^{\frac{1}{2}})^* K x \\
&= K^* K x.
\end{aligned}$$

So we have $\langle S_\Theta x, x \rangle = \langle K^* K x, x \rangle = \langle K x, K x \rangle = \|K x\|^2$. Also, the synthesis operator associated with $\{\theta_j\}_{j=1}^M$ is $K^* S_\Phi^{-\frac{1}{2}} P_{S_\Phi^{\frac{1}{2}}(R(K))} T_\Phi^*$ and the synthesis operator associated with $\{\gamma_j\}_{j=1}^M$ is $S_\Phi^{\frac{1}{2}} (K^\dagger |_{R(K)}))^* T_\Psi^*$. Since

$$\begin{aligned}
T_\Theta^* (\{c_j\}_{j=1}^M) &= \sum_{j=1}^M c_j K^* S_\Phi^{-\frac{1}{2}} P_{S_\Phi^{\frac{1}{2}}(R(K))} \varphi_j \\
&= K^* S_\Phi^{-\frac{1}{2}} P_{S_\Phi^{\frac{1}{2}}(R(K))} \sum_{j=1}^M c_j \varphi_j \\
&= K^* S_\Phi^{-\frac{1}{2}} P_{S_\Phi^{\frac{1}{2}}(R(K))} T_\Phi^*.
\end{aligned}$$

We know that $K^\dagger |_{R(K)}: R(K) \longrightarrow \mathscr{H}^N$, thus $(K^\dagger |_{R(K)})^* : \mathscr{H}^N \longrightarrow R(K)$. So

$$\begin{aligned}
T_\Theta^* T_\Gamma &= K S_\Phi^{-\frac{1}{2}} P_{S_\Phi^{\frac{1}{2}}(R(K))} T_\Phi^* (S_\Phi^{\frac{1}{2}} (K^\dagger |_{R(K)})^* T_\Psi^*)^* \\
&= K S_\Phi^{-\frac{1}{2}} P_{S_\Phi^{\frac{1}{2}}(R(K))} T_\Phi^* T_\Psi K^\dagger |_{R(K)} S_\Phi^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= KS_\Phi^{-\frac{1}{2}} P_{S_\Phi^{\frac{1}{2}}(R(K))} K I_{N\times N} K^\dagger |_{R(K)} S_\Phi^{\frac{1}{2}} \\
&= KS_\Phi^{-\frac{1}{2}} P_{S_\Phi^{\frac{1}{2}}(R(K))} S_\Phi^{\frac{1}{2}} \\
&= K.
\end{aligned}
$$

Therefore $\Theta = \{\theta_j\}_{j=1}^M$ and $\Gamma = \{\gamma_j\}_{j=1}^M$ are K-dual frame pairs. Moreover, we have $T_\Theta^* T_\Gamma = K = T_\Phi^* T_\Psi$, that gives

$$
\operatorname{diag}(T_\Phi^* T_\Psi) = \operatorname{diag}(T_\Theta^* T_\Gamma) = \nu = \sum_{j=1}^M \langle \varphi_j, \psi_j \rangle = Tr(K). \qquad \square
$$

In the following results we characterize K-duals of a K-frame.

Proposition 3.5. *Let $\Phi = \{\varphi_j\}_{j=1}^M$ be a K-frame for $\mathscr{H}^N$. Then $\Psi = \{\psi_j\}_{j=1}^M$ is a K-dual for Φ if and only if $R(T_\Phi) \perp R(T_\Theta)$, where T_Θ is the analysis operator of the sequence $\Theta = \{\theta_j\}_{j=1}^M = \{\psi_j - K^* S_\Phi^{-1} P_{S_\Phi(R(K))} \varphi_j\}_{j=1}^M$.*

Proof. Let $\Phi = \{\varphi_j\}_{j=1}^M$ be a K-frame for $\mathscr{H}^N$ with K-dual $\Psi = \{\psi_j\}_{j=1}^M$, then

$$
\begin{aligned}
Kx &= \sum_{j=1}^M \langle x, \psi_j \rangle \varphi_j \\
&= \sum_{j=1}^M \langle x, \psi_j - K^* S_\Phi^{-1} P_{S_\Phi(R(K))} \varphi_j + K^* S_\Phi^{-1} P_{S_\Phi(R(K))} \varphi_j \rangle \varphi_j \\
&= \sum_{j=1}^M \langle x, \theta_j \rangle \varphi_j + \sum_{j=1}^M \langle x, K^* S_\Phi^{-1} P_{S_\Phi(R(K))} \varphi_j \rangle \varphi_j \\
&= \sum_{j=1}^M \langle x, \theta_j \rangle \varphi_j + \sum_{j=1}^M \langle Kx, S_\Phi^{-1} P_{S_\Phi(R(K))} \varphi_j \rangle \varphi_j \\
&= T_\Phi^* T_\Theta x + Kx. \qquad \square
\end{aligned}
$$

Recall that two K-frames $\Phi = \{\phi_j\}_{j=1}^M$ and $\Psi = \{\psi_j\}_{j=1}^M$ in a Hilbert space $\mathscr{H}^N$ are isomorphic K-frames if there exists an invertible operator $U : \mathscr{H}^N \longrightarrow \mathscr{H}^N$ so that $U\phi_j = \psi_j$ for all $1 \le j \le M$.

Proposition 3.6. *Let $K \in B(\mathscr{H}^N)$ and $\Phi = \{\varphi_j\}_{j=1}^M$ be a K-frame for $R(K^*)$ with the two different K-dual frames $\Psi = \{\psi_j\}_{j=1}^M$ and $\Gamma = \{\gamma_j\}_{j=1}^M$. Then Ψ and Γ are not isomorphic.*

Proof. Assume that $\Psi = \{\psi_j\}_{j=1}^M$ and $\Gamma = \{\gamma_j\}_{j=1}^M$ are isomorphic. Hence there exists an invertible operator $U \in B(\mathscr{H}^N)$ satisfying $U\psi_j = \gamma_j$, $j = 1, 2, \ldots, M$. Now, for any $x \in R(K^*)$ we can write

$$
KU^* x = \sum_{j=1}^M \langle U^* x, \gamma_j \rangle \varphi_j = \sum_{j=1}^M \langle x, U\gamma_j \rangle \varphi_j = \sum_{j=1}^M \langle x, \psi_j \rangle \varphi_j = Kx.
$$

Therefore $U|_{R(K^*)} = Id$ which is a contradiction. □

Proposition 3.7. *Let $K \in B(H^N)$. Let $\Phi = \{\varphi_j\}_{j=1}^M$ be a K-frame for $R(K^*)$, then the only K-dual frame of Φ, which is isomorphic to Φ is $\{K^* S_\Phi^{-1} P_{S_\Phi(R(K))}\varphi_j\}_{j=1}^M$.*

Proof. Suppose that $\Psi = \{\psi_j\}_{j=1}^M$ is a K-dual frame of Φ and there is an invertible operator U so that $\psi_j = UK^* S_\Phi^{-1} P_{S_\Phi(R(K))}\varphi_j$ for all i = 1, 2, M. Then, for every $x \in R(K^*)$ we have

$$\begin{aligned} KU^*x &= \sum_{j=1}^M \langle KU^*x, S_\Phi^{-1} P_{S_\Phi(R(K))}\varphi_j\rangle \varphi_j \\ &= \sum_{j=1}^M \langle x, UK^* S_\Phi^{-1} P_{S_\Phi(R(K))}\varphi_j\rangle \varphi_j \\ &= \sum_{j=1}^M \langle x, \psi_j\rangle \varphi_j = Kx. \end{aligned}$$

Hence $U|_{R(K^*)} = I_{N\times N}$. Thus

$$\psi_j = UK^* S_\Phi^{-1} P_{S_\Phi(R(K))}\varphi_j = K^* S_\Phi^{-1} P_{S_\Phi(R(K))}\varphi_j.$$ □

Oblique dual frames were proposed by several researchers. In particular, oblique dual frames in finite dimensional Hilbert space were studied in [18]. In the last part of this section, we study this notion for K-frames.

Definition 3.8. Let $\mathcal{U}$ and $\mathcal{W}$ be two subspaces of $\mathcal{H}^N$ and suppose that $\Phi = \{\varphi_j\}_{j=1}^M$ and $\Psi = \{\psi_j\}_{j=1}^M$ are in $\mathcal{H}^N$ and $\mathcal{W} = \text{span}\{\varphi_j : j = 1, 2, \ldots, M\}$, $\mathcal{U} = \text{span}\{\psi_j : j = 1, 2, \ldots, M\}$. The sequence $\Psi = \{\psi_j\}_{j=1}^M$ is an oblique K-dual frame of the K-frame $\Phi = \{\varphi_j\}_{j=1}^M$ on $\mathcal{W}$ if

$$Kx = \sum_{j=1}^M \langle x, \psi_j\rangle \varphi_j, \quad (x \in \mathcal{W}). \tag{3.5}$$

Note that (3.5) implies that $R(K|_{\mathcal{W}}) \subseteq \mathcal{W}$.

In the following two propositions a characterization of the oblique K-dual frames pair that are obtained by adding a pair of vector sequences to a given Bessel sequence is given. Also, characterize the uniqueness of the oblique K-dual frame pair.

Proposition 3.9. *Suppose that $\mathcal{W}$ is a subspace of $\mathcal{H}^N$ and sequences $\Phi = \{\varphi_j\}_{j=1}^M$, $\Psi = \{\psi_j\}_{j=1}^L$ and $\Gamma = \{\gamma_j\}_{j=1}^L$ in $\mathcal{H}^N$ satisfy that $span(\Phi \cup \Gamma) = \mathcal{W}$. Then the following statements are equivalent:*

(i) *$\Phi \cup \Psi$ is an oblique K-dual frame of $\Phi \cup \Gamma$ on $\mathcal{W}$.*

(ii) *For any $x \in \mathcal{W}$, $(K - S_\Phi)x = \sum_{j=1}^L \langle x, \psi_j\rangle \gamma_j$.*

Proof. (i) and (ii) are equivalent, since $\Phi \cup \Psi$ is an oblique K-dual frame of $\Phi \cup \Gamma$ on $\mathscr{W}$, if and only for all $x \in \mathscr{W}$

$$\begin{aligned} Kx &= \sum_{j=1}^{M} \langle x, \varphi_j \rangle \varphi_j + \sum_{j=1}^{L} \langle x, \psi_j \rangle \gamma_j \\ &= S_\Phi x + \sum_{j=1}^{L} \langle x, \psi_j \rangle \gamma_j, \end{aligned}$$

so

$$(K - S_\Phi)x = \sum_{j=1}^{L} \langle x, \psi_j \rangle \gamma_j. \qquad \square$$

Proposition 3.10. *If $\Psi = \{\psi_j\}_{j=1}^{M}$ is an oblique K-dual frame of $\Phi = \{\varphi_j\}_{j=1}^{M}$ on $\mathscr{W}$ and Φ is K-minimal, then the oblique K-dual frame of Φ on $\mathscr{W}$ is unique in the sense that if $\Gamma = \{\gamma_j\}_{j=1}^{M}$ is another oblique K-dual frame of Φ, then $\psi_j = \gamma_j$, $j = 1, \dots, M$, where Ψ, Γ are restricted in $\mathscr{W}$.*

Proof. Due to the fact that $\Psi = \{\psi_j\}_{j=1}^{M}$ and $\Gamma = \{\gamma_j\}_{j=1}^{M}$ are oblique K-dual frames of Φ on $\mathscr{W}$, then we can write

$$Kx = \sum_{j=1}^{M} \langle x, \psi_j \rangle \varphi_j = \sum_{j=1}^{M} \langle x, \gamma_j \rangle \varphi_j, \ (x \in \mathscr{W}).$$

Hence $\sum_{j=1}^{M} \langle x, \psi_j - \gamma_j \rangle \varphi_j = 0, x \in \mathscr{W}$. Now by K-minimality of Φ, we have

$$\langle x, \psi_j - \gamma_j \rangle = 0, \ j = 1, 2, \dots, M, \ x \in \mathscr{W},$$

and therefore $\psi_j = \gamma_j$, $j = 1, \dots, M$. $\square$

Here, we state that if Φ is a K-frame for $R(K)$, then we can make an oblique K-dual frame of algebraic multiplicity of $\{\varphi_j\}_{j=1}^{M} \cup \{e_j\}_{j \neq j_0}$ where $\{e_j\}_{j=1}^{d}$ is an orthonormal eigenbasis of the frame operator S_Φ with associated eigenvalues $\{\lambda_j\}_{j=1}^{d}$.

Theorem 3.11. *Let $K \in B(\mathscr{H}^N)$ and $\Phi = \{\varphi_j\}_{j=1}^{M}$ be a K-frame for $\mathscr{W} = R(K)$ with $dim \mathscr{W} = d$. Let $\{e_j\}_{j=1}^{d}$ be an orthonormal eigenbasis of the frame operator S_Φ with associated eigenvalues $\{\lambda_j\}_{j=1}^{d}$. Then for any eigenvalue $0 \neq \lambda_{j_0}$, the sequence $\{\frac{1}{\sqrt{\lambda_{j_0}}} K^* \varphi_j\}_{j=1}^{M} \cup \{\frac{(\lambda_{j_0} - \lambda_j)^{\frac{1}{3}}}{\sqrt{\lambda_{j_0}}} K^* e_j + K^* \gamma_j\}_{j \neq j_0}$, is an oblique K-dual frame of $\{\frac{1}{\sqrt{\lambda_{j_0}}} \varphi_j\}_{j=1}^{M} \cup \{\frac{(\lambda_{j_0} - \lambda_j)^{\frac{2}{3}}}{\sqrt{\lambda_{j_0}}} e_j\}_{j \neq j_0}$ on $\mathscr{W}$, where $\{\gamma_j\}_{j_0 \neq j=1}^{d} \subset \mathscr{H}^N$ satisfies*

$$\sum_{j_0 \neq j=1}^{d} \langle x, K^* \gamma_j \rangle e_j = 0, \ (x \in \mathscr{W}).$$

Proof. Let Φ be a K-frame for $\mathscr{W} = R(K)$ with the frame operator S_Φ. Also $\{e_j\}_{j=1}^{d}$ is an orthonormal eigenbasis, so for any $x \in \mathscr{W}$ we can write $Kx = \sum_{j=1}^{d} \langle Kx, e_j \rangle e_j$. Thus

$$S_\Phi Kx = \sum_{j=1}^{M} \langle Kx, \varphi_j \rangle \varphi_j = S_\Phi \sum_{j=1}^{d} \langle Kx, e_j \rangle e_j$$

$$= \sum_{j=1}^{d} \langle Kx, e_j\rangle S_{\Phi} e_j = \sum_{j=1}^{d} \langle Kx, e_j\rangle \lambda_j e_j$$
$$= \sum_{j=1}^{d} \lambda_j \langle Kx, e_j\rangle e_j.$$

Now for all $x \in \mathscr{W}$ we have

$$\begin{aligned}
Kx &= \frac{1}{\lambda_{j_0}} \sum_{j=1}^{d} \lambda_{j_0} \langle Kx, e_j\rangle e_j \\
&= \frac{1}{\lambda_{j_0}} (\lambda_{j_0} \langle Kx, e_{j_0}\rangle e_{j_0} + \sum_{j\neq j_0} \lambda_{j_0} \langle Kx, e_j\rangle e_j) \\
&= \frac{1}{\lambda_{j_0}} (\lambda_{j_0} \langle Kx, e_{j_0}\rangle e_{j_0} + \sum_{j\neq j_0} \lambda_j \langle Kx, e_j\rangle e_j - \sum_{j\neq j_0} \lambda_j \langle Kx, e_j\rangle e_j \\
&\quad + \sum_{j\neq j_0} \lambda_{j_0} \langle Kx, e_j\rangle e_j) \\
&= \frac{1}{\lambda_{j_0}} (\lambda_{j_0} \langle Kx, e_{j_0}\rangle e_{j_0} + \sum_{j\neq j_0} \lambda_j \langle Kx, e_j\rangle e_j \\
&\quad + \sum_{j\neq j_0} (\lambda_{j_0} - \lambda_j) \langle Kx, e_j\rangle e_j) \\
&= \frac{1}{\lambda_{j_0}} (\sum_{j=1}^{d} \lambda_j \langle Kx, e_j\rangle e_j + \sum_{j\neq j_0} (\lambda_{j_0} - \lambda_j) \langle Kx, e_j\rangle e_j) \\
&= \frac{1}{\lambda_{j_0}} \sum_{j=1}^{M} \langle Kx, \varphi_j\rangle \varphi_j + \sum_{j\neq j_0} (\frac{\lambda_{j_0} - \lambda_j}{\lambda_{j_0}}) \langle Kx, e_j\rangle e_j \\
&= \frac{1}{\lambda_{j_0}} \sum_{j=1}^{M} \langle Kx, \varphi_j\rangle \varphi_j + \sum_{j\neq j_0} \frac{(\lambda_{j_0} - \lambda_j)^{\frac{1}{3}}}{\sqrt{\lambda_{j_0}}} \frac{(\lambda_{j_0} - \lambda_j)^{\frac{2}{3}}}{\sqrt{\lambda_{j_0}}} \langle Kx, e_j + \gamma_j\rangle e_j \\
&= \frac{1}{\lambda_{j_0}} \sum_{j=1}^{M} \langle Kx, \varphi_j\rangle \varphi_j + \sum_{j\neq j_0} \langle Kx, \frac{(\lambda_{j_0} - \lambda_j)^{\frac{1}{3}}}{\sqrt{\lambda_{j_0}}} e_j + \gamma_j\rangle \frac{(\lambda_{j_0} - \lambda_j)^{\frac{2}{3}}}{\sqrt{\lambda_{j_0}}} e_j \\
&= \sum_{j=1}^{M} \langle Kx, \frac{1}{\sqrt{\lambda_{j_0}}} \varphi_j\rangle \frac{1}{\sqrt{\lambda_{j_0}}} \varphi_j \\
&\quad + \sum_{j\neq j_0} \langle Kx, \frac{(\lambda_{j_0} - \lambda_j)^{\frac{1}{3}}}{\sqrt{\lambda_{j_0}}} e_j + \gamma_j\rangle \frac{(\lambda_{j_0} - \lambda_j)^{\frac{2}{3}}}{\sqrt{\lambda_{j_0}}} e_j \\
&= \sum_{j=1}^{M} \langle x, \frac{1}{\sqrt{\lambda_{j_0}}} K^* \varphi_j\rangle \frac{1}{\sqrt{\lambda_{j_0}}} \varphi_j \\
&\quad + \sum_{j\neq j_0} \langle x, \frac{(\lambda_{j_0} - \lambda_j)^{\frac{1}{3}}}{\sqrt{\lambda_{j_0}}} K^* e_j + K^* \gamma_j\rangle \frac{(\lambda_{j_0} - \lambda_j)^{\frac{2}{3}}}{\sqrt{\lambda_{j_0}}} e_j,
\end{aligned}$$

which complete the proof. □

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