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REPRESENTATIONS OF COMPLETE REGULAR LOCAL NOETHER LATTICES

E. W. JOHNSON AND JOHNNY A. JOHNSON

Abstract. In this paper, we prove the uniqueness of the the ring of representation for any complete regular local Noether lattice. We investigate conditions for representability and obtain a form of necessary and sufficient condition.

1. Introduction

Throughout this paper, the term lattice will be used to denote a complete, modular lattice L with least element 0 and greatest element I . A multiplicative lattice is also assumed commutative with $IA = A$, for all $A \in L$ and with $A(\bigvee_{\alpha\in S} B_\alpha) = \bigvee_{\alpha\in S} AB_\alpha$, for all $A \in L$ and subsets $\{B_\alpha \in L \mid \alpha \in S\}$ of L.

Lattices in general form a natural abstraction of the lattice of ideals of a ring, and multiplicative lattices even more so. However, lacking a good notion of "principal" it is impossible to get very deep results [4]. Dilworth overcame this in [9] with a new notion of a principal element. Basically, an element E of a multiplicative lattice L is said to be principal if $A \wedge E = (A : E)E$ and $AE : E = A \vee (0 : E)$, for all $A \in L$. Here the residual quotient of two elements A, B is denoted by $A : B$, so $A : B = \bigvee\{X \in L \mid XB \le A\}$. If L is a commutative multiplicative lattice and E is a principal element of L, then $E \vee A$ is a principal element of $L / A = [I, A]$ with the multiplication $(F \vee A)(G \vee A) = (FG \vee A)$ under this definition of principal element (which otherwise would differ slightly from Dilworth's). This result was obtained by the first author and also reportedly by Dilworth. A proof was recorded in [8].

Dilworth's paper introducing principal elements also defined a *Noether Lattice* to be a (commutative) Noetherian multiplicative lattice which is principally generated, in the sense that every element is the join of principal elements. Dilworth went on to show that Noether lattices satisfy the Intersection Theorem and the Principal Ideal Theorem, thereby establishing that his definition of principal elements was a good one. It is known that in a local ring, Dilworth's definitions characterize principal ideals, and that outside of the local case, they do not. For example, if D is any Dedekind domain, then every ideal of D is a principal element but not necessarily a principal ideal. In fact, the lattice of ideals of D is isomorphic to the lattice of ideals of a PID, as was shown in [13], which can be interpreted to mean that principal ideals can not be distinguished lattice theoretically from principal elements.

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2. Regular Local Noether Lattices

Following Dilworth's seminal paper on the subject, work followed on regular local Noether lattices (see [7], [10]) and complete local Noether lattices . Work outside of the Noetherian case was also actively pursued, especially by D.D. Anderson [2], [3]. A Local Noether lattice $(L, \mathfrak{m})$ is said to be regular if the rank r of $\mathfrak{m}$ is equal to the number d of elements in a minimal set of (principal) generators. By the Principal Element Theorem, $d \geq r$. Bogart looked at distributive local Noether lattices, and especially at distributive regular local Noether lattices. Local Noether lattices have Hilbert polynomials with the same basic properties as those for rings.

3. Completions of Local Noether Lattices

A metric on a metric space induces a metric on the space of closed subsets of the space, called the Hausdorff topology. These authors have looked at the Hausdorff topology and the analogous topology on a local Noether lattice $(L, \mathfrak{m})$. This gives $d(A, B) = 2^{-s}$ if $A \vee \mathfrak{m}^s = B \vee \mathfrak{m}^s$ and $A \vee \mathfrak{m}^{s+1} \neq B \vee \mathfrak{m}^{s+1}$. We note that $A \vee \mathfrak{m}^s = B \vee \mathfrak{m}^s$ for all s implies $A = B$ by the Intersection Theorem [9]. We will call this the $\mathfrak{m}$-adic topology. A Noether lattice is said to be complete if it is complete in the topology of the Jacobson radical. For historical reasons, one would suspect that complete local Noether lattices would be most amenable to representation.

Theorem 1.(E. W. Johnson and J. A. Johnson [12]) *Every local Noether lattice $(L, \mathfrak{m})$ has a completion, and the lattice of ideals of a local ring $(R, \mathfrak{m})$ is complete iff for each decreasing sequence $\{C_i\}$ of ideals with intersection C_0, and for each natural number s there exists a natural number N such that $C_i \subseteq C_0 + \mathfrak{m}^s$, for all $i \geq N$. The same condition defines the completeness of $(L, \mathfrak{m})$, i.e., $C_i \leq C_0 \vee \mathfrak{m}^s$, for all $i \geq N$, where $C_0 = \bigwedge_i C_i$.*

A local ring $(R, \mathfrak{m})$ is said to be quasi-complete if the ideal lattice $(\mathfrak{L}(R), \mathfrak{m})$ is complete.

If $(L, \mathfrak{m})$ is a local Noether lattice and $(R, \mathfrak{n})$ a local ring, we denote the completions in the natural topologies by $(\widehat{L}, \widehat{\mathfrak{m}})$ and $(\widehat{R}, \hat{\mathfrak{n}})$.We denote the lattice of subspaces of an R-module M by $\mathfrak{L}_R(M)$ and we denote $\mathfrak{L}_R(R)$ by $\mathfrak{L}(R)$.

We call a local Noether ring $(R, \mathfrak{m})$ *large* if $char(R/\mathfrak{m}) = 0$.

4. Complete Regular Local Noether Lattices

In this section, we consider the representation of a complete, regular local Noether lattice $(L,\mathfrak{m})$ as the lattice of ideals of a local ring $(R,\mathfrak{n})$. We begin by noting that if $(L,\mathfrak{m})$ is representable as the lattice of ideals of a local ring $(R,\mathfrak{n})$ then R need not be complete but $\mathfrak{L}(R)$ satisfies the condition that for each decreasing sequence $\{C_i\}$ of elements with intersection C_0, and for each natural number s there exists a natural number N such that $C_i \subseteq C_0 + \mathrm{m}^s$, for all $i \geq N$. It follows from [11] that we have $\mathfrak{L}(R) \approx \mathfrak{L}(\widehat{R})$. The only principal elements of the lattice of ideals of a local ring are the principal ideals, so necessarily $(R,\mathfrak{n})$ is regular if $(L,\mathfrak{m})$ is regular and $\phi : \mathfrak{L}(R) \to L$ is an isomorphism. Hence we can consider from the beginning only representations of the form $\mathfrak{L}(R)$, for R a complete, regular local ring.

We will have occasion to deal with a "bilinear isomorphism" ψ from an R-module M onto an S-module N. By this we mean a ring isomorphism $a \mapsto \hat{a}$ of R onto S and group isomorphism ψ of M onto M with $\psi(ar) = \hat{a}\psi(r)$, for all $r \in R$.We will simply say that M and N are isomorphic in this case, as it is more intuitive.

If $(L,\mathfrak{m})$ is a regular, local Noether lattice of dimension 0 or 1 (and therefore necessarily complete), then L has the representation $L \approx \mathfrak{L}(R)$, where R is any regular local ring of the same dimension. There is no uniqueness to R, but its existence is trivial. But if L has dimension $d > 1$, the situation is turned around. Assume $(L,\mathfrak{m})$ is a complete local Noether lattice. If $\mathfrak{m}$ is the maximal element of L and n is chosen so that $U = m^n/m^{n+1}$ has dimension $r > 2$, then either U is the lattice of subspaces of a vector space over an infinite field $F_L = k$ or it is not. If it is, then k is unique, by the Fundamental Theorem of Projective Geometry. If then $(L,\mathfrak{m})$ has a representation as the lattice of ideals of an equicharacteristic regular local ring $(R,\mathfrak{n})$,the local ring is necessarily quasi complete, and so can be assumed to be complete. By the structure theorem for complete regular local rings, $R \approx k[[x_1,\ldots,x_d]]$, so this gives us the representation $(L,\mathfrak{m}) \approx L(R)$.Summarizing, we have the following two results.

Theorem 2.(Representation Test Criterion) *Let $(L,\mathfrak{m})$ be a complete regular local Noether lattice of dimension d. Assume $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ has a representation as the lattice of subspaces of a vector space U of dimension $r > 2$ over a field k of characteristic zero, i.e., $\mathfrak{m}^i/\mathfrak{m}^{i+1} \approx \mathfrak{L}_k(V)$.Then $(L,\mathfrak{m})$ is representable as the lattice of ideals of a large local ring iff L is isomorphic to the lattice of ideals of $k[[x_1,\ldots,x_d]]$.*

Theorem 3.(Uniqueness of Large Representation) *Let $(L,\mathfrak{m})$ be a complete, regular local Noether lattice of Krull dimension $d \geq 2$. Let $(S,\mathfrak{u})$ and $(T,\mathfrak{v})$ be complete local rings which represent L. Assume S is large. Then $S \approx T$.*

We note that there is not uniqueness to the representing ring without the assumption of completeness.

While it is enticing to guess that every complete regular local lattice is representable as the ideal lattice of a regular local ring, the following leads to examples to show this is not true.

Theorem 4. *Let* $(L, \mathfrak{m})$ *be a distributive local Noether lattice of dimension* d. *Then* L *is complete in the* m*-adic topology.*

Proof. It suffices to show that if $\{C_i\}_{i>0}$ is a decreasing sequence in L with meet C_0 then for each s, $C_i \leq C_0 \vee \mathfrak{m}^s$, for large i.Consider the decreasing sequence $\{C_i \vee \mathfrak{m}^s\}_{i\geq 1}$. As $L/\mathfrak{m}^s$ is finite dimensional, there exists an N with $C_i \vee \mathfrak{m}^s$ constant for $i \geq N$. If E is any principal element $E \leq C_i \vee \mathfrak{m}^s$, then either $E \leq C_i$ or $E \leq m^s$.Otherwise, $E = E \wedge (C_i \vee \mathfrak{m}^s) = (E \wedge Ci) \vee (E \wedge \mathfrak{m}^s) = (C_i : E)E \vee (\mathfrak{m}^s : E)E \leq \mathfrak{m}E$, so $E = 0 \leq C_i \wedge \mathfrak{m}^s$. If E is principal with $E \leq C_N \vee \mathfrak{m}^s$ then $E \leq \mathfrak{m}^s$ implies $E \leq (\bigwedge_i C_i) \vee \mathfrak{m}^s$, and $E \nleq \mathfrak{m}^s$ implies $E \leq C_i$, for all $i \geq N$, implies $E \leq (\bigwedge\limits_i C_i) \vee \mathfrak{m}^s = C_0 \vee \mathfrak{m}^s$.Hence $C_N \leq C_0 \vee \mathfrak{m}^s$.

We note the following additional result in closing. A Noether lattice is said to be small if it has a finite number of prime elements.

Theorem 5. *Let* $(L, \mathfrak{m})$ *be a small regular local Noether lattice with* s *rank one primes. Then* L *is a complete, regular local Noether lattice but* L *is not representable as the lattice of ideals of a ring.*

Proof. Let $(L, \mathfrak{m})$ be such a local Noether lattice. Anderson and E.W.Johnson have shown that small regular local Noether lattices are distributive [6]. It follows from the previous theorem that L is complete. Clearly L is not representable if $s > 1$, by the prime avoidance theorem.

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Santa Maria, CA, 93458, USA.

E-mail: ejohnson@math.uiowa.edu

Department of Mathematics, University of Houston, Houston, TX, 77204-3008, USA.

E-mail: jjohnson@math.uh.edu