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A REFINEMENT OF HÖLDER'S INTEGRAL INEQUALITY

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Abstract. The purpose of this note is to show that there is monotonic continuous function $p(t)$ such that

$$\int_a^b \left(\prod_{i=1}^n f_i(x)\right) dx \leq p(t) \leq \prod_{i=1}^n \left(\int_a^b f_i^{r_i}(x)dx\right)^{\frac{1}{r_i}},$$

where $f_1,\ f_2,\ldots,f_n$ are real positive continuous functions on $[a,b]$ and $r_1,\ r_2,\ldots,r_n$ are real positive numbers with $\sum_{i=1}^n \frac{1}{r_i}=1$.

It is well known that if $f_1,\ f_2,\ldots,f_n$ are real positive continuous functions on $[a,b]$ and $r_1,\ r_2,\ldots,r_n$ are real positive numbers with $\sum_{i=1}^n \frac{1}{r_i}=1$ then we have the Hölder's integral inequality

$$\int_a^b \left(\prod_{i=1}^n f_i(x)\right) dx \leq \prod_{i=1}^n \left(\int_a^b f_i^{r_i}(x)dx\right)^{\frac{1}{r_i}},$$

with equality holding if and only if $f_1^{r_1},\ f_2^{r_2},\ldots,f_n^{r_n}$ are effectively proportional.

Our aim is to give a continuous strictly increasing function $p(t)$ on $[0,1]$ such that

$$\int_a^b \left(\prod_{i=1}^n f_i(x)\right) dx \leq p(t) \leq \prod_{i=1}^n \left(\int_a^b f_i^{r_i}(x)dx\right)^{\frac{1}{r_i}}. \tag{1}$$

for all $t\in[0,1]$.

Lemma.([1]) *Let f and g be increasing functions on $[0,+\infty)$. Let v and h be non-negative measurable functions, and let a and b, $a<b$, be real numbers. Then*

$$\int_a^b f(v(t))g(v(t))h(t)dt\int_a^b h(t)dt \geq \int_a^b f(v(t))h(t)dt\int_a^b g(v(t))h(t)dt. \tag{2}$$

Remark. By Theorem 10 of 2.5 in [2], we see that equality holds in (2) if and only if $v(t)$ is a constant.

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Theorem. *Let f_1, $f_2, \ldots, f_n$ be real positive continuous functions on $[a,b]$. Let r_1, $r_2, \ldots, r_n$ be real positive numbers with $\sum_{i=1}^n \frac{1}{r_i} = 1$.*

Define a function p by

$$p(t) = \prod_{k=1}^n \left\{ \int_a^b \left[\prod_{i=1}^n f_i(x) \right]^{1-t} [f_k^{r_k}(x)]^t dx \right\}^{\frac{1}{r_k}}, \quad t \in (-\infty, +\infty). \tag{3}$$

Then $p'(t) \geq 0$ for $t > 0$, $p'(t) \leq 0$ for $t < 0$, and $p'(t) = 0$ if and only if $t = 0$ or $f_1^{r_1}$, $f_2^{r_2}, \ldots, f_n^{r_n}$ are effectively proportional.

Proof. It is easy to find that for each $x \in [a,b]$ we have

$$\sum_{k=1}^n \frac{1}{r_k} \log \frac{f_k^{r_k}(x)}{\prod_{i=1}^n f_i(x)} = \sum_{k=1}^n \frac{1}{r_k} \left[r_k \log f_k(x) - \sum_{i=1}^n \log f_i(x) \right] = 0,$$

and hence

$$\sum_{k=1}^n \frac{1}{r_k} \int_a^b \left(\prod_{i=1}^n f_i(x) \right) \log \frac{f_k^{r_k}(x)}{\prod_{i=1}^n f_i(x)} dx = 0. \tag{4}$$

Let $g(x) = \prod_{i=1}^n f_i(x)$, $h_k(x) = \frac{f_k^{r_k}(x)}{g(x)}$, $k = 1, 2, \ldots, n$.

Then $g(x)$ and $h_1(x)$, $h_2(x), \ldots, h_n(x)$ are all real positive continuous functions on $[a,b]$ and we can write (3) as

$$p(t) = \prod_{k=1}^n \left[\int_a^b g(x) h_k^t(x) dx \right]^{\frac{1}{r_k}}.$$

Let $P(t) = \log p(t)$. Observe that (4) can write as

$$\sum_{k=1}^n \frac{1}{r_k} \int_a^b g(x) \log h_k(x) dx = 0,$$

We obtain

$$\begin{aligned} P'(t) = \frac{p'(t)}{p(t)} &= \sum_{k=1}^n \frac{1}{r_k} \frac{\int_a^b g(x) h_k^t(x) \log h_k(x) dx}{\int_a^b g(x) h_k^t(x) dx} \\ &= \sum_{k=1}^n \frac{1}{r_k} \frac{\int_a^b g(x) h_k^t(x) \log h_k(x) dx}{\int_a^b g(x) h_k^t(x) dx} - \sum_{k=1}^n \frac{1}{r_k} \frac{\int_a^b g(x) \log h_k(x) dx}{\int_a^b g(x) dx} \\ &= \sum_{k=1}^n \frac{1}{r_k} \frac{\int_a^b g(x) h_k^t(x) \log h_k(x) dx \int_a^b g(x) dx - \int_a^b g(x) h_k^t(x) dx \int_a^b g(x) \log h_k(x) dx}{\int_a^b g(x) dx \int_a^b g(x) h_k^t(x) dx} \end{aligned}$$

Consequently, the results follow from the lemma and its remark.

Corollary. *Let $f_1,\ f_2,\dots,f_n$ be real positive continuous functions on $[a,b]$. Let r_1, $r_2,\dots,r_n$ be real positive numbers with $\sum_{i=1}^n \frac{1}{r_i}=1$. Then the function p defined by* (3) *has the following properties:*

(i) *$p(t)$ is continuous strictly increasing for $t\geq 0$ and continuous strictly decreasing for $t<0$ or a constant.*

(ii) *$p(t)$ is a constant if and only if $f_1^{r_1},\ f_2^{r_2},\dots,f_n^{r_n}$ are effectively proportional.*

(iii) *$p(0)=\int_a^b(\prod_{i=1}^n f_i(x))dx$ and $p(1)=\prod_{i=1}^n(\int_a^b f_i^{r_i}(x)dx)^{\frac{1}{r_i}}$.*

(iv) *The inequalities* (1) *are valid for all $t\in[0,1]$ with equalities holding if and only if $f_1^{r_1},\ f_2^{r_2},\dots,f_n^{r_n}$ are effectively proportional.*

References

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