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TAUBERIAN THEOREMS FOR STATISTICAL CONVERGENCE

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Abstract. The Tauberian theorems for statistical limitable method are proved by both Fridy and Khan [2] and Móricz [10]. Here we generalize these theorems to (C; i) statistical limitable method.

1. Introduction

Let S be a subset of an ordered set of $\mathbb{R}$ or $\mathbb{C}$ numbers. Consider the set

$$S(n) := \{i \le n | i \in S\}.$$

We say that$S$ has density $D(S)$, if the limit

$$D(S) := \lim_{n\to\infty} \frac{|S(n)|}{n+1},$$

exists. Here $|S(n)|$ means the cardinality of the set $S(n)$.
Let (u_n) be a sequence in S and consider the set

$$S_\varepsilon(\ell) := \{k \in S : |u_k - \ell| \ge \varepsilon\}$$

for every $\varepsilon > 0$ and $k = 0, 1, 2, 3...$. Hence $D(S_\varepsilon(\ell))$ denotes the density of the set $S_\varepsilon(\ell)$.

Definition 1.1. A sequence $u = (u_n)$ of real (or complex) numbers is said to be statistically convergent to ℓ if $D(S_\varepsilon(\ell)) = 0$ for every $\varepsilon > 0$ and we write st-lim $u_n = \ell$.

For $i \in \mathbb{N}$ and $n \in \mathbb{N}^*$, define

$$\sigma_n^i(u) = \begin{cases} \dfrac{1}{n+1} \displaystyle\sum_{k=0}^{n} \sigma_k^{i-1}(u) & \text{if} \quad i \ge 1, \\ u_n & \text{if} \quad i = 0. \end{cases}$$

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Definition 1.2. We say that a sequence $u = (u_n)$ is statistically summable (C,i) to ℓ for all $i \in \mathbb{N}$ if

$$st - \lim \sigma_n^i(u) = \ell.$$

By [11], it is known that a sequence (u_n) of real (or complex) numbers is slowly oscillating if for any given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ and $N = N(\varepsilon)$ such that

$$|u_m - u_n| < \varepsilon \quad \text{if} \quad n \geq N(\varepsilon) \quad \text{and} \quad n \leq m \leq (1+\delta)n,$$

and a sequence (u_n) of real numbers is slowly deacreasing if

$$\liminf (u_m - u_n) \geq 0 \quad \text{whenever} \quad m > n \to \infty, \ \frac{m}{n} \to 1.$$

By [8], we say that a sequence (u_n) is statistically slowly decreasing if for each $\varepsilon > 0$,

$$\inf_{\lambda > 1} \limsup_{N \to \infty} \frac{1}{N+1} \left| \left\{ n \leq N : \min_{n < m \leq \lambda_n} (u_m - u_n) \leq -\varepsilon \right\} \right| = 0,$$

or equivalently

$$\inf_{0 < \lambda < 1} \limsup_{N \to \infty} \frac{1}{N+1} \left| \left\{ n \leq N : \min_{\lambda_n < m \leq n} (u_n - u_m) \leq -\varepsilon \right\} \right| = 0$$

and also, (u_n) is statistically slowly oscillating if for each $\varepsilon > 0$,

$$\inf_{\lambda > 1} \limsup_{N \to \infty} \frac{1}{N+1} \left| \left\{ n \leq N : \max_{n < m \leq \lambda_n} |u_m - u_n| \geq \varepsilon) \right\} \right| = 0,$$

or equivalently

$$\inf_{0 < \lambda < 1} \limsup_{N \to \infty} \frac{1}{N+1} \left| \left\{ n \leq N : \max_{\lambda_n < m \leq n} |u_n - u_m| \geq \varepsilon) \right\} \right| = 0.$$

Definition 1.3 ([3])**.** A sequence (u_n) of real numbers is called Abel convergent (or Abel summable) to ℓ if the series $\Sigma_{k=0}^{\infty} u_k x^k$ is convergent for $0 \leq x < 1$ and

$$\lim_{x \to 1^-} (1-x) \sum_{k=0}^{\infty} u_k x^k = \ell.$$

In this case, we write $Abel - \lim u_n = \ell$.

Moreover, by [5], the sequence (u_n) is Borel summable to ℓ provided that

$$\lim_{t \to \infty} e^{-t} \sum_{k=0}^{\infty} \frac{u_k t^k}{k!} = \ell.$$

Consider the summability matrix $B^* = (b_{nk})$ is given by

$$b_{nk} = \frac{e^{-n} n^k}{k!}$$

and the Abel matrix $A_\lambda = [a_{ij}]$ associated with a strictly increasing sequence $\lambda = (\lambda_n)$ of real number with $\lambda_0 \geq 1$ is defined by

$$a_{ij} = \frac{1}{\lambda_i}\Big(1 - \frac{1}{\lambda_i}\Big)^j, \qquad j = 0, 1, 2, \dots.$$

It will be convenient for us to extend λ to be defined over the interval $[1, \infty)$ by making it linear and continuous over $[j, j+1]$ for $j = 1, 2, \dots$.

By [7], a sequence of real (or complex) numbers (u_n) is said to be summable $(L, 1)$ to ℓ if $\lim \varphi_n = \ell$ where

$$\tau_n(u) = \frac{1}{h_n}\sum_{k=1}^{n}\frac{u_k}{k} \quad \text{and} \quad h_n = \sum_{k=1}^{n}\frac{1}{k} \sim \log n, \quad n = 1, 2, 3, \dots.$$

Throughout this paper, the symbols $\Delta u_n = u_n - u_{n-1} = o(1)$ and $u_n = O(1)$ mean that $u_n \to 0$ as $n \to \infty$ and that (u_n) is bounded for large enough n, respectively.

Theorem 1.4 ([2], [9], [10])**.** *With preceding notation, we have:*

(1) *If st-*$\lim \sigma_n^1(u) = \ell$ *and* $n\Delta u_n = n(u_n - u_{n-1}) = O(1)$ *then* $\lim u_n = \ell$.

(2) *If st-*$\lim u_n = \ell$ *or st-*$\lim \sigma_n^1(u) = \ell$ *and* $n\Delta u_n \geq -c$ *for a positive number c then* $\lim u_n = \ell$.

(3) *Let* λ_n *be a strictly increasing sequence of real numbers tends to infinity such that*

$$\lim_{\delta \searrow 0}\lim_{n}\frac{\lambda_{n+\delta n}}{\lambda_n} = 1.$$

If st-$\lim A_\lambda u_n = \ell$ *and* (u_n) *satisfies the slow decrease condition, then* $\lim u_n = \ell$.

(4) *Let* B^* *be the Borel summability matrix associated with* (u_n) *such that satisfies* (B^*) *st-*$\lim B^* u_n = \ell$ *and* $\Delta u_n = O(\frac{1}{\sqrt{n}})$*, then* $\lim u_n = \ell$.

(5) *If st-* $\lim \tau_n(u) = \ell$ *and* (u_n) *slowly decreasig then* $\lim u_n = \ell$.

(6) *If st-* $\lim u_n = \ell$ *and* (u_n) *slowly decreasig then* $\lim u_n = \ell$.

Now, we will prove that the hypothesis st-$\lim \sigma_n^1(u) = \ell$ and st-$\lim A_\lambda u_n = \ell$ can be replaced by st-$\lim \sigma_n^i(u) = \ell$ and st-$\lim A_\lambda \sigma_n^i(u) = \ell$. Similarly, the hypothesis st-$\lim u_n = \ell$ and st- $\lim \tau_n(u) = \ell$ can be replaced by st-$\lim \sigma_n^i(u) = \ell$ and st- $\lim \tau_n(\sigma_n^i(u)) = \ell$. Moreover we prove a different formulation of (2) above. Before proving our statements, we recall more results that we will need in the sequel.

Theorem 1.5 ([8], [11],[10])**.**

(i) *Let a sequence* (u_n) *of real numbers be statistically slowly decreasing. Then*

$$st - \lim \sigma_n^1(u) = \ell \quad \textit{implies} \quad st - \lim u_n = \ell.$$

(ii) *Let a sequence* (u_n) *of complex numbers be statistically slowly oscillating. Then*

$$st-\lim \sigma_n^1(u)=\ell \quad implies \quad st-\lim u_n=\ell.$$

(iii) *Let a sequence* (u_n) *of real numbers be slowly decreasing. Then*

$$Abel-\lim u_n=\ell \quad implies \quad \lim u_n=\ell.$$

(iv) *If st-* $\lim u_n=\ell$ *and* (u_n) *slowly decreasing, then* $\lim u_n=\ell$.

(v) *If st-* $\lim u_n=\ell$ *and* (u_n) *slowly oscillating, then* $\lim u_n=\ell$.

2. Main results

Lemma 2.1. *If* (u_n) *is slowly oscillating then* $(\sigma_n^i(u))$ *for all* $i\geq 1$ *is slowly oscillating.*

Proof. By hypothesis, since (u_n) is slowly oscillating we write $|u_m-u_n|\leq\varepsilon$ whenever $m>n\to\infty$, $\frac{m}{n}\to 1$. Hence, we have $|u_m-u_n|=|\sum_{k=n+1}^{m}\Delta u_k|\leq\varepsilon$. We claim that $(\sigma_n^i(u))$ for all $i\geq 1$ is slowly oscillating that is $|\sigma_m^i(u)-\sigma_n^i(u)|\leq\varepsilon$ whenever $m>n\to\infty$, $\frac{m}{n}\to 1$. We will prove this by using mathematical induction. We show that our claims true for $i=1$.

$$\begin{aligned}
|\sigma_m^1(u)-\sigma_n^1(u)| &= \Big|\sum_{k=n+1}^{m}(\sigma_k^1(u)-\sigma_{k-1}^1(u))\Big| = \Big|\sum_{k=n+1}^{m}\frac{k}{k}\Big\{\frac{1}{k}\sum_{p=1}^{k}u_p-\frac{1}{k-1}\sum_{p=1}^{k-1}u_p\Big\}\Big| \\
&= \Big|\sum_{k=n+1}^{m}\frac{1}{(k-1)k}\Big\{k-1\sum_{p=1}^{k}u_p-k\sum_{p=1}^{k-1}u_p\Big\}\Big| \\
&= \Big|\sum_{k=n+1}^{m}\frac{1}{(k-1)k}\Big\{(k-1)u_p-\sum_{p=1}^{k-1}u_p\Big\}\Big| = \Big|\sum_{k=n+1}^{m}\frac{1}{(k-1)k}\sum_{p=1}^{k-1}(u_k-u_p)\Big| \\
&= \Big|\sum_{k=n+1}^{m}\frac{1}{(k-1)k}\sum_{p=1}^{k-1}\sum_{j=p+1}^{k}\Delta u_j\Big| \leq \sum_{k=n+1}^{m}\frac{1}{(k-1)k}\sum_{p=1}^{k-1}\Big|\sum_{j=p+1}^{k}\Delta u_j\Big| \\
&\leq \sum_{k=n+1}^{m}\frac{1}{(k-1)k}\sum_{p=1}^{k-1}\varepsilon = \varepsilon\sum_{k=n+1}^{m}\frac{1}{(k-1)k}\sum_{p=1}^{k-1} = \varepsilon\sum_{k=n+1}^{m}\frac{1}{(k-1)k}(k-1) \\
&= \varepsilon\sum_{k=n+1}^{m}\frac{1}{k} = \varepsilon\Big(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{m}\Big) \leq \varepsilon\Big(\frac{1}{n}+\frac{1}{n}+\cdots+\frac{1}{n}\Big) \\
&= \varepsilon\frac{m-n}{n} = \varepsilon\Big(\frac{m}{n}-1\Big)\leq\varepsilon, \quad \text{by} \quad \frac{m}{n}\to 1.
\end{aligned}$$

Assume that it is true for $i=t-1$, and we will prove that it is true for $i=t$. By assumption, since $(\sigma_n^{t-1}(u))$ is slowly oscillating we write $|\sigma_m^{t-1}(u)-\sigma_n^{t-1}(u)|\leq\varepsilon$ whenever $m>n\to\infty$, $\frac{m}{n}\to 1$. Hence, we have

$$|\sigma_m^{t-1}(u)-\sigma_n^{t-1}(u)| = \Big|\sum_{k=n+1}^{m}\sigma_n^{t-1}(u)\Big|\leq\varepsilon. \ \ For \ \ i=t,$$

$$\begin{aligned}
|\sigma^t_m(u)-\sigma^t_n(u)| &= \Big|\sum_{k=n+1}^{m}(\sigma^t_k(u)-\sigma^t_{k-1}(u))\Big| \\
&= \Big|\sum_{k=n+1}^{m}\frac{k}{k}\Big\{\frac{1}{k}\sum_{p=1}^{k}\sigma^{t-1}_p(u)-\frac{1}{k-1}\sum_{p=1}^{k-1}\sigma^{t-1}_p(u)\Big\}\Big| \\
&= \Big|\sum_{k=n+1}^{m}\frac{1}{(k-1)k}\Big\{k-1\sum_{p=1}^{k}\sigma^{t-1}_p(u)-k\sum_{p=1}^{k-1}\sigma^{t-1}_p(u)\Big\}\Big| \\
&= \Big|\sum_{k=n+1}^{m}\frac{1}{(k-1)k}\{(k-1)\sigma^{t-1}_p(u)-\sum_{p=1}^{k-1}\sigma^{t-1}_p(u)\}\Big| \\
&= \Big|\sum_{k=n+1}^{m}\frac{1}{(k-1)k}\sum_{p=1}^{k-1}(\sigma^{t-1}_k(u)-\sigma^{t-1}_p(u))\Big| \\
&= \Big|\sum_{k=n+1}^{m}\frac{1}{(k-1)k}\sum_{p=1}^{k-1}\sum_{j=p+1}^{k}\Delta\sigma^{t-1}_j(u)\Big| \le \sum_{k=n+1}^{m}\frac{1}{(k-1)k}\sum_{p=1}^{k-1}\Big|\sum_{j=p+1}^{k}\Delta\sigma^{t-1}_j(u)\Big| \\
&\le \sum_{k=n+1}^{m}\frac{1}{(k-1)k}\sum_{p=1}^{k-1}\varepsilon=\varepsilon\sum_{k=n+1}^{m}\frac{1}{(k-1)k}\sum_{p=1}^{k-1}=\varepsilon\sum_{k=n+1}^{m}\frac{1}{(k-1)k}(k-1) \\
&= \varepsilon\sum_{k=n+1}^{m}\frac{1}{k}=\varepsilon\Big(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{m}\Big)\le\varepsilon\Big(\frac{1}{n}+\frac{1}{n}+\cdots+\frac{1}{n}\Big) \\
&= \varepsilon\frac{m-n}{n}=\varepsilon\Big(\frac{m}{n}-1\Big)\le\varepsilon,\quad \text{by}\quad \frac{m}{n}\to 1.
\end{aligned}$$

Thus proof is done. □

Lemma 2.2. *If (u_n) is slowly decreasing then $(\sigma^i_n(u))$ for all $i\ge 1$ is slowly decreasing.*

Proof. Proof is similar to one of Lemma 2.1. □

Lemma 2.3. *If (u_n) is slowly decreasing then $(\sigma^i_n(u))$ for all $i\ge 1$ is statistically slowly decreasing.*

Proof. By Lemma 2.2, as (u_n) is slowly decreasing $(\sigma^i_n(u))$ for all $i\ge 1$ is slowly decreasing. Since $(\sigma^i_n(u))$ is slowly decreasing we write for large enough n, $n>N_1$, and

$$\sigma^i_m(u)-\sigma^i_n(u)=\sum_{k=n+1}^{m}\Delta\sigma^i_k(u)\ge-\varepsilon,\ \textit{whenever}\ m>n\to\infty,\frac{m}{n}\to 1.$$

Since for $N_1<n<m\le\lambda_n$, for $N>N_1$ the set

$$\{N_1<n\le N:\min_{n<m\le\lambda_n}(\sigma^i_m(u)-\sigma^i_n(u))\le-\varepsilon\}$$

is empty. It follows that $(\sigma^i_n(u))$ is statistically slowly decreasing. □

Lemma 2.4. *If (u_n) is slowly oscillating then $(\sigma^i_n(u))$ for all $i\ge 1$ is statistically slowly oscillating.*

Proof. Proof is similar to one of Lemma 2.3. □

The following theorem generalizes (1) above which is given by Fridy and Khan in [2].

Theorem 2.5. *If* $st-\lim\sigma_n^i(u)=\ell$ *and* $n\Delta u_n=O(1)$ *then* $\lim u_n=\ell$.

Proof. If $n\Delta u_n=O(1)$ for a positive number c then $n\Delta\sigma_n^i(u)=O(1)$; we will this by using mathematical induction. For $i=1$ and $n>1$,

$$\begin{aligned}
n\Delta\sigma_n^1(u) &= n\Big\{\frac{1}{n}\sum_{k=1}^{n}u_k-\frac{1}{n-1}\sum_{k=1}^{n-1}u_k\Big\}=\frac{1}{n-1}\Big\{n-1\sum_{k=1}^{n}u_k-n\sum_{k=1}^{n-1}u_k\Big\}\\
&=\frac{1}{n-1}\Big\{(n-1)u_k-\sum_{k=1}^{n-1}u_k\Big\}=\frac{1}{n-1}\sum_{k=1}^{n-1}(u_n-u_k)=\frac{1}{n-1}\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}\Delta u_j\\
&=\frac{1}{n-1}\sum_{j=2}^{n}(j-1)\Delta u_j=\frac{1}{n-1}\sum_{j=2}^{n}O(1)=O(1).
\end{aligned}$$

Assume that it is true for $i=t-1$, and we prove that it is true for $i=t$. By assumption, we write $n\Delta\sigma_n^{t-1}(u)=O(1)$. For $i=t$ and $n>1$,

$$\begin{aligned}
n\Delta\sigma_n^t(u) &= n\Big\{\frac{1}{n}\sum_{k=1}^{n}\sigma_k^{t-1}(u)-\frac{1}{n-1}\sum_{k=1}^{n-1}\sigma_k^{t-1}(u)\Big\}\\
&=\frac{1}{n-1}\Big\{n-1\sum_{k=1}^{n}\sigma_k^{t-1}(u)-n\sum_{k=1}^{n-1}\sigma_k^{t-1}(u)\Big\}=\frac{1}{n-1}\Big\{(n-1)\sigma_k^{t-1}(u)-\sum_{k=1}^{n-1}\sigma_k^{t-1}(u)\Big\}\\
&=\frac{1}{n-1}\sum_{k=1}^{n-1}(\sigma_n^{t-1}(u)-\sigma_k^{t-1}(u))=\frac{1}{n-1}\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}\Delta\sigma_j^{t-1}(u)\\
&=\frac{1}{n-1}\sum_{j=2}^{n}(j-1)\Delta\sigma_j^{t-1}(u)=\frac{1}{n-1}\sum_{j=2}^{n}O(1)=O(1).
\end{aligned}$$

Hence, we have $n\Delta\sigma_n^{i-1}(u)=O(1)$ is a Tauberian condition for statistical convergence. Therefore, st-$\lim\sigma_n^i(u)=\ell$ implies that $\lim\sigma_n^{i-1}(u)=\ell$. By the fact that every sequence (C,i-1) limitable is (C,i-1) statistical limitable, we have st- $\lim\sigma_n^{i-1}(u)=\ell$. From this it follows that $st-\lim\sigma_n^1(u)=\ell$. By (1) above, one obtains $\lim u_n=\ell$. □

The following theorem extends (2) above.

Theorem 2.6. *If* $st-\lim\sigma_n^i(u)=\ell$ *and* $n\Delta u_n\geq -c$ *for a positive number c and for every n, then* $\lim u_n=\ell$.

Proof. If $n\Delta u_n\geq -c$ for a positive number c satifies then satisfies $n\Delta\sigma_n^i(u)\geq -c$; for, by the proof of Theorem 2.5,

$$n\Delta\sigma_n^i(u)=\frac{1}{n-1}\sum_{j=2}^{n}(j-1)\Delta\sigma_j^{i-1}(u)\geq\frac{-(n-1)c}{n-1}=-c.$$

So by (ii) above, $\lim \sigma_n^{i-1}(u) = \ell$. By the fact that every sequence (C,i-1) limitable is (C,i-1) statistical limitable, we have st- $\lim \sigma_n^{i-1}(u) = \ell$. Continuing in this way, we obtain $st{-}\lim \sigma_n^1(u) = \ell$. By (ii) above, $\lim u_n = \ell$. □

Theorem 2.7 extends the (6) above which is given by Móricz [10].

Theorem 2.7. *Let (u_n) be a sequence for real numbers be slowly decreasing. Then*

$$st-\lim \sigma_n^i(u) = \ell \quad \Rightarrow \quad st-\lim u_n = \ell \quad \Rightarrow \quad \lim u_n = \ell.$$

Proof. By Lemma 2.3, as (u_n) is slowly decreasing $(\sigma_n^{i-1}(u))$ for all $i \geq 1$ is statistically slowly decreasing. Since $\sigma_n^i(u)$ is statistically convergence to ℓ, $st-\lim \sigma_n^{i-1}(u) = \ell$, by (i) above. It follows that $(\sigma_n^{i-2}(u))$ is statistically slowly decreasing $st-\lim \sigma_n^{i-2}(u) = \ell$. Continuing in this way, we obtain $st{-}\lim \sigma_n^1(u) = \ell$. By (i) above, $st{-}\lim u_n = \ell$. Furthermore, since (u_n) is slowly decreasing, $\lim u_n = \ell$ by (iv) above. □

Corollary 2.8. *If $st-\lim \sigma_n^i(u) = \ell$ and $n\Delta u_n \geq -c$ for a positive number c then* $\lim u_n = \ell$.

Proof. The condition $n\Delta u_n \geq -c$ for some $c > 0$ implies that (u_n) is slowly decreasing (See [8]). Thus the proof follows immediately from Theorem 2.7. □

Note that, if the sequence (u_n) is bounded and statistically slowly decreasing, we obtain an analogue to (i).

Corollary 2.9. *Let (u_n) be a sequence of real numbers which is statistically slowly decreasing and bounded. Then*

$$st-\lim \sigma_n^i(u) = \ell \quad \textit{implies} \quad st-\lim u_n = \ell.$$

Proof. The proof is similar to one of Theorem 2.7. □

Remark 2.10. Theorem 2.7 remains true if the term "decreasing" is replaced by "increasing." Furthermore, condition where (u_n) is slowly increasing [8] can be replaced if there exists a positive constant c such that $n\Delta u_n \leq c$ for all n large enough. Thus, we give a Tauberian condition for the sequences of complex numbers in connection (2) above and the following theorem extends the result of Móricz in [10].

Theorem 2.11. *Let (u_n) be a sequence for complex numbers which is slowly oscillating. Then*

$$st-\lim \sigma_n^i(u) = \ell \quad \Rightarrow \quad st-\lim u_n = \ell \quad \Rightarrow \quad \lim u_n = \ell.$$

Proof. By Lemma 2.1, as (u_n) is slowly oscillating, $(\sigma_n^{i-1}(u))$ for all $i \geq 1$ is slowly oscillating. This by Lemma 2.4 implies that $(\sigma_n^{i-1}(u))$ is statistically slowly oscillating. Since $\sigma_n^i(u)$ is statistically convergent to ℓ, we have $st-\lim \sigma_n^{i-1}(u) = \ell$, by (ii) above. If we continue in that way, we obtain $st-\lim \sigma_n^1(u) = \ell$. By (ii) above, $st-\lim u_n = \ell$. Furthermore, since (u_n) is slowly oscillating, we have $\lim u_n = \ell$ by (v) above. □

Remark 2.12. The condition $n\Delta u_n = n(u_n - u_{n-1}) = O(1)$ implies that (u_n) is slowly oscillating [8]. Thus, Theorem 2.11 generalizes (1) above which is given by Fridy and Khan [2].

Corollary 2.13. *If $st-\lim \sigma_n^i(u) = \ell$ and $n\Delta u_n = O(1)$ then $\lim u_n = \ell$.*

Proof. The proof is obvious from Theorem 2.11. □

Note that, if the sequence (u_n) is bounded and statistically slowly decreasing, we obtain an analogue to (ii).

Corollary 2.14. *Let (u_n) be a sequence of complex numbers which is statistically slowly oscillating and bounded. Then*

$$st-\lim \sigma_n^i(u) = \ell \quad \textit{implies} \quad st-\lim u_n = \ell.$$

Proof. The proof is clear after Theorem 2.11. □

Our next theorem generalizes (3) above.

Theorem 2.15. *Let λ_n be a strictly increasing sequence of real numbers tends to infinity such that*

$$\lim_{\delta \searrow 0} \lim_n \frac{\lambda_{n+\delta n}}{\lambda_n} = 1.$$

If the sequence $(\sigma_n^i(u))$ satisfies st-$\lim A_\lambda \sigma_n^i(u) = \ell$ and (u_n) satisfies the slowly decreasing condition, then $\lim u_n = \ell$.

Proof. By Lemma 2.2 slowly decreasing of $u = (u_n)$ implies slowly decreasing of $(\sigma_n^i(u))$ for all $i \geq 1$. By Lemma 2.2 in [2], we see that $(A_\lambda \sigma_n^i(u))$ obeys slowly decreasing Tauberian condition. By (2) above, we have $\lim A_\lambda \sigma_n^i(u) = \ell$. Theorem 5 in [1] implies that $\sigma_n^i(u)$ is Abel summable to ℓ. Since $Abel-\lim \sigma_n^i(u) = \ell$, $\lim \sigma_n^i(u) = \ell$, by (iii) above. By the fact that every sequence (C,i) limitable is (C,i) statistical limitable, we have st- $\lim \sigma_n^i(u) = \ell$. Since (u_n) is slowly decreasing, $\lim u_n = \ell$, by Theorem 2.7. □

We now give Tauberian theorem for Borel summability.

Theorem 2.16. *If the sequence (u_n) satisfies (B^*) st-$\lim B^* \sigma_n^i(u) = \ell$ and $\Delta u_n = O(\frac{1}{n})$, then* $\lim u_n = \ell$.

Proof. $\Delta u_n = O(\frac{1}{n})$ implies both $\Delta u_n = O(\frac{1}{\sqrt{n}})$ and $\Delta\sigma^i_n(u) = O(\frac{1}{\sqrt{n}})$. By Lemma 3.1 in [2], we obtain $\Delta B^*\sigma^i_n(u) = O(\frac{1}{\sqrt{n}})$, which allows us to apply (4) above that $\lim\sigma^i_n(u) = \ell$. By the fact that every sequence (C, 1) limitable is Abel limitable, we have $Abel - \lim\sigma^{i-1}_n(u) = \ell$. By the proof of Theorem 2.5, $\Delta u_n = O(\frac{1}{n})$ implies $\Delta\sigma^{i-1}_n(u) = O(\frac{1}{n})$. Since $(\sigma^{i-1}_n(u))$ is Abel summability to ℓ, $\lim\sigma^{i-1}_n(u) = \ell$ by Hardy's theorem in [4]. Continuing in this way, we have $Abel - \lim u_n = \ell$. By Hardy's theorem in [4], $\lim u_n = \ell$. □

Our last result generalizes (5) above which is given by Móricz [10].

Theorem 2.17. *If $(\sigma^i_n(u_n))$ is statistical summable (L, 1) to ℓ and (u_n) is slowly decreasig then* $\lim u_n = \ell$.

Proof. By Lemma 2.2, as (u_n) is slowly decreasing $(\sigma^i_n(u))$ for all $i \geq 1$ is slowly decreasing. Since $(\sigma^i_n(u))$ is statistical summable $(L,1)$, $\lim\sigma^i_n(u) = \ell$, by theorem (5) above. By the fact that every sequence (C,i) limitable is (C,i) statistical limitable, we have st- $\lim\sigma^i_n(u) = \ell$. Since (u_n) is slowly decreasing, $\lim u_n = \ell$, by Theorem 2.7. □

Corollary 2.18. *Let (u_n) be a sequence of complex numbers. If $(\sigma^i_n(u_n))$ is statistical summable $(L,1)$ and (u_n) is slowly oscillating, then* $\lim u_n = \ell$.

Proof. It's proof is similar to one of Theorem 2.17. □

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