TAMKANG JOURNAL OF MATHEMATICS
Volume 34, Number 4, Winter 2003

DEGREE OF APPROXIMATION OF CONJUGATE OF LIP α CLASS FUNCTION BY K^λ-SUMMABILITY MEANS OF CONJUGATE SERIES OF A FOURIER SERIES

SHYAM LAL AND GOPAL KRISHNA SINGH

Abstract. In this paper the degree of approximation of conjugate of a function belonging to Lip α class by K^λ-summability means of conjugate series of its Fourier series has been determined.

1. Introduction

The method K^λ was first introduced by Karamata [4]. Lotosky [6] reintroduced the special case $\lambda = 1$. Only after the paper of Agnew [1], an intensive study of these and similar methods took place. Vuĉkoviĉ [14] applied this method for summability of Fourier series. Kathal [5] extended Vuĉkoviĉ's result. Working in the same direction Ojha [8], Tripathi and Lal [13] have studied K^λ-summability of Fourier series under different conditions. For the function $f \in \text{Lip } \alpha$, the degree of approximation by Cesàro means and by Nörlund means of the Fourier series of f have been studied by Alexits [2], Sahney and Goel [12], Chandra [3], Qureshi [9, 10], Qureshi and Neha [11] and many other. But till now nothing seems to have been done for determining the degree of approximation of conjugate of Lip α function by K^λ-summability means of conjugate series of a Fourier series. In an attempt to make a study in this direction, in this paper, the degree of approximation of conjugate of Lipschitz function has been determined.

2. Definitions and Notations

Let us define, for $n = 0, 1, 2, 3, \ldots$, the numbers $\begin{bmatrix} n \\ m \end{bmatrix}$, for $0 \le m \le n$, by

$$\prod_{v=0}^{n-1}(x+v) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} x^m = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)(x+2)\cdots(x+n-1). \tag{2.1}$$

Received September 3, 2002.
2000 *Mathematics Subject Classification*. 42B05, 42B08, 42A24, 42A30, 42A50.
Key words and phrases. Degree of approximation, Fourier series, K^λ-summability, Lipschitz function.

The numbers $\begin{bmatrix} n \\ m \end{bmatrix}$ are known as the absolute value of stirling number of first kind. Let $\{S_n\}$ be the sequence of partial sums of an infinite series $\sum a_n$ and let us write

$$S_n^\lambda = \frac{\Gamma\lambda}{\Gamma(\lambda+n)} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \lambda^m S_m, \tag{2.2}$$

to denote the n^{th} K^λ-mean of order $\lambda > 0$. If $S_n^\lambda \to S$ as $n \to \infty$, where S, is a fixed finite number then the sequence $\{S_n\}$ or the series $\sum a_n$ is said to be summable by Karamata method K^λ of order $\lambda > 0$ to the sum S and we can write

$$S_n^\lambda \to s(K^\lambda), \quad \text{as } n \to \infty. \tag{2.3}$$

The method K^λ is regular for $\lambda > 0$.

Let $f : R \to R$ be 2π-periodic and Lip α, $0 < \alpha \leq 1$, so that

$$|f(x+t) - f(x)| = O(|t|^\alpha), \quad \text{for all } x,\ t. \tag{2.4}$$

Then f has its Fourier series, with the conjugate series.

$$\sum_{n=1}^{\infty} (a_n \sin nx - b_n \cos nx), \tag{2.5}$$

where a_n, b_n are the Fourier coefficients of f over $[-\pi, \pi]$. Writing

$$\psi_x(t) = f(x+t) - f(x-t) \quad \text{for all } x, t,$$

f has also its conjugate function $\bar{f}$, [15], given by

$$\overline{f} = -\frac{1}{2\pi} \int_0^\pi \psi_x(t) \cot\left(\frac{t}{2}\right) dt. \tag{2.6}$$

The degree of approximation of a function $g : [-\pi, \pi] \to R$ by a trigonometric polynomial T_n of order n is defined by, Zygmund [15]

$$\|T_n - g\|_\infty = \sup\{|T_n(x) - g(x)| : -\pi \leq x \leq \pi\}.$$

We write

$$\psi(t) = f(x+t) - f(x-t)$$

$$k_n(t) = \frac{\displaystyle\sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \lambda^m \cos(m + \frac{1}{2})t}{\Gamma(\lambda+n) \sin(\frac{t}{2})}$$

$$\bar{f}(x) = -\frac{1}{2\pi} \int_0^\pi \psi(t) \cot\frac{t}{2} dt.$$

3. Main Theorem

In this paper, the degree of approximation of conjugate function $\bar{f}$ by K^λ-summability means of conjugate series of Fourier series of f is determined in the following form:

Theorem. *If $f : R \to R$ is 2π-periodic and Lip α then the degree of approximation of its conjugate function $\bar{f}$ by K^λ-means of the conjugate series of f satisfies.*

$$\|\bar{S}_n^\lambda - \bar{f}\|_\infty = O\left[\left(\frac{\log(n+1)e}{(n+1)^{\alpha+2}}\right) + \frac{1}{(n+1)^\alpha}\left(1 + \frac{1}{\Gamma(\lambda+n)}\right)\right],$$
$$\text{for } 0 < \alpha \le 1,\ n = 0, 1, 2, 3, \dots,$$

where $\bar{S}_n^\lambda$ are K^λ-means of series (2.5).

4. Lemma

For the proof of our theorem following lemma is required:

Lemma (Vuĉkoviĉ [14]). *Let $\lambda > 0$ and $0 < t < \frac{\pi}{2}$,*
Then

$$\frac{Im\ \Gamma(\lambda e^{it} + n)}{\Gamma(\lambda \cos t + n)\sin(\frac{t}{2})} = \frac{|\sin(\lambda \log(n+1)\sin t)|}{\sin(\frac{t}{2})} + O(1), \quad \text{as } n \to \infty, \text{ uniformly in } t.$$

5. Proof of the Main Theorem

Following Lal [7] the n^{th} partial sum $\bar{S}_m(x)$ of series (2.6) at $t = x$ is given by

$$\bar{S}_m(x) - \left[-\frac{1}{2\pi}\int_0^\pi \psi(t)\cot\frac{1}{2}tdt\right] = \frac{1}{2\pi}\int_0^\pi \psi(t)\frac{\cos(m+\frac{1}{2})t}{\sin\frac{1}{2}t}dt.$$

Therefore

$$\frac{\Gamma(\lambda)}{\Gamma(\lambda+n)}\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\lambda^m\left\{\bar{S}_m(x) - \left(-\frac{1}{2\pi}\int_0^\pi \psi(t)\cot\frac{1}{2}tdt\right)\right\}$$
$$= \frac{1}{2\pi}\int_0^\pi \psi(t)\frac{\Gamma(\lambda)}{\Gamma(\lambda+n)}\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\lambda^m\frac{\cos(m+\frac{1}{2})t}{\sin\frac{1}{2}t}dt$$

$$\begin{aligned}\bar{S}_n^\lambda(x) - (\bar{f}(x)) &= \frac{\Gamma(\lambda)}{2\pi}\int_0^\pi \psi(t)k_n(t)dt \\ &= \left[\left\{\int_0^{1/n+1} + \int_{1/n+1}^\pi\right\}|\psi(t)|\,|k_n(t)|dt\right] \\ &= O(I_1) + O(I_2).\end{aligned} \tag{5.1}$$

Now by (1)

$$k_n(t) = \frac{\operatorname{Re}\left\{e^{it/2}\cdot\frac{\Gamma(\lambda e^{it}+n)}{\Gamma(\lambda e^{it})}\right\}}{\Gamma(\lambda+n)\cdot\sin(\frac{t}{2})}, \quad \text{by (2.1)}$$
$$= O\left|\frac{\operatorname{Re}\left\{e^{it/2}\cdot\frac{\Gamma(\lambda e^{it}+n)}{\Gamma(\lambda e^{it})}\right\}}{\Gamma(\lambda+n)\cdot\sin(\frac{t}{2})}\right|$$
$$= O\left[\frac{\operatorname{Re}\left\{e^{it/2}\cdot\frac{\Gamma(\lambda e^{it}+n)}{\Gamma(\lambda e^{it})}\right\}}{\Gamma(\lambda+n)\cdot\sin(\frac{t}{2})}\right] + O\left[\frac{\operatorname{Im}\,\Gamma(\lambda e^{it}+n)}{\Gamma(\lambda+n)}\right]$$
$$= O\left[\frac{\Gamma(\lambda\cos t+n)}{\Gamma(\lambda+n)\sin(\frac{t}{2})}\right] + O\left[\frac{\Gamma(\lambda\cos t+n)}{\Gamma(\lambda+n)}\cdot\frac{\operatorname{Im}\,\Gamma(\lambda e^{it}+n)}{\Gamma(\lambda\cos t+n)}\right]$$

For $0 < t < \frac{1}{n}$

$$\frac{\Gamma(\lambda\cos t+n)}{\Gamma(\lambda+n)} = O[n^{-\lambda(1-\cos t)}]$$
$$= O[e^{-\lambda(1-\cos t)\log n}]$$
$$= O\left[e^{-\frac{\lambda}{2}t^2\log n}\right].$$

Since, for $0 < t < \frac{1}{n}$, $0 < 1-\cos t < \frac{t^2}{2}$,
therefore,

$$|I_1| = \int_0^{1/n+1} |\psi(t)|k_n(t)dt$$
$$= O\left[\int_0^{1/n+1} e^{-\lambda(1-\cos t)\log n}\frac{\operatorname{Im}\,\Gamma(\lambda e^{it}+n)}{\Gamma(\lambda\cos t+n)}|\psi(t)|dt\right]$$
$$+O\left[\int_0^{1/n+1}\frac{e^{-\frac{\lambda}{2}t^2\log n}}{\sin\frac{t}{2}}|\psi(t)|dt\right]$$
$$= O(I_{1.1}) + O(I_{1.2}), \quad \text{say} \tag{5.2}$$

$$I_{1.1} = \int_0^{1/n+1} e^{-\lambda(1-\cos t)\log(n+1)}\frac{\operatorname{Im}\,\Gamma(\lambda e^{it}+n)}{\Gamma(\lambda\cos t+n)}|\psi(t)|dt$$

Applying lemma

$$= \int_0^{1/n+1} e^{-\frac{\lambda}{2}(t^2-\log(n+1))}|\sin(\lambda\log(n+1)t\sin t)|\,|\psi(t)|dt$$
$$+O\left[\int_0^{1/n+1} e^{-\frac{\lambda}{2}t^2\log(n+1)}\left|\sin\left(\frac{t}{2}\right)\right|\,|\psi(t)|dt\right]$$

$$= \int_0^{1/n+1} e^{-\frac{\lambda}{2}(t^2-\log(n+1))} |\sin(\lambda\log(n+1)t\sin t)|\,|\psi(t)|dt$$
$$+O\left[\int_0^{1/n+1} e^{-\frac{\lambda}{2}t^2\log(n+1)} t|\psi(t)|dt\right]$$

$$I_{1.1} = O(\lambda\log(n+1))\int_0^{1/n+1} t|\psi(t)|dt + \int_0^{1/n+1} t|\psi(t)|dt$$
$$= O(I_{1.11}) + O(I_{1.12}), \quad \text{say} \tag{5.3}$$

we have

$$\psi(t) = f(x+t) - f(x) + f(x) - f(x-t)$$

or

$$|\psi(t)| \le |f(x+t) - f(x)| + |f(x-t) - f(x)|$$
$$\le O(|t|^\alpha) + O(|t|^\alpha) \quad (\because f \in \text{Lip } \alpha)$$
$$|\psi(t)| \le O(|t|^\alpha)$$

Now,

$$I_{1.11} = (\lambda\log(n+1))\int_0^{1/(n+1)} t|\psi(t)|dt$$
$$= \lambda\log(n+1)\int_0^{1/(n+1)} O(|t|^{\alpha+1})dt$$
$$= (\lambda\log(n+1))\left[\left|\frac{t^{\alpha+2}}{\alpha+2}\right|\right]_0^{\frac{1}{n+1}} = O\left(\frac{\lambda\log(n+1)}{(n+1)^{\alpha+2}}\right) \tag{5.4}$$

$$I_{1.12} = \int_0^{1/(n+1)} t|\psi(t)|dt$$
$$= \int_0^{1/(n+1)} O(|t|^{\alpha+1})dt$$
$$= O\left(\frac{t^{\alpha+2}}{\alpha+2}\right)_0^{\frac{1}{n+1}}$$
$$= O\left(\frac{1}{(n+1)^{\alpha+2}}\right)$$

$$I_{1.1} = O\left(\frac{\log(n+1)}{(n+1)^{\alpha+2}}\right) + O\left(\frac{1}{(n+1)^{\alpha+2}}\right)$$
$$= O\left(\frac{\log(n+1)+1}{(n+1)^{\alpha+2}}\right) = O\left(\frac{\log(n+1)e}{(n+1)^{\alpha+2}}\right)$$

$$I_{1.2} = \int_0^{\frac{1}{n+1}} \frac{e^{-\frac{\lambda}{2}t^2\log n}}{\sin(\frac{t}{2})}|\psi(t)|dt$$

$$= \int_0^{\frac{1}{n+1}} \frac{e^{-\frac{\lambda}{2}t^2 \log n}}{\sin(\frac{t}{2})} |\psi(t)| dt = O\left[e^{-\frac{\lambda}{2}\frac{\log n}{n^2}}\right] \int_0^{\frac{1}{n+1}} \frac{\psi(t)}{t} dt$$

By second mean value theorem

$$= O\left[e^{-\frac{\lambda}{2}\frac{\log n}{n^2}}\right] \int_\epsilon^{\frac{1}{n+1}} \frac{|\psi(t)|}{t} dt$$

$$= O\left[e^{-\frac{\lambda}{2}\frac{\log n}{n^2}}\right] \int_\epsilon^{\frac{1}{n+1}} \frac{O(|t|^\alpha)}{t} = O(1) \int_\epsilon^{\frac{1}{n+1}} (t^{\alpha-1}) dt$$

$$= O(1) \left(\frac{t^\alpha}{\alpha}\right)_\epsilon^{\frac{1}{n+1}}$$

$$I_{1.2} = O\left(\frac{1}{(n+1)^\alpha}\right). \tag{5.5}$$

For $\frac{1}{n} < t < \pi$,

$$k_n(t) = O\left(\frac{1}{\Gamma(\lambda+n)\cdot\sin(\frac{t}{2})}\right) = O\left(\frac{1}{\Gamma(\lambda+n)\cdot\sin(\frac{t}{2})}\right)$$

$$= O\left(\frac{1}{\Gamma(\lambda+n)t}\right)$$

$$= O\left(\frac{1}{\Gamma(\lambda+n)t}\right).$$

At last

$$I_2 = \int_{\frac{1}{n+1}}^{\pi} O(t^\alpha) \frac{1}{\Gamma(\lambda+n)t} dt$$

$$= O \int_{\frac{1}{n+1}}^{\pi} \frac{t^{\alpha-1}}{\Gamma(\lambda+n)} dt$$

$$= O\left(\frac{1}{\Gamma(\lambda+n)}\right) \left(\frac{t^\alpha}{\alpha}\right)_{\frac{1}{n+1}}^{\pi}$$

$$= O\left(\frac{1}{\Gamma(\lambda+n)(n+1)^\alpha}\right). \tag{5.6}$$

Collecting the equations (5.1) to (5.6) we have

$$\bar{S}_n^\lambda - \bar{f}(x) = O\left(\frac{\log(n+1)e}{(n+1)^{\alpha+2}}\right) + O\left(\frac{1}{(n+1)^\alpha}\right) + O\left(\frac{1}{\Gamma(\lambda+n)(n+1)^\alpha}\right)$$

$$= O\left(\frac{\log(n+1)e}{(n+1)^{\alpha+2}}\right) + O\left[\frac{1}{(n+1)^\alpha}\left(1 + \frac{1}{\Gamma(\lambda+n)}\right)\right].$$

Then

$$\|\bar{S}_n^\lambda - \bar{f}(x)\|_\infty = \sup\{|\bar{S}_n^\lambda - \bar{f}(x)| : -\pi \le x \le \pi\}$$

$$= \text{Sup}\left[\left(\frac{\log(n+1)e}{(n+1)^{\alpha+2}}\right) + \frac{1}{(n+1)^{\alpha}}\left(1+\frac{1}{\Gamma(\lambda+n)}\right)\right], \quad -\pi \leq x \leq \pi.$$

Thus we obtain that

$$\|\bar{S}_n^{\lambda} - \bar{f}(x)\|_{\infty} = O\left[\left(\frac{\log(n+1)e}{(n+1)^{\alpha+2}}\right) + \frac{1}{(n+1)^{\alpha}}\left(1+\frac{1}{\Gamma(\lambda+n)}\right)\right], \quad \text{for } 0<\alpha\leq 1,\ n=0,1,2,\ldots.$$

This completes the proof of the theorem.

Acknowledgement

The authors are thankful to Dr L. M. Tripathi Ex. Professor & Head, Department of Mathematics, Banaras Hindu University, Varanasi (INDIA) for his valuable suggestions in prepartation of this paper. Shyam Lal, one of the authors is thankful to University Grants Commission, New Delhi, for providing financial assistance in the form of a minor research project vide letter no. F3.3 (58)/1999-2000/MRP/NR dated 31/3/2000.

References

[1] R. P. Agnew, *The Lotosky method for evaluation of series*, Michigan, Math. Journal **4**(1957), 105.

[2] G. Alexits, Convergence Problems of Orthogonal Series, Pergamon Press, London, 1961.

[3] Prem Chandra, *On degree of approximation of functions belonging to the Lipschitz class*, Nanta Math. **8**(1975), 88-89.

[4] J. Karamata, Theorems Sur la Sommabilite exponentielle etd, autres Sommabilities S'y rattachant, Mathematica (Cluj) **9**(1993), 164.

[5] P. D. Kathal, *A new criteria for Karamata summability of Fourier series*, Riv. Math. Univ. Parma, Italy **10**(1969), 33-38.

[6] A. V. Lotosky, *On a linear transformation of sequences* (*Russian*), Ivanov. Gos, Res. Inst. Uchen. Zap. **4**(1963), 61.

[7] Shyam Lal, *On K^{λ}-summability of conjugate series of Fourier series*, Bull. Cal. Math. Soc. **89**(1997), 97-104.

[8] M. K. Ojha, Ph.D. Thesis, Banaras Hindu University, Varanasi, India (1982), 120-128.

[9] K. Quereshi, *On degree of approximation of functions belonging to the class Lip α*, Indian J. Pure Appl. Math. **13**(1982), 898.

[10] K. Quereshi, *On degree of approximation of a periodic function f by almost Nörlund means*, Tamkang J. Math. **12**(1981), 35.

[11] K. Querishi and H. K. Neha, *A class of function and their degree of approximation Ganita*, **41**(1990), 37-42.

[12] B. N. Sahney and D. S. Goel , *On the degree of approximation of continuous functions*, Ranchi Univ. Maths. J. **4**(1973), 50.

[13] L. M. Tripathi and Shyam Lal, K^{λ}-*summability of Fourier series*, Journn. Sc. Res. **34**(1984), 69-74.
[14] V. Vuĉkoviĉ, *The summability of Fourier series by Karamata methods*, Math. Zeitchr. **89**(1965), 192.
[15] A. Zygmund, Trigenometric Series, 2nd Rev. Ed., Combridge Univ. Press, Cambridge, 1968.

Department of Mathmatics, Faculty of Science, University of Allahabad, Allahabad - 211002, India.