TAMKANG JOURNAL OF MATHEMATICS
Volume 34, Number 4, Winter 2003

A SUMMABILITY TYPE FACTOR THEOREM

B. E. RHOADES AND EKREM SAVAŞ

Abstract. We obtain sufficient conditions for the series $\sum a_n\epsilon_n$ to be absolutely summable of order k by a wieghted mean method.

The concept of absolute summability of order k was defined by Flett [2] as follows. Let $\sum a_n$ be a given infinite series with partial sums s_n, and let σ_n^α denote the n-th Cesáro means of order α, $\alpha > -1$, of the sequence $\{s_n\}$. The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$, $\alpha > -1$ if

$$\sum_{n=1}^{\infty} n^{k-1}|\Delta\sigma_{n-1}^\alpha|^k < \infty, \tag{1}$$

where, for any sequence $\{b_n\}$, $\Delta b_n = b_n - b_{n+1}$.

In defining absolute summability of order k for weighted mean methods Bor [1] and others used the definition

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|\Delta u_{n-1}|^k < \infty, \tag{2}$$

where

$$u_n := \sum_{\nu=0}^{n} p_\nu s_\nu.$$

In using (2) as the definition, it was apparently assumed that the n in (1) represented the reciprocal of the nth main diagonal term of $(C, 1)$. But this interpretation cannot be correct. For, if it were, then the Cesáro methods (C, α), for $\alpha \neq 1$ would have to satisfy the condition

$$\sum_{n=1}^{\infty}(n^\alpha)^{k-1}|\Delta\sigma_{n-1}^\alpha|^k < \infty.$$

However, Flett [2] stays with n for all values of $\alpha > -1$.

In a recent paper [3], Sulaiman proved the following two results.

Received September 5, 2002.
2000 *Mathematics Subject Classification*. Primary: 40G99; Secondary: 40G05, 40D15.
Key words and phrases. Absolute summability, weighted mean matrix, Cesáro matrix, summability factor.

Theorem 1. (A) *Let* $\{p_n\}$ *be a sequence of positive numbers. Let* T_n *be the* $(\bar{N}, p_n)$*-mean of the series* $\sum a_n$*. If*

$$\sum_{n=1}^{\infty} n^{k-1}|\epsilon_n|^k|\Delta T_{n-1}|^k < \infty,$$

$$\sum_{n=1}^{\infty} n^{k-k\alpha-1}\Big(\frac{P_n}{p_n}\Big)^k|\epsilon_n|^k|\Delta T_{n-1}|^k < \infty, \quad (0 < \alpha < 1)$$

$$\sum_{n=1}^{\infty} n^{-1}\Big(\frac{P_n}{p_n}\Big)^k|\epsilon_n|^k|\Delta T_{n-1}|^k < \infty, \quad (\alpha \geq 1)$$

$$\sum_{n=1}^{\infty} n^{-1}\Big(\frac{P_n}{p_n}\Big)^k|\Delta\epsilon_n|^k|\Delta T_{n-1}|^k < \infty$$

then the series $\sum a_n\epsilon_n$ *is summable* $|C, \alpha|_k$*,* $k \geq 1$*,* $\alpha > 0$*.*

(B) *Let* $\{p_n\}$ *be a sequence of positive numbers satisfying*

$$\begin{array}{ll} (i) & np_n = O(P_n), \\ (ii) & P_n = O(np_n). \end{array} \tag{3}$$

Let $\{\lambda_n\}$*,* $\{\epsilon_n\}$ *be such that* $\{\lambda_n\}$ *is nonnegative, nondecreasing,* $n^{1-\alpha}\lambda_n|\epsilon_n| = O(1)$ *for* $0 < \alpha < 1$*,* $\lambda_n|\epsilon_n| = O(1)$*,* $\epsilon_n = o(1)$ *for* $\alpha \geq 1$*,* $\Delta\epsilon_n = O(n^{-1}|\epsilon_n|)$*, and*

$$\sum_{n=1}^{m} \Big(\frac{P_n}{p_n}\Big)^{k-1}|\Delta T_{n-1}|^k = O(\lambda_m^k), \quad m \to \infty.$$

Then, in order to have the series $\sum a_n\epsilon_n$ *summable* $|C, \alpha|_k$*, it is sufficient that*

$$\sum_{n=1}^{\infty} n^{2-\alpha}\lambda_n|\Delta^2\epsilon_n| < \infty, \quad (0 < \alpha < 1)$$

and

$$\sum_{n=1}^{\infty} n\lambda_n|\Delta^2\epsilon_n| < \infty, \quad (\alpha \geq 1).$$

Theorem 2. (A) *Let* $\{p_n\}$ *be a sequence of positive numbers. Let* t_n^1 *be the nth* $(C, 1)$*-mean of the sequence* $\{na_n\}$*. If*

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n}|\epsilon_n|^k|t_n^1|^k < \infty,$$

$$\sum_{n=1}^{\infty} \frac{1}{n^k}\Big(\frac{P_n}{p_n}\Big)^{k-1}|\epsilon_n|^k|t_n^1|^k < \infty,$$

$$\sum_{n=1}^{\infty} \Big(\frac{P_n}{p_n}\Big)^{k-1}|\Delta\epsilon_n|^k|t_n^1|^k < \infty,$$

then the series $\sum a_n\epsilon_n$ *is summable* $|\bar{N},p_n|_k$, $k \geq 1$.

(B) *Let* $\{p_n\}$ *be a sequence of positive numbers such that* (3) *holds. Let* $\{\lambda_n\},\{\epsilon_n\}$ *be such that* $\{\lambda_n\}$ *is nonnegative, nondecreasing,* $\lambda_n|\epsilon_n| = O(1)$, $\epsilon_n = o(1)$, $\Delta\epsilon_n = O(n^{-1}|\epsilon_n|)$, *and*

$$\sum_{n=1}^{m} n^{-1}|t_n^1|^k = O(\lambda_n^k).$$

Then, in order to have the series $\sum a_n\epsilon_n$ *summable* $|\bar{N},p_n|_k$, $k \geq 1$, *it is sufficient that*

$$\sum_{n=1}^{\infty} n\lambda_n|\Delta^2\epsilon_n| < \infty.$$

In the proof of Theorem 1, Sulaiman uses the correct definition for absolute Cesáro summability. However, in proving Theorem 2, he reverts to (2) for absolute weighted mean summability.

In this paper we shall prove the corresponding version of Theorem 2, using the correct definition (1).

Theorem 3. (A) *Let* $\{p_n\}$ *be a positive sequence such that*

$$\sum_{n=\nu+1}^{\infty} \frac{n^{k-1}}{P_{n-1}}\Big(\frac{p_n}{P_n}\Big)^k = O\Big(\frac{\nu^{k-1}p_\nu^{k-1}}{P_\nu^k}\Big). \tag{4}$$

Let t_n^1 *denote the nth* $(C,1)$ *mean of* $\{na_n\}$. *If*

$$\sum_{\nu=1}^{\infty} \nu^{k-1}|\epsilon_\nu|^k|t_\nu^1|^k < \infty, \tag{5}$$

$$\sum_{\nu=1}^{\infty} \nu^{k-1}|\Delta\epsilon_\nu|^k|t_\nu^1|^k < \infty, \tag{6}$$

then the series $\sum a_n\epsilon_n$ *is summable* $|\bar{N},p_n|_k$, $k \geq 1$.

(B) *Let* $\{p_n\}$ *be a positive sequence satisfying* (3). *Let* $\{\lambda_n\}$, $\{\epsilon_n\}$ *be such that* $\{\lambda_n\}$ *is nonnegative, nondecreasing,* $n\lambda_n|\epsilon_n| = O(1)$, $\epsilon_n = o(1)$, $\Delta\epsilon_n = O(n^{-1}|\epsilon_n|)$ *and*

$$\sum_{\nu=1}^{n} \nu^{-1}|t_n^1|^k = O(\lambda_n^k). \tag{7}$$

Then, in order to have $\sum a_n\epsilon_n$ *summable* $|\bar{N},p_n|_k$, $k \geq 1$, *it is sufficient that*

$$\sum_{\nu=1}^{\infty} \nu^2\lambda_\nu|\Delta^2\epsilon_\nu| < \infty. \tag{8}$$

Proof. Part (A). Let Q_n denote the $(\bar{N}, p_n)$-mean of the series $\sum a_n\epsilon_n$. Then, as in [3],

$$Q_n = \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu \sum_{r=0}^{\nu} a_r\epsilon_r = \frac{1}{P_n}\sum_{\nu=0}^{n}(P_n - P_{\nu-1})a_\nu\epsilon_\nu.$$

$$\begin{aligned} Q_n - Q_{n-1} &= \frac{p_n}{P_nP_{n-1}}\sum_{\nu=1}^{n} P_{\nu-1}a_\nu\epsilon_\nu \\ &= \frac{p_n}{P_nP_{n-1}}\sum_{\nu=1}^{n} \nu a_\nu \frac{P_{\nu-1}\epsilon_\nu}{\nu} \\ &= \frac{p_n}{P_nP_{n-1}}\left[\sum_{\nu=1}^{n-1}(\nu+1)t_\nu^1\Big\{-\frac{p_\nu\epsilon_\nu}{\nu} + \frac{P_\nu\epsilon_\nu}{\nu(\nu+1)} + \frac{P_\nu\Delta\epsilon_\nu}{\nu+1}\Big\} + \frac{n+1}{n}P_{n-1}\epsilon_n t_n^1\right] \\ &= Q_{n1} + Q_{n2} + Q_{n3} + Q_{n4}. \end{aligned}$$

Using Hölder's inequality and (5),

$$\begin{aligned} \sum_{n=2}^{m+1} n^{k-1}|Q_{n1}|^k &= \sum_{n=2}^{m+1} n^{k-1}\Big|\frac{p_n}{P_nP_{n-1}}\sum_{\nu=1}^{n-1}\Big(1+\frac{1}{\nu}\Big)p_\nu\epsilon_\nu t_\nu^1\Big|^k \\ &= O(1)\sum_{n=2}^{m+1}\frac{n^{k-1}}{P_{n-1}}\Big(\frac{p_n}{P_n}\Big)^k\left[\sum_{\nu=1}^{n-1}p_\nu|\epsilon_\nu|^k|t_\nu^1|^k\right]\left[\frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1}p_\nu\right]^{k-1} \\ &= O(1)\sum_{n=2}^{m+1}\frac{n^{k-1}}{P_{n-1}}\Big(\frac{p_n}{P_n}\Big)^k\sum_{\nu=1}^{n-1}p_\nu|\epsilon_\nu|^k|t_\nu^1|^k \\ &= O(1)\sum_{\nu=1}^{m}p_\nu|\epsilon_\nu|^k|t_\nu^1|^k\sum_{n=\nu+1}^{m+1}\frac{n^{k-1}}{P_{n-1}}\Big(\frac{p_n}{P_n}\Big)^k \\ &= O(1)\sum_{\nu=1}^{m}p_\nu|\epsilon_\nu|^k|t_\nu^1|^k\frac{\nu^{k-1}p_\nu^{k-1}}{P_\nu^k} \\ &= O(1)\sum_{\nu=1}^{m}\nu^{k-1}\Big(\frac{p_\nu}{P_\nu}\Big)^k|\epsilon_\nu|^k|t_\nu^1|^k \\ &\le O(1)\sum_{\nu=1}^{m}\nu^{k-1}|\epsilon_\nu|^k|t_n^1|^k = O(1). \end{aligned}$$

By Hölder's inequality, (4), and (5),

$$\begin{aligned} \sum_{n=2}^{m} n^{k-1}|Q_{n2}|^k &= \sum_{n=2}^{m+1} n^{k-1}\Big|\frac{p_n}{P_nP_{n-1}}\sum_{\nu=1}^{n-1}\frac{P_\nu\epsilon_\nu(\nu+1)t_\nu^1}{\nu(\nu+1)}\Big|^k \\ &\le \sum_{n=2}^{m+1}\frac{n^{k-1}}{P_{n-1}}\Big(\frac{p_n}{P_n}\Big)^k\left[\sum_{\nu=1}^{n-1}\Big(\frac{P_\nu}{\nu p_\nu}\Big)^k p_\nu|\epsilon_\nu|^k|t_\nu^1|^k\right]\left[\frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1}p_\nu\right]^{k-1} \end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{\nu=1}^{m+1}\Big(\frac{P_\nu}{\nu p_\nu}\Big)^k p_\nu|\epsilon_\nu|^k|t_\nu^1|^k\sum_{n=\nu+1}^{m+1}\frac{n^{k-1}}{P_{n-1}}\Big(\frac{p_n}{P_n}\Big)^k\\
&= O(1)\sum_{\nu=1}^{m+1}\Big(\frac{P_\nu}{\nu p_\nu}\Big)^k p_\nu|\epsilon_\nu|^k|t_\nu^1|^k\Big(\frac{\nu^{k-1}p_\nu^{k-1}}{P_\nu^k}\Big)\\
&= O(1)\sum_{\nu=1}^{m+1}\nu^{k-1}|\epsilon_\nu|^k|t_\nu^1|^k<\infty.
\end{aligned}
$$

Using Hölder's inequality,(4), and (6),

$$
\begin{aligned}
\sum_{n=2}^{m}n^{k-1}|Q_{n3}|^k &= \sum_{n=2}^{m+1}n^{k-1}\Big|\frac{p_n}{P_nP_{n-1}}\sum_{\nu=1}^{n-1}P_\nu\Delta\epsilon_\nu t_\nu^1\Big|^k\\
&\le \sum_{n=2}^{m}\frac{n^{k-1}}{P_{n-1}}\Big(\frac{p_n}{P_n}\Big)\left[\sum_{\nu=1}^{n-1}\Big(\frac{P_\nu}{p_\nu}\Big)^k p_\nu|\Delta\epsilon_\nu|^k|t_\nu^1|^k\right]\left[\frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1}p_\nu\right]^{k-1}\\
&= O(1)\sum_{\nu=1}^{m}\Big(\frac{P_\nu}{p_\nu}\Big)^k p_\nu|\Delta\epsilon_\nu|^k|t_\nu^1|^k\sum_{n=\nu+1}^{m+1}\frac{n^{k-1}}{P_{n-1}}\Big(\frac{p_n}{P_n}\Big)^k\\
&= O(1)\sum_{\nu=1}^{m}\Big(\frac{P_\nu}{p_\nu}\Big)^k p_\nu|\Delta\epsilon_\nu|^k|t_\nu^1|^k\,\frac{\nu^{k-1}p_\nu^{k-1}}{P_\nu^k}\\
&= O(1)\sum_{\nu=1}^{m}\nu^{k-1}|\Delta\epsilon_\nu|^k|t_\nu^1|^k=O(1).
\end{aligned}
$$

From (5),

$$
\begin{aligned}
\sum_{n=2}^{m}n^{k-1}|Q_{n4}|^k &= \sum_{n=2}^{m+1}n^{k-1}\Big|\frac{p_n}{P_n}\epsilon_n t_n^1\Big|^k\\
&= \sum_{n=2}^{m+1}n^{k-1}\Big(\frac{p_n}{P_n}\Big)^k|\epsilon_n|^k|t_n^1|^k\\
&\le \sum_{n=2}^{m+1}n^{k-1}|\epsilon_n|^k|t_n^1|^k=O(1).
\end{aligned}
$$

Part (B). It is sufficient to show that conditions (5) and (6) are satisfied, since any weighted mean matrix satisfying (3) automatically satisfies (4).

Since $\Delta\epsilon_n = O(1)(n^{-1}|\epsilon_n|)$, (6) becomes

$$
\begin{aligned}
\sum_{\nu=1}^{m}\nu^{k-1}O(1)\Big(\frac{|\epsilon_\nu|}{\nu}\Big)^k|t_\nu^1|^k &= O(1)\sum_{\nu=1}^{m}\frac{1}{\nu^k}\nu^{k-1}|\epsilon_\nu|^k|t_\nu^1|^k\\
&\le O(1)\sum_{\nu=1}^{m}\nu^{k-1}|\epsilon_\nu|^k|t_\nu^1|^k.
\end{aligned}
$$

To prove (5),

$$\begin{aligned}\sum_{\nu=1}^{n}\nu^{k-1}|\epsilon_\nu|^k|t_\nu^1|^k &= \sum_{\nu=1}^{n}\nu^k|\epsilon_\nu|^k\left[\sum_{i=1}^{\nu}\frac{1}{i}|t_i^1|^k-\sum_{i=1}^{\nu-1}\frac{1}{i}|t_i^1|^k\right]\\ &= n^k|\epsilon_n|^k\sum_{i=1}^{n}\frac{1}{i}|t_i^1|^k+\sum_{\nu=1}^{n-1}\sum_{i=1}^{\nu}\frac{1}{i}|t_i^1|^k[\nu^k|\epsilon_\nu|^k-(\nu+1)^k|\epsilon_{\nu+1}|^k]\\ &\le n^k|\epsilon_n|^k\sum_{i=1}^{n}\frac{1}{i}|t_i^1|^k+\sum_{\nu=1}^{n}\sum_{i=1}^{\nu}\frac{1}{i}|t_i^1|^k(\nu+1)^k\Delta|\epsilon_\nu|^k\\ &= I_1+I_2.\end{aligned}$$

From (7), $I_1 = O(n^k|\epsilon_n|^k\lambda_n^k) = O(1)$

Since $\Delta|\epsilon_\nu|^k = k(|\epsilon_\nu|-|\epsilon_{\nu+1}|)\xi^{k-1}$ for some ξ between $|\epsilon_\nu|$ and $|\epsilon_{\nu+1}|$ by the mean value theorem,

$$\begin{aligned}\Delta|\epsilon_\nu|^k &\le k|\Delta\epsilon_\nu|\xi^{k-1}\\ &= O(|\epsilon_\nu|^{k-1}|\Delta\epsilon_\nu|) \quad \text{since} \quad \epsilon_n = o(1).\end{aligned}$$

Therefore

$$\begin{aligned}I_2 &= O(1)\sum_{\nu=1}^{n}\lambda_\nu^k(\nu+1)^k\Delta|\epsilon_\nu|^k\\ &= O(1)\sum_{\nu=1}^{n}\lambda_\nu^k(\nu+1)^k|\epsilon_\nu|^{k-1}|\Delta\epsilon_\nu|.\end{aligned}$$

Since $\nu\lambda_\nu|\epsilon_\nu| = O(1)$,

$$\begin{aligned}I_2 &= O(1)\sum_{\nu=1}^{n}\lambda_\nu(\nu+1)|\Delta\epsilon_\nu|\\ &= O(1)\sum_{\nu=1}^{n}|\Delta\epsilon_\nu|\left[\sum_{i=1}^{\nu}(i+1)\lambda_i-\sum_{i=1}^{\nu-1}(i+1)\lambda_i\right]\\ &= O(1)\left[|\Delta\epsilon_n|\sum_{i=1}^{n}(i+1)\lambda_i+\sum_{\nu=1}^{n-1}\sum_{i=1}^{\nu}(i+1)\lambda_i\Delta(|\Delta\epsilon_\nu|)\right].\end{aligned}$$

$$\begin{aligned}|\Delta\epsilon_n|\sum_{i=1}^{n}(i+1)\lambda_i &= O(n^{-1}|\epsilon_n|)O((n+1)\lambda_n\, n)\\ &= O((n+1)\lambda_n|\epsilon_n|) = O(1).\end{aligned}$$

Thus

$$I_2 \le O(1)+\sum_{\nu=1}^{n-1}\sum_{i=1}^{\nu}(i+1)\lambda_i|\Delta^2\epsilon_\nu|$$

$$\leq O(1) + \sum_{\nu=1}^{n-1} \lambda_\nu |\Delta^2 \epsilon_\nu| \sum_{i=1}^{\nu} (i+1)$$

$$= O(1) + O(1) \sum_{\nu=1}^{n-1} \lambda_\nu \nu^2 |\Delta^2 \epsilon_\nu| = O(1),$$

by (8).

References

[1] H. Bor, *On two summability methods*, Math. Proc. Cambrige Phil. Soc. **98**(1985), 147-149
[2] T. M. Flett, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. London Math. Soc. **7**(1957), 113-141.
[3] W. T. Sulaiman, *A study on a relation between two summability methods*, Proc. Amer. Math. Sox. **115**(1992), 303-312.

Department of Mathematics, Indiana University, Bloomington, IN 47405-7106.

E-mail: rhoades@indiana.edu

Department of Mathematics, Yüzüncü Yil University, Van, Turkey.

E-mail: ekremsavas@yahoo.com