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ENTIRE SOLUTION ORIGINATING FROM THREE FRONTS FOR A DISCRETE DIFFUSIVE EQUATION

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Abstract. In this paper, we study a discrete diffusive equation with a bistable nonlinearity. For this equation, there are three types of traveling fronts. By constructing some suitable pairs of super-sub-solutions, we show that there are only two types of entire solutions originating from three fronts of this equation. These results show us some new dynamics of this discrete diffusive equation.

1. Introduction

In this work, we study the following discrete diffusive equation

$$u_t(x,t) = d(u(x+1,t)+u(x-1,t)-2u(x,t)) + f(u(x,t)), \quad x\in\mathbb{R},\ t\in\mathbb{R}, \tag{1.1}$$

where the function $f(u)\in C^2(\mathbb{R})$ satisfies

$$f(0)=f(1)=0, \quad f'(0),\ f'(1)<0, \tag{1.2}$$

$$f(a)=0,\ f'(a)>0,\ a\in(0,1), \quad f(u)\neq 0 \text{ for } u\in(0,a)\cup(a,1), \tag{1.3}$$

$$\int_0^1 f(s)\,ds>0. \tag{1.4}$$

Here, d is a positive constant. By (1.2)-(1.3), $u=0$ and $u=1$ are stable and $u=a$ is unstable for the equation (1.1) when $d=0$.

The equation (1.1) is the continuum version of the following lattice dynamical system

$$\dot{u}_j(t) = d(u_{j+1}(t)+u_{j-1}(t)-2u_j(t)) + f(u_j(t)), \quad j\in\mathbb{Z},\ t\in\mathbb{R}. \tag{1.5}$$

where the dot denotes the derivative with respect to t. This system was studied extensively in past years. One of the main concerns is the existence of the traveling wave solution. A

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solution $\{u_j(t)\}$ of (1.5) is called a traveling wave solution if $u_j(t) = U(j+ct)$, $j \in \mathbb{Z}$, $t \in \mathbb{R}$, for some function $U \in C^2(\mathbb{R})$ and some constant c. Here U is the wave profile and c is the wave speed. Moreover, if a traveling wave solution connects two different constant states, we call it traveling front. In [16, 17], Zinner showed there is a $d_* > 0$ such that there exists a unique (up to translations) traveling front of (1.5) connecting 0 and 1 with positive speed if $d > d_*$. For d small, the non-existence of the traveling wave connecting 0 and 1 was shown by Keener [11]. Now we denote the wave profile and the wave speed of the traveling front connecting 0 and 1 by U_0 and $c_0 > 0$, respectively. Set $\xi = j + c_0 t$. Then $U_0(\xi)$ satisfies

$$c_0 U_0'(\xi) = d[U_0(\xi+1) + U_0(\xi-1) - 2U_0(\xi)] + f(U_0(\xi)),\ U_0'(\xi) > 0, \quad \xi \in \mathbb{R},$$
$$U_0(-\infty) = 0, \quad U_0(\infty) = 1.$$

On the other hand, from the results shown in [1, 2], there exists a $c_{1,max} < 0$ such that there exists a traveling front $U_1(j + c_1 t)$ of (1.5) connecting 0 and a for each $c_1 \le c_{1,max}$. By setting $\xi = j + c_1 t$, $U_1(\xi)$ satisfies

$$c_1 U_1'(\xi) = d[U_1(\xi+1) + U_1(\xi-1) - 2U_1(\xi)] + f(U_1(\xi)),\ U_1'(\xi) > 0, \quad \xi \in \mathbb{R},$$
$$U_1(-\infty) = 0, \quad U_1(\infty) = a.$$

Similarly, there exists a $c_{2,min} > 0$ such that there exists a traveling front $U_2(j + c_2 t)$ of (1.5) connecting a and 1 for each $c_2 \ge c_{2,min}$. For $\xi = j + c_2 t$, $U_2(\xi)$ satisfies

$$c_2 U_2'(\xi) = d[U_2(\xi+1) + U_2(\xi-1) - 2U_2(\xi)] + f(U_2(\xi)),\ U_2'(\xi) > 0, \quad \xi \in \mathbb{R},$$
$$U_2(-\infty) = a, \quad U_2(\infty) = 1.$$

For (1.1), the traveling front $u(x,t)$ with the speed c exists if $u(x,t) = U(x+ct)$ for some function $U \in C^2(\mathbb{R})$ and it connects two different constant states. Now we set $\xi := x + ct$ and substitute $U(\xi)$ into (1.1). Then $U(\xi)$ is the solution of the following equation

$$cU'(\xi) = d[U(\xi+1) + U(\xi-1) - 2U(\xi)] + f(U(\xi)), \quad \xi \in \mathbb{R},$$
$$U(-\infty) = \alpha, \quad U(\infty) = \omega.$$

where $\{\alpha, \omega\} \subset \{0, a, 1\}$ and $\alpha \neq \omega$. From the results shown in the above, (1.1) has three traveling fronts $u(x,t) = U_0(x + c_0 t)$, $U_1(x + c_1 t)$, $U_2(x + c_2 t)$.

A classical solution $u(x,t)$ of (1.1) defined for all $(x,t) \in \mathbb{R}^2$ is called an entire solution of (1.1). Obviously, the traveling front is an entire solution. In past years, the other types of entire solutions were studied in the following reaction-diffusion equation

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R},\ t \in \mathbb{R}. \tag{1.6}$$

Indeed, the equation (1.1) is the discrete version of (1.6). From the results shown in [15, 5, 7, 10, 12], there are entire solutions which behave as two traveling fronts of (1.6) on the left x-axis and right x-axis as $t \to -\infty$. We call this type of entire solution by *entire solution originating from two fronts*. In [7], Guo and Morita proved the existence of the entire solution originating from two fronts for the discrete KPP equation. Later on, under some conditions of the wave speed, there are three types of the entire solution originating from two fronts shown in [9]. More precisely, they behave as (i) $U_0(-x+c_0t)$ and $U_0(x+c_0t)$, (ii) $U_1(x+c_1t)$ and $U_2(x+c_2t)$ and (iii) $U_0(-x+c_0t)$ and $U_1(x+c_1t)$ on the left x-axis and right x-axis as $t \to -\infty$. According to these two works, all types of the entire solution originating from two fronts of (1.1) were studied. For the entir e solution originating from two fronts of the other discrete diffusive equation or lattice dynamical system, we refer the readers to [6, 8, 14, 13].

In this work, we would like to show the existence of the entire solution originating from three fronts of (1.1). Here, we define this type of entire solution as follows.

Definition 1.1. Let (ϕ_i, v_i), $i = 1,2,3$ be the traveling fronts of (1.1). If the entire solution $u(x,t)$ of (1.1) satisfies

$$\limsup_{t\to-\infty}\left\{\sum_{1\le i\le 3}\sup_{d_{i-1}(t)<x<d_i(t)}|u(x,t)-\phi_i(x+v_it+\theta_i)|\right\}=0 \tag{1.7}$$

where

$$v_1 < v_2 < v_3, \tag{1.8}$$

$\theta_1, \theta_2, \theta_3$ are some constants, $d_i(t) := -(v_i + v_{i+1})t/2$, $d_0(t) = -\infty$ and $d_3(t) = \infty$, it is called the *entire solution originating from three fronts* of (1.1).

By (1.8), the continuity of entire solutions and the symmetry with respect to reflection, there are only two possible types of the entire solution originating from three fronts of (1.1). We state them in the following two main theorems.

Theorem 1.1. *Consider* $(\phi_1(\xi), v_1) = (U_0(-\xi), -c_0)$, $(\phi_i(\xi), v_i) = (U_{i-1}(\xi), c_{i-1})$, $i = 2,3$ *where* (U_i, c_i), $i = 0,1,2$ *are the traveling fronts described as above. If*

$$v_1 < v_2, \tag{1.9}$$

then there exists an entire solution originating from three fronts of (1.1).

Theorem 1.2. *Consider* $(\phi_1(\xi), v_1) = (U_0(-\xi), -c_0)$, $(\phi_2(\xi), v_2) = (U_1(\xi), c_1)$, $(\phi_3(\xi), v_3) = (U_1(-\xi), -c_1)$, *where* (U_i, c_i), $i = 0,1$ *are the traveling fronts described as above. If*

$$v_1 < v_2,$$

then there exists an entire solution originating from three fronts of (1.1)*. Moreover,*

$$\lim_{t\to\infty}\sup_{x\in\mathbb{R}}|u(x,t)-\phi_1(x+v_1t+\theta)|=0 \tag{1.10}$$

holds for some constant θ*.*

To show these two theorems, since the comparison principle can be applied, we only need to construct a suitable pair of super-solution and sub-solution. In the study of the entire solution originating from two fronts of (1.6), Morita and Ninomiya find some auxiliary rational functions with certain properties to help them to construct super-sub-solutions (see [12]). In [9], this method was used to prove some types of entire solution originating from two fronts of (1.1). Therefore, we would like to apply it to obtain Theorem 1.1 and Theorem 1.2. In fact, for (1.6), we already obtained the same results as Theorem 1.1 in [4]. Hence, we follow the ideas used in [9] and [4] to prove Theorem 1.1. Similarly, Theorem 1.2 can be shown by using a different auxiliary rational function to construct a suitable pair of super-sub-solution.

The remainder of this paper is organized as follows. We first show the proof of Theorem 1.1 in §2. In §3, we prove Theorem 1.2.

2. The Proof of Theorem 1.1

Without loss of generality, we assume that $d=1$. Recall that $(\phi_1(\xi),v_1)=(U_0(-\xi),-c_0)$, $(\phi_i(\xi),v_i)=(U_{i-1}(\xi),c_{i-1})$, $i=2,3$. We know that

$$\begin{cases} D[\phi_i](\xi)-v_i\phi_i'(\xi)+f(\phi_i(\xi))=0, & \xi\in\mathbb{R},\\ \phi_i(-\infty)=\alpha_i, \quad \phi_i(\infty)=\omega_i, \end{cases} \tag{2.1}$$

where $(\alpha_1,\omega_1,\alpha_2,\omega_2,\alpha_3,\omega_3)=(1,0,0,a,a,1)$ and $D[\phi_i](\xi)=\phi_i(\xi+1)+\phi_i(\xi-1)-2\phi_i(\xi)$. From the results shown in [3], there are positive constants β_i,γ_i, $i=1,2,3$, and $K>0$ such that

$$\begin{cases} |\phi_i'(\xi)|\le K\exp(\beta_i\xi), & \xi\le 0,\\ |\phi_i'(\xi)|\le K\exp(-\gamma_i\xi), & \xi\ge 0. \end{cases} \tag{2.2}$$

Also, there are constants $m,M>0$ such that

$$\begin{cases} m\le\dfrac{|\phi_1(\xi)-1|}{|\phi_1'(\xi)|}\le M,\ \xi\le 1, & m\le\dfrac{|\phi_1(\xi)-0|}{|\phi_1'(\xi)|}\le M,\ \xi\ge -1,\\ m\le\dfrac{|\phi_2(\xi)-0|}{|\phi_2'(\xi)|}\le M,\ \xi\le 1, & m\le\dfrac{|\phi_2(\xi)-a|}{|\phi_2'(\xi)|}\le M,\ \xi\ge -1,\\ m\le\dfrac{|\phi_3(\xi)-a|}{|\phi_3'(\xi)|}\le M,\ \xi\le 1, & m\le\dfrac{|\phi_3(\xi)-1|}{|\phi_3'(\xi)|}\le M,\ \xi\ge -1. \end{cases} \tag{2.3}$$

Furthermore, we have the following lemma.

Lemma 2.1. *There exists a positive constant* $\widetilde{M} = (Me^{1/m})/m$ *such that*

$$|\phi_i'(\xi+\theta)| \le \widetilde{M}|\phi_i'(\xi)|, \quad i = 1,2,3 \tag{2.4}$$

holds for all $\xi \in \mathbb{R}$, $\theta \in [-1,1]$.

Proof. For $i = 1$, given an arbitrary $\theta_0 \in [-1,1]$, we have

$$\begin{aligned}\left|\frac{\phi_1'(\xi+\theta_0)}{\phi_1'(\xi)}\right| &= \left|\frac{\phi_1'(\xi+\theta_0)}{1-\phi_1(\xi+\theta_0)}\right| \cdot \frac{1-\phi_1(\xi+\theta_0)}{1-\phi_1(\xi)} \cdot \left|\frac{1-\phi_1(\xi)}{\phi_1'(\xi)}\right| \\ &\le \frac{1}{m}\exp\left(-\int_\xi^{\xi+\theta_0} \frac{\phi_1'(\zeta)}{1-\phi_1(\zeta)}d\zeta\right) M \le \widetilde{M}\end{aligned}$$

for $\xi \le 0$ by (2.3). For $\xi \ge 0$, by (2.3), we obtain

$$\begin{aligned}\left|\frac{\phi_1'(\xi+\theta_0)}{\phi_1'(\xi)}\right| &= \left|\frac{\phi_1'(\xi+\theta_0)}{\phi_1(\xi+\theta_0)}\right| \cdot \frac{\phi_1(\xi+\theta_0)}{\phi_1(\xi)} \cdot \left|\frac{\phi_1(\xi)}{\phi_1'(\xi)}\right| \\ &\le \frac{1}{m}\exp\left(\int_\xi^{\xi+\theta_0} \frac{\phi_1'(\zeta)}{\phi_1(\zeta)}d\zeta\right) M \le \widetilde{M}.\end{aligned}$$

Thus, (2.4) holds for $i = 1$.

By the similar argument as above, the proof of the other cases can be done. Therefore, we get the conclusion. □

Now we consider the solution $u(x,t) = U(\eta,t)$ of (1.1) with $\eta := x+\overline{v}t$ and $\overline{v} := (v_1+v_2)/2$. Then $U(\eta,t)$ satisfies the following equation

$$U_t(\eta,t) = D[U](\eta,t) - \overline{v}U_\eta(\eta,t) + f(U(\eta,t)),\ (\eta,t) \in \mathbb{R}^2. \tag{2.5}$$

where $D[U](\eta,t) := U(\eta+1,t) + U(\eta-1,t) - 2U(\eta,t)$. Obviously, $U(\eta,t) = \phi_1(\eta - s_1 t), \phi_2(\eta + s_1 t), \phi_3(\eta + s_2 t)$ with $s_1 = (v_2 - v_1)/2 > 0$ and $s_2 = v_3 - \overline{v} > s_1$ are traveling fronts of (2.5).

Now we start to construct a pair of super-sub-solution of (2.5) by using the ideas in [4]. We first take the auxiliary rational function $Q(y,z,w)$ as follows.

$$Q(y,z,w) = z + (1-z)\frac{(1-y)z(w-a) + y(a-z)(1-w)}{(1-y)z(1-a) + (a-z)(1-w)}. \tag{2.6}$$

Then we define the functions $\overline{U}(\eta,t)$ and $\underline{U}(\eta,t)$ by

$$\overline{U}(\eta,t) := Q(\phi_1(\eta - p_1(t)), \phi_2(\eta + p_1(t)), \phi_3(\eta + p_2(t))), \tag{2.7}$$

$$\underline{U}(\eta,t) := Q(\phi_1(\eta - r_1(t)), \phi_2(\eta + r_1(t)), \phi_3(\eta + r_2(t))). \tag{2.8}$$

The functions $p_i(t)$, $r_i(t)$, $i = 1,2$ are the solutions of the following initial value problems

$$\dot{p}_1 = s_1 + Le^{\kappa p_1}, \quad -\infty < t < 0, \quad p_1(0) = p_0; \tag{2.9}$$

$$\dot{p}_2 = s_2 + Le^{\kappa p_1}, \quad -\infty < t < 0, \quad p_2(0) = p_0; \tag{2.10}$$

$$\dot{r}_1 = s_1 - Le^{\kappa r_1}, \quad -\infty < t < 0, \quad r_1(0) = r_0; \tag{2.11}$$

$$\dot{r}_2 = s_2 - Le^{\kappa r_1}, \quad -\infty < t < 0, \quad r_2(0) = r_0, \tag{2.12}$$

where $L > 2K\rho$ is a positive constant,

$$\kappa := \min\left\{\gamma_1, \gamma_2, \beta_2, \beta_3, \frac{(s_2 - s_1)\gamma_2}{4s_1}, \frac{(s_2 - s_1)\beta_3}{4s_1}\right\}.$$

and p_0 and r_0 satisfying

$$p_0 = -\frac{1}{\kappa}\log\left(e^{-\kappa r_0} - \frac{2L}{s_1}\right) < -\delta, \quad r_0 < -\frac{1}{\kappa}\log\left(\frac{2L}{s_1} + e^{\kappa\delta}\right).$$

Here, δ is a given sufficiently large positive constant and ρ is a positive constant to be determined later. Also, there exists a positive constant N such that

$$0 < p_1(t) - r_1(t) = p_2(t) - r_2(t) \le Ne^{\kappa s_1 t} \text{ for all } t \le 0, \tag{2.13}$$

and $p_1(t), p_2(t), r_1(t), r_2(t) \le -\delta$ for all $t \le 0$.

Set

$$\mathscr{L}[U] := U_t - D[U] + \overline{v}U_\eta - f(U).$$

Then U is a super-solution (sub-solution, resp.) of (2.5) for $t \le T$ with some constant T if $\mathscr{L}[U] \ge 0$ ($\mathscr{L}[U] \le 0$, resp.) holds for $t \le T$ for some constant T. Now, we claim that $\overline{U}(\eta, t)$ is a super-solution of (2.5) for $t \le t_0$ with some constant $t_0 < 0$. To simplify the notation, we define

$$\begin{aligned}
&p_1 = p_1(t),\ p_2 = p_2(t),\\
&\overline{U}(\eta, t) = Q(\phi_1, \phi_2, \phi_3),\ \phi_1 = \phi_1(\eta - p_1),\ \phi_2 = \phi_2(\eta + p_1),\ \phi_3 = \phi_3(\eta + p_2),\\
&\widetilde{\phi_1}'(\tau) = \phi_1'(\eta - p_1 + \tau), \quad \widetilde{\phi_2}'(\tau) = \phi_2'(\eta + p_1 + \tau), \quad \widetilde{\phi_3}'(\tau) = \phi_3'(\eta + p_2 + \tau),\\
&\widetilde{Q}_{yy}(\eta_1, \eta_2, \eta_3) = Q_{yy}(\phi_1(\eta - p_1 + \eta_1), \phi_2(\eta + p_1 + \eta_2), \phi_3(\eta + p_2 + \eta_3)),\\
&\widetilde{Q}_{zz}(\eta_1, \eta_2, \eta_3) = Q_{zz}(\phi_1(\eta - p_1 + \eta_1), \phi_2(\eta + p_1 + \eta_2), \phi_3(\eta + p_2 + \eta_3)),\\
&\widetilde{Q}_{ww}(\eta_1, \eta_2, \eta_3) = Q_{ww}(\phi_1(\eta - p_1 + \eta_1), \phi_2(\eta + p_1 + \eta_2), \phi_3(\eta + p_2 + \eta_3)),\\
&\widetilde{Q}_{yz}(\eta_1, \eta_2, \eta_3) = Q_{yz}(\phi_1(\eta - p_1 + \eta_1), \phi_2(\eta + p_1 + \eta_2), \phi_3(\eta + p_2 + \eta_3)),\\
&\widetilde{Q}_{yw}(\eta_1, \eta_2, \eta_3) = Q_{yw}(\phi_1(\eta - p_1 + \eta_1), \phi_2(\eta + p_1 + \eta_2), \phi_3(\eta + p_2 + \eta_3)),\\
&\widetilde{Q}_{zw}(\eta_1, \eta_2, \eta_3) = Q_{zw}(\phi_1(\eta - p_1 + \eta_1), \phi_2(\eta + p_1 + \eta_2), \phi_3(\eta + p_2 + \eta_3)).
\end{aligned}$$

Then, by using (2.9), (2.10) and the mean value theorem, we obtain that

$$\mathscr{L}[\overline{U}(\eta, t)] = -Q_y\phi_1'(\dot{p}_1 - s_1) + Q_z\phi_2'(\dot{p}_1 - s_1) + Q_w\phi_3'(\dot{p}_2 - s_2) - G(\phi_1, \phi_2, \phi_3) - H(\phi_1, \phi_2, \phi_3)$$

$$= F(\phi_1,\phi_2,\phi_3)\cdot Le^{\kappa p_1} - G(\phi_1,\phi_2,\phi_3) - H(\phi_1,\phi_2,\phi_3)$$

where

$$F(\phi_1,\phi_2,\phi_3) := -Q_y(\phi_1,\phi_2,\phi_3)\phi_1' + Q_z(\phi_1,\phi_2,\phi_3)\phi_2' + Q_w(\phi_1,\phi_2,\phi_3)\phi_3',$$

$$\begin{aligned}
&G(\phi_1,\phi_2,\phi_3)\\
&:= \sigma_1\widetilde{Q}_{yy}(a_{11}\sigma_1,a_{12}\sigma_2,a_{13}\sigma_3)\widetilde{\phi_1}'(b_{11}\sigma_1)\widetilde{\phi_1}'(d_1)\\
&\quad+\sigma_2\widetilde{Q}_{yz}(a_{11}\sigma_1,a_{12}\sigma_2,a_{13}\sigma_3)\widetilde{\phi_2}'(b_{12}\sigma_2)\widetilde{\phi_1}'(d_1)\\
&\quad+\sigma_3\widetilde{Q}_{yw}(a_{11}\sigma_1,a_{12}\sigma_2,a_{13}\sigma_3)\widetilde{\phi_3}'(b_{13}\sigma_3)\widetilde{\phi_1}'(d_1)\\
&\quad+\sigma_1\widetilde{Q}_{yz}(a_{21}\sigma_1,a_{22}\sigma_2,a_{23}\sigma_3)\widetilde{\phi_1}'(b_{21}\sigma_1)\widetilde{\phi_2}'(d_2)\\
&\quad+\sigma_2\widetilde{Q}_{zz}(a_{21}\sigma_1,a_{22}\sigma_2,a_{23}\sigma_3)\widetilde{\phi_2}'(b_{22}\sigma_2)\widetilde{\phi_2}'(d_2)\\
&\quad+\sigma_3\widetilde{Q}_{zw}(a_{21}\sigma_1,a_{22}\sigma_2,a_{23}\sigma_3)\widetilde{\phi_3}'(b_{23}\sigma_3)\widetilde{\phi_2}'(d_2)\\
&\quad+\sigma_1\widetilde{Q}_{yw}(a_{31}\sigma_1,a_{32}\sigma_2,a_{33}\sigma_3)\widetilde{\phi_1}'(b_{31}\sigma_1)\widetilde{\phi_3}'(d_3)\\
&\quad+\sigma_2\widetilde{Q}_{zw}(a_{32}\sigma_2,a_{32}\sigma_2,a_{33}\sigma_3)\widetilde{\phi_2}'(b_{33}\sigma_3)\widetilde{\phi_3}'(d_3)\\
&\quad+\sigma_3\widetilde{Q}_{ww}(a_{31}\sigma_1,a_{32}\sigma_2,a_{33}\sigma_3)\widetilde{\phi_3}'(b_{33}\sigma_1)\widetilde{\phi_3}'(d_3)\\
&\quad+\tau_1\widetilde{Q}_{yy}(-a_{41}\tau_1,-a_{42}\tau_2,-a_{43}\tau_3)\widetilde{\phi_1}'(-b_{41}\tau_1)\widetilde{\phi_1}'(-d_4)\\
&\quad+\tau_2\widetilde{Q}_{yz}(-a_{41}\tau_1,-a_{42}\tau_2,-a_{43}\tau_3)\widetilde{\phi_2}'(-b_{42}\tau_2)\widetilde{\phi_1}'(-d_4)\\
&\quad+\tau_3\widetilde{Q}_{yw}(-a_{41}\tau_1,-a_{42}\tau_2,-a_{43}\tau_3)\widetilde{\phi_3}'(-b_{43}\tau_3)\widetilde{\phi_1}'(-d_4)\\
&\quad+\tau_1\widetilde{Q}_{yz}(-a_{51}\tau_1,-a_{52}\tau_2,-a_{53}\tau_3)\widetilde{\phi_1}'(-b_{51}\tau_1)\widetilde{\phi_2}'(-d_5)\\
&\quad+\tau_2\widetilde{Q}_{zz}(-a_{51}\tau_1,-a_{52}\tau_2,-a_{53}\tau_3)\widetilde{\phi_2}'(-b_{52}\tau_2)\widetilde{\phi_2}'(-d_5)\\
&\quad+\tau_3\widetilde{Q}_{zw}(-a_{51}\tau_1,-a_{52}\tau_2,-a_{53}\tau_3)\widetilde{\phi_3}'(-b_{53}\tau_3)\widetilde{\phi_2}'(-d_5)\\
&\quad+\tau_1\widetilde{Q}_{yw}(-a_{61}\tau_1,-a_{62}\tau_2,-a_{63}\tau_3)\widetilde{\phi_1}'(-b_{61}\tau_1)\widetilde{\phi_3}'(-d_6)\\
&\quad+\tau_2\widetilde{Q}_{zw}(-a_{61}\tau_1,-a_{62}\tau_2,-a_{63}\tau_3)\widetilde{\phi_2}'(-b_{62}\tau_2)\widetilde{\phi_3}'(-d_6)\\
&\quad+\tau_3\widetilde{Q}_{ww}(-a_{61}\tau_1,-a_{62}\tau_2,-a_{63}\tau_3)\widetilde{\phi_3}'(-b_{63}\tau_3)\widetilde{\phi_3}'(-d_6),
\end{aligned}$$

$$H(\phi_1,\phi_2,\phi_3) := f(Q) - Q_y f(\phi_1) - Q_z f(\phi_2) - Q_w f(\phi_3).$$

for constants $a_{ij}, b_{ij}, d_i, \sigma_j, \tau_j \in [0,1]$, $i = 1,\cdots,6$, $j = 1,2,3$.

From [4], we have the following lemmas.

Lemma 2.2 ([4], (iii) of Lemma 2.1)**.** *There exist functions R_j, $j = 1,\cdots,16$, such that*

$$\begin{aligned}
Q_{yy}(y,z,w) &= zR_1(y,z,w) = (a-z)R_2(y,z,w) = (1-w)R_3(y,z,w),\\
Q_{zz}(y,z,w) &= (1-y)R_4(y,z,w) = (1-w)R_5(y,z,w)\\
&= yR_6(y,z,w) + (w-a)R_7(y,z,w),\\
Q_{ww}(y,z,w) &= (1-y)R_8(y,z,w) = zR_9(y,z,w) = (a-z)R_{10}(y,z,w),
\end{aligned}$$

$$
\begin{aligned}
Q_{yz}(y,z,w) &= (1-w)R_{11}(y,z,w), \quad Q_{zw}(y,z,w) = (1-y)R_{12}(y,z,w),\\
Q_{yw}(y,z,w) &= (1-y)R_{13}(y,z,w) = zR_{14}(y,z,w)\\
&= (a-z)R_{15}(y,z,w) = (1-w)R_{16}(y,z,w).
\end{aligned}
$$

Lemma 2.3 ([4], Lemma 2.2)**.** *There exist positive constants ϵ_1, ϵ_2 and ϵ_3 such that*

$$
\begin{aligned}
&Q_y(\phi_1(\eta-p_1),\phi_2(\eta+p_1),\phi_3(\eta+p_2)) \ge \epsilon_1 && \text{for } \eta \le -p_1,\\
&Q_z(\phi_1(\eta-p_1),\phi_2(\eta+p_1),\phi_3(\eta+p_2)) \ge \epsilon_2 && \text{for } p_1 \le \eta \le -p_2,\\
&Q_w(\phi_1(\eta-p_1),\phi_2(\eta+p_1),\phi_3(\eta+p_2)) \ge \epsilon_3 && \text{for } \eta \ge -p_1.
\end{aligned}
$$

Lemma 2.4 ([4], Lemma 2.3)**.** *The following statements hold.*

$$
\begin{aligned}
&F(\phi_1,\phi_2,\phi_3) > 0 && \text{for } \eta \in \mathbb{R},\\
&F(\phi_1,\phi_2,\phi_3) \ge \frac{1}{2}Q_y|\phi_1'(\eta-p_1)| && \text{for } \eta \le p_1,\\
&F(\phi_1,\phi_2,\phi_3) \ge \frac{1}{2}\Big[Q_y|\phi_1'(\eta-p_1)| + Q_z|\phi_2'(\eta+p_1)|\Big] && \text{for } p_1 \le \eta \le -p_1,\\
&F(\phi_1,\phi_2,\phi_3) \ge \frac{1}{2}\Big[Q_z|\phi_2'(\eta+p_1)| + Q_w|\phi_3'(\eta+p_2)|\Big] && \text{for } -p_1 \le \eta \le -p_2,\\
&F(\phi_1,\phi_2,\phi_3) \ge \frac{1}{2}Q_w|\phi_3'(\eta+p_2)| && \text{for } \eta \ge -p_2,
\end{aligned}
$$

Since the proof of Lemma 2.2-2.4 are the same as in [4], we omit them here. Also, for any $\eta_1,\eta_2,\eta_3 \in [-1,1]$, we know that there exists a positive constant C such that

$$
|\widetilde{R_j}(\eta_1,\eta_2,\eta_3)| \le C, \tag{2.14}
$$

for $\eta \in \mathbb{R}$, $j = 1,\cdots,16$ where

$$
\widetilde{R_j}(\eta_1,\eta_2,\eta_3) := R_j(\phi_1(\eta-p_1+\eta_1),\phi_2(\eta+p_1+\eta_2),\phi_3(\eta+p_2+\eta_3)).
$$

By using the results shown in the above, we obtain the following key lemma.

Lemma 2.5 ([4], Lemma 2.4)**.** *There is a positive constant ρ such that*

$$
\left|\frac{H(\phi_1,\phi_2,\phi_3)+G(\phi_1,\phi_2,\phi_3)}{F(\phi_1,\phi_2,\phi_3)}\right| \le
\begin{cases}
\rho(|\phi_2'|+|\phi_3'|) & \text{for } \eta \le 0,\\
\rho(|\phi_1'|+|\phi_3'|) & \text{for } 0 \le \eta \le -\dfrac{p_1+p_2}{2},\\
\rho(|\phi_1'|+|\phi_2'|) & \text{for } \eta \ge -\dfrac{p_1+p_2}{2}.
\end{cases} \tag{2.15}
$$

Proof. For the estimation of $|H(\phi_1,\phi_2,\phi_3)/F(\phi_1,\phi_2,\phi_3)|$, since the proof is the same as in the proof of [4, Lemma 2.4], we do not repeat it here. Thus, we only estimate

$$
|G(\phi_1,\phi_2,\phi_3)/F(\phi_1,\phi_2,\phi_3)|.
$$

To simplify the notation, we omit (ϕ_1,ϕ_2,ϕ_3) for the functions $G(\phi_1,\phi_2,\phi_3)$, $F(\phi_1,\phi_2,\phi_3)$ and so on.

For $\eta \le p_1$, we have $F \ge \epsilon_1|\phi_1'|/2$ by Lemma 2.3 and Lemma 2.4. Then, for any $\eta_1,\eta_2,\eta_3 \in [-1,1]$, we derive

$$\begin{aligned}|\widetilde{Q}_{yy}(\eta_1,\eta_2,\eta_3)| &= |\phi_2(\eta+p_1+\eta_2)||\widetilde{R_1}(\eta_1,\eta_2,\eta_3)| \le C|\phi_2(\eta+p_1+\eta_2)|\\ &\le CM|\phi_2'(\eta_2)| \le CM\widetilde{M}|\phi_2'|\end{aligned}$$

by Lemma 2.1, Lemma 2.2 and (2.3). Similarly, we have

$$\begin{aligned}&|\widetilde{Q}_{zz}(\eta_1,\eta_2,\eta_3)|, |\widetilde{Q}_{ww}(\eta_1,\eta_2,\eta_3)|, |\widetilde{Q}_{zw}(\eta_1,\eta_2,\eta_3)| \le CM\widetilde{M}|\phi_1'|,\\ &|\widetilde{Q}_{yz}(\eta_1,\eta_2,\eta_3)|, |\widetilde{Q}_{yw}(\eta_1,\eta_2,\eta_3)| \le C,\end{aligned}$$

for any $\eta_1,\eta_2,\eta_3 \in [-1,1]$. Therefore, we obtain

$$\begin{aligned}\left|\frac{G}{F}\right| &\le 4\left[\frac{CM\widetilde{M}^3|\phi_2'||\phi_1'|^2 + CM\widetilde{M}^3|\phi_1'||\phi_2'|^2 + CM\widetilde{M}^3|\phi_1'||\phi_3'|^2}{\epsilon_1|\phi_1'|}\right]\\ &\quad +8\left[\frac{C\widetilde{M}^2|\phi_1'||\phi_2'| + C\widetilde{M}^2|\phi_1'||\phi_3'| + CM\widetilde{M}^3|\phi_1'||\phi_2'||\phi_3'|}{\epsilon_1|\phi_1'|}\right]\\ &= \frac{4C\widetilde{M}^2}{\epsilon_1}[M\widetilde{M}K|\phi_2'| + M\widetilde{M}K|\phi_2'| + M\widetilde{M}K|\phi_3'| + 2|\phi_2'| + 2|\phi_3'| + 2M\widetilde{M}K|\phi_2'|]\end{aligned}$$

Similarly, we can show the estimation of $|G/F|$ for $\eta \ge p_1$ and we get the conclusion. □

By the choice of κ, p_1, p_2, (2.2) and Lemma 2.5, there exist a $t_0 < 0$ such that

$$|H(\phi_1,\phi_2,\phi_3) + G(\phi_1,\phi_2,\phi_3)| \le F(\phi_1,\phi_2,\phi_3)\cdot 2K\rho e^{\kappa p_1}.$$

for all $t \le t_0$. Since

$$\begin{aligned}\mathscr{L}[\overline{U}(\eta,t)] &= F(\phi_1,\phi_2,\phi_3)\cdot Le^{\kappa p_1} - G(\phi_1,\phi_2,\phi_3) - H(\phi_1,\phi_2,\phi_3)\\ &\ge F(\phi_1,\phi_2,\phi_3)(L-2K\rho)e^{\kappa p_1} \ge 0\end{aligned}$$

for $t \le t_0$, $\overline{U}(\eta,t)$ is a super-solution of (2.5) for $t \le t_0$.

By using the similar argument as above, $\underline{U}(\eta,t)$ is a sub-solution of (2.5) for $t \le t_0$. Moreover, by (2.13), Lemma 2.4, the function F is bounded above and

$$\begin{aligned}\overline{U}(\eta,t) - \underline{U}(\eta,t) &= Q(\phi_1(\eta-p_1(t)),\phi_2(\eta+p_1(t)),\phi_3(\eta+p_2(t)))\\ &\quad -Q(\phi_1(\eta-r_1(t)),\phi_2(\eta+r_1(t)),\phi_3(\eta+r_2(t)))\\ &= \int_0^1 F(\phi_1(\eta-\theta p_1-(1-\theta)r_1),\phi_2(\eta+\theta p_1+(1-\theta)r_1),\end{aligned}$$

$$\phi_3(\eta+\theta p_2+(1-\theta)r_2))d\theta\cdot(p_1-r_1),$$

we have

$$\overline{U}(\eta,t)\geq\underline{U}(\eta,t)\quad\text{for}\quad\eta\in\mathbb{R},\ t\leq t_0, \tag{2.16}$$

$$\sup_{\eta\in\mathbb{R}}\{\overline{U}(\eta,t)-\underline{U}(\eta,t)\}\leq\mu e^{\kappa s_1 t}\quad\text{for}\quad t\leq t_0, \tag{2.17}$$

for some constant $\mu>0$.

By using the same method as in [5, 7], there exists an unique entire solution $u(x,t)$ of (1.1) such that

$$\underline{U}(x+\overline{v}t,t)\leq u(x,t)\leq\overline{U}(x+\overline{v}t,t)$$

for all $x\in\mathbb{R}$ and $t\leq t_0$ where the functions $\underline{U}$ and $\overline{U}$ are defined as (2.7) and (2.8). Then we take

$$\theta_1=\frac{1}{\kappa}\log\left(e^{-\kappa r_0}-\frac{L}{s_1}\right),\quad\theta_2=\theta_3=-\theta_1,$$

and the entire solution $u(x,t)$ of (1.1) shown in the above satisfies (1.7) (see [4, Theorem 3.3]). The proof of Theorem 1.1 has been completed.

3. The Proof of Theorem 1.2

First, we consider the auxiliary rational function $Q(y,z,w)$ as follows.

$$Q(y,z,w)=z+\frac{(1-y)z(a-w)(-z)+y(a-z)w(1-z)}{(1-y)za+(a-z)w}.$$

Then we take the functions $p_i(t)$, $r_i(t)$, $i=1,2$ are the solutions of the following initial value problems

$$\begin{aligned}
\dot{p}_1&=s_1+Le^{\kappa p_1},\quad -\infty<t<0,\quad p_1(0)=p_0,\\
\dot{r}_1&=s_1-Le^{\kappa r_1},\quad -\infty<t<0,\quad r_1(0)=r_0,\\
\dot{p}_2&=s_2-Le^{\kappa p_1},\quad -\infty<t<0,\quad p_2(0)=r_0,\\
\dot{r}_2&=s_2+Le^{\kappa r_1},\quad -\infty<t<0,\quad r_2(0)=p_0,
\end{aligned}$$

where p_0, r_0 are the same as in §2, $L>2K\rho$ is a positive constant and

$$\kappa:=\min\left\{\gamma_1,\gamma_2,\beta_2,\beta_3,\frac{(s_2-s_1)\gamma_2}{4s_1},\frac{(s_2-s_1)\beta_3}{4s_1}\right\}.$$

Define the functions $\overline{u}(x,t)$ and $\underline{u}(x,t)$ as follows.

$$\overline{u}(x,t) = Q(\phi_1(x+\overline{v}t - p_1(t)), \phi_2(x+\overline{v}t + p_1(t)), \phi_3(x+\overline{v}t + p_2(t)))$$
$$\underline{u}(x,t) = Q(\phi_1(x+\overline{v}t - r_1(t)), \phi_2(x+\overline{v}t + r_1(t)), \phi_3(x+\overline{v}t + r_2(t))).$$

By the similar argument as in §2, the functions $\overline{u}(x,t)$ and $\underline{u}(x,t)$ are a pair of super-sub-solution of (1.1) for $t \le t_0$ with some constant $t_0 < 0$. Hence, the existence and uniqueness of the entire solution $u(x,t)$ of (1.1) can be shown and $u(x,t)$ satisfies

$$\underline{u}(x,t) \le u(x,t) \le \overline{u}(x,t)$$

for all $x \in \mathbb{R}$ and $t \le t_0$. Moreover, it is not difficult to check the entire solution $u(x,t)$ satisfies (1.7) by using the similar argument as in [4, Theorem 3.3] and taking

$$\theta_1 = -\theta_2 := \frac{1}{\kappa}\log\left(e^{-\kappa r_0} - \frac{L}{s_1}\right), \quad \theta_3 := \frac{1}{\kappa}\log\left(e^{-\kappa r_0} - \frac{L}{s_1}\right) + p_0 + r_0.$$

Finally, (1.10) holds from the result shown in [16, Theorem 1.1]. Therefore, the proof of Theorem 1.2 has been done.

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