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NEVANLINNA'S FIVE-VALUE THEOREM FOR DERIVATIVES OF ALGEBROID FUNCTIONS ON ANNULI

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Abstract. In this paper, we first obtain the famous Xiong Inequality for algebroid functions on annuli and also generalise Nevanlinna's five-value theorem for derivatives of algebroid functions by considering weaker assumptions of sharing five values and small functions to partially sharing $k(\geq 5)$ values and small functions on annuli. As a particular cases of our results, we deduce several results.

1. Introduction

The uniqueness theory of algebroid functions is an interesting problem in the value distribution theory. The uniqueness problem of algebroid functions was firstly considered by Valiron, afterwards some scholars have got several uniqueness theorems of algebroid functions in the complex plane $\mathbb{C}$ (see [2, 3, 5, 9, 10, 11]). In 2005, A. Ya. Khrystiyanyn and A. A. Kondratyuk have proposed on the Nevanlinna Theory for meromorphic functions on annuli (see [7, 8]). In 2009, Cao and Yi [1] investigated the uniqueness of meromorphic functions sharing some values on annuli. In 2015,Yang Tan [12], Yang Tan and Yue Wang [13] proved some interesting results on the multiple values and uniqueness of algebroid functions on annuli. Thus it is interesting to consider the uniqueness problem of algebroid functions in multiply connected domains. By Doubly connected mapping theorem [17] each doubly connected domain is conformally equivalent to the annulus $\{z : r < |z| < R\}, 0 \leq r < R \leq +\infty$. We consider only two cases : $r = 0$, $R = +\infty$ simultaneously and $0 \leq r < R \leq +\infty$. In the latter case the homothety $z \mapsto \frac{z}{rR}$ reduces the given domain to the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right) = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $R_0 = \sqrt{\frac{R}{r}}$. Thus, in two cases every annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$.

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2. Basic notations and definitions

We assume that the reader is familiar with the Nevanlinna theory of meromorphic functions and algebroid functions (see [4] and [15]).

Let $A_v(z), A_{v-1}(z), \dots, A_0(z)$ be a group of analytic functions which have no common zeros and define on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$,

$$\psi(z,W) = A_v(z)W^v + A_{v-1}(z)W^{v-1} + \cdots + A_1(z)W + A_0(z) = 0. \tag{2.1}$$

Then irreducible equation (2.1) defines a v-valued algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$.

Let $W(z)$ be a v-valued algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$, we use the following notations

$$m(r,W) = \frac{1}{v}\sum_{j=1}^{v} m(r,w_j) = \frac{1}{v}\sum_{j=1}^{v}\frac{1}{2\pi}\int_0^{2\pi}\log^+|w_j(re^{i\theta})|d\theta,$$

$$N_1(r,W) = \frac{1}{v}\int_{\frac{1}{r}}^{1}\frac{n_1(t,W)}{t}dt, \quad N_2(r,W) = \frac{1}{v}\int_1^r\frac{n_2(t,W)}{t}dt,$$

$$\overline{N}_1\left(r,\frac{1}{W-a}\right) = \frac{1}{v}\int_{\frac{1}{r}}^{1}\frac{\overline{n}_1\left(t,\frac{1}{W-a}\right)}{t}dt, \quad \overline{N}_2\left(r,\frac{1}{W-a}\right) = \frac{1}{v}\int_1^r\frac{\overline{n}_2\left(t,\frac{1}{W-a}\right)}{t}dt,$$

$$m_0(r,W) = m(r,W) + m\left(\frac{1}{r},W\right) - 2m(1,W), \quad N_0(r,W) = N_1(r,W) + N_2(r,W),$$

$$\overline{N}_0\left(r,\frac{1}{W-a}\right) = \overline{N}_1\left(r,\frac{1}{W-a}\right) + \overline{N}_2\left(r,\frac{1}{W-a}\right),$$

where $w_j(z)(j=1,2,\dots,v)$ is one valued branch of $W(z)$, $n_1(t,W)$ is the counting function of poles of the function $W(z)$ in $\{z : t < |z| \le 1\}$ and $n_2(t,W)$ is the counting function of poles of the function $W(z)$ in $\{z : 1 < |z| \le t\}$ (both counting multiplicity).$\overline{n}_1\left(t,\frac{1}{W-a}\right)$ is the counting function of poles of the function $\frac{1}{W-a}$ in $\{z : t < |z| \le 1\}$ and $\overline{n}_2\left(t,\frac{1}{W-a}\right)$ is the counting function of poles of the function $\frac{1}{W-a}$ in $\{z : 1 < |z| \le t\}$ (both ignoring multiplicity). $\overline{n}_1^{k)}\left(t,\frac{1}{W-a}\right)\left(\overline{n}_1^{(k}\left(t,\frac{1}{W-a}\right)\right)$ is the counting function of poles of the function $\frac{1}{W-a}$ with multiplicity $\le k$ (or $> k$) in $\{z : t < |z| \le 1\}$, each point count only once; $\overline{n}_2^{k)}\left(t,\frac{1}{W-a}\right)\left(\overline{n}_2^{(k}\left(t,\frac{1}{W-a}\right)\right)$ is the counting function of poles of the function $\frac{1}{W-a}$ with multiplicity $\le k$ (or $> k$) in $\{z : 1 < |z| \le t\}$, each point count only once, respectively.

Let $W(z)$ be a v-valued algebroid function which determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$, when $a \in \mathbb{C}$, $n_0\left(r,\frac{1}{W-a}\right) = n_0\left(r,\frac{1}{\psi(z,a)}\right)$, $N_0\left(r,\frac{1}{W-a}\right) = \frac{1}{v}N_0\left(r,\frac{1}{\psi(z,a)}\right)$. In particular, when $a = 0$, $N_0\left(r,\frac{1}{W}\right) = \frac{1}{v}N_0\left(r,\frac{1}{A_0}\right)$. When $a = \infty$, $N_0(r,W) = \frac{1}{v}N_0\left(r,\frac{1}{A_v}\right)$; where $n_0\left(r,\frac{1}{W-a}\right)$ and $n_0\left(r,\frac{1}{\psi(z,a)}\right)$ are the counting function of zeros of $W(z) - a$ and $\psi(z,a)$ on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$, respectively.

Definition 2.1 ([12]). Let $W(z)$ be an algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, the function

$$T_0(r, W) = m_0(r, W) + N_0(r, W), \qquad 1 \leq r < R_0$$

is called *Nevanlinna characteristic of* $W(z)$.

3. Some lemmas

Lemma 3.1 ([7] (Jensen theorem for meromorphic function on annuli)). *Let f be a meromorphic function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, then*

$$N_0\left(r, \frac{1}{f}\right) - N_0(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(\frac{1}{r}e^{i\theta}\right)\right| d\theta$$
$$- \frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|d\theta,$$

where $1 \leq r < R_0$.

Lemma 3.2 ([13] (The first fundamental theorem on annuli)). *Let $W(z)$ be v-valued algebroid function which is determined by* (2.1) *on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, $a \in \mathbb{C}$*

$$m_0(r, a) + N_0(r, a) = T_0(r, W) + O(1).$$

Lemma 3.3 ([13] (The second fundamental theorem on annuli)). *Let $W(z)$ be v-valued algebroid function which is determined by* (2.1) *on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, a_k $(k = 1, 2, \ldots, p)$ are p distinct complex numbers (finite or infinite), then we have*

$$(p - 2v)T_0(r, W) \leq \sum_{k=1}^{p} N_0\left(r, \frac{1}{W - a_k}\right) - N_1(r, W) + S_0(r, W) \tag{3.1}$$

$N_1(r, W)$ is the density index of all multiple values including finite or infinite, every τ multiple value counts $\tau - 1$, and

$$S_0(r, W) = m_0\left(r, \frac{W'}{W}\right) + \sum_{j=1}^{p} m_0\left(r, \frac{W'}{W - a_k}\right) + O(1).$$

The remainder of the second fundamental theorem is the following formula

$$S_0(r, W) = O\left(\log T_0(r, W)\right) + O(\log r),$$

outside a set of finite linear measure, if $r \to R_0 = +\infty$, while

$$S_0(r, W) = O\left(\log T_0(r, W)\right) + O\left(\log \frac{1}{R_0 - r}\right),$$

outside a set E of r such that $\int_E \frac{dr}{R_0 - r} < +\infty$, when $r \to R_0 < +\infty$.

Remark 3.1 ([13]). The second fundamental theorem on annuli has other forms, as the following:

$$(p-1)T_0(r,W) \le N_0(r,W) + \sum_{k=1}^{p} N_0\left(r,\frac{1}{W-a_k}\right) - N_1(r) + Q_1(r,W), \tag{3.2}$$

$$N_1(r,W) = 2N_0(r,W) - N_0(r,W') + N_0\left(r,\frac{1}{W'}\right),$$
$$Q_1(r,W) = \sum_{k=0}^{p} m_0\left(r,\frac{W'}{W-a_k}\right) + O(1), a_0 = 0.$$

We notice that the following formula is true,

$$\sum_{k=1}^{p} N_0\left(r,\frac{1}{W-a_k}\right) - N_1(r,W) \le \sum_{k=1}^{p} \overline{N}_0\left(r,\frac{1}{W-a_k}\right). \tag{3.3}$$

$\overline{N}_0\left(r,\frac{1}{W-a_k}\right)$ is the reduced counting function of zeros(ignoring multiplicity). Then the second fundamental theorem can be rewritten as the following

$$(p-2v)T_0(r,W) \le \sum_{k=1}^{p} \overline{N}_0\left(r,\frac{1}{W-a_k}\right) + S_0(r,W). \tag{3.4}$$

Lemma 3.4 ([13]). *Let $W(z)$ be v-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$, if the following conditions are satisfied*

$$\underline{\lim}_{r\to\infty} \frac{T_0(r,W)}{\log r} < \infty, \quad R_0 = +\infty,$$
$$\underline{\lim}_{r\to R_0^-} \frac{T_0(r,W)}{\log\frac{1}{(R_0-r)}} < \infty, \quad R_0 < +\infty,$$

then $W(z)$ is an algebraic function.

Remark 3.2 ([13]). Let $W(z)$ be a v-valued algebroid function which is determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$, where $1<R_0\le+\infty$ and $\widehat{W}(z)$ be a μ-valued algebroid functions which is determined by the following equation on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$, where $1<R_0\le+\infty$,

$$\varphi(z,\widehat{W}) = B_\mu(z)\widehat{W}^\mu + B_{\mu-1}(z)\widehat{W}^{\mu-1} + \cdots + B_1(z)\widehat{W} + B_0(z) = 0.$$

Without loss of generality, let $\mu\le v$, $\overline{n}_\Delta(r,a)$ denotes the counting function of the common values of $W(z)=a$ and $\widehat{W}(z)=a$ on the annulus $\mathbb{A}\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$, ignoring multiplicity. And let

$$\overline{N}_\Delta(r,a) = \frac{\mu+v}{2\mu v}\int_{\frac{1}{r}}^{1} \frac{\overline{n}_{\Delta_1}(t,a)}{t}dt + \frac{\mu+v}{2\mu v}\int_{1}^{r} \frac{\overline{n}_{\Delta_2}(t,a)}{t}dt$$
$$\overline{N}_{12}(r,a) = \overline{N}_0\left(r,\frac{1}{W-a}\right) + \overline{N}_0\left(r,\frac{1}{\widehat{W}-a}\right) - 2\overline{N}_\Delta(r,a).$$

4. Main results

Let $W(z)$ be an algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$, where $1 < R_0 \leq +\infty$, and a be a complex number in the extended complex plane. Write $E(a, W) = \{z \in \mathbb{A} : W(z) - a = 0\}$, where each zero with multiplicity m is counted m times. If we ignore the multiplicity, then the set is denoted by $\overline{E}(a, W)$. We use $\overline{E}_{k)}(a, W)$ to denote the set of zeros of $W - a$ with multiplicities not greater than k, in which each zero is counted only once.

In this paper, we say that two algebroid functions on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, share a function $a(z)$ if we have $W(z) - a(z) = 0$ if and only if $\widehat{W} - a(z) = 0$. Now we consider the case that two algebroid function partially share small functions.

Definition 4.1. Let $W(z)$ be an algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ and $a(z)$ be a small function of $W(z)$. We define

$$\overline{W}(a, W) = \{z | W(z) - a(z) = 0\}$$

in which each zero is counted only once.

We say that an algebroid function $W(z)$ partially shares a value a with an algebroid function $\widehat{W}(z)$ on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ if

$$\overline{E}(a, W) \subseteq \overline{E}(a, \widehat{W})$$

To prove our main theorem, we need to get the following Xiong inequality for algebroid functions on annuli.

Theorem 4.1. *Let $W(z)$ be a ν-valued algebroid function determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, respectively and b_j $(j = 1, 2, \ldots, q)$ be distinct finite non zero complex numbers. Then for any positive integer n, we have*

$$\begin{aligned} qT_0(r, W) < \overline{N}_0(r, W) + qN_0\left(r, \frac{1}{W}\right) + \sum_{j=1}^{q} N_0\left(r, \frac{1}{W^{(n)} - b_j}\right) \\ - \left[(q-1)N_0\left(r, \frac{1}{W^{(n)}}\right) + N_0\left(r, \frac{1}{W^{(n+1)}}\right)\right] + S_0(r, W). \end{aligned} \tag{4.1}$$

Proof. We have

$$\begin{aligned} T_0(r, W') &= T_0\left(r, W\frac{W'}{W}\right) \leq T_0(r, W) + T_0\left(r, \frac{W'}{W}\right) + O(1) \\ &\leq T_0(r, W) + m_0\left(r, \frac{W'}{W}\right) + N_0\left(r, \frac{W'}{W}\right) + O(1) \\ &= T_0(r, W) + \overline{N}_0(r, W) + S_0(r, W) \end{aligned}$$

$$= 2\nu T_0(r,W) + S_0(r,W). \tag{4.2}$$

Hence, by Lemma 3.3 and (4.2), we have

$$S_0(r,W^{(k)}) = O(\log r\, T_0(r,W^{(k)})) = O(\log r\, T_0(r,W)) = S_0(r,W). \tag{4.3}$$

$$m_0\left(r,\frac{W^{(k)}}{W-a_i}\right) = S_0(r,W). \tag{4.4}$$

From Lemma 3.3, (4.3) and (4.4), we have

$$m_0\left(r,\frac{W^{(k)}}{\prod\limits_{i=1}^{p}(W-a_i)}\right) = S_0(r,W^{(k)}),\quad m_0\left(r,\frac{W^{(k+1)}}{W^{(k)}\prod\limits_{j=1}^{q}(W^{(k)}-b_j)}\right) = S_0(r,W^{(k)})$$

and

$$\frac{1}{\prod\limits_{i=1}^{p}(W-a_i)^n} = \left\{\frac{W^{(k)}}{\prod\limits_{i=1}^{p}(W-a_i)}\right\}^n \frac{W^{(k+1)}}{W^{(k)}\prod\limits_{j=1}^{q}(W^{(k)}-b_j)}\,\frac{\prod\limits_{j=1}^{q}(W^{(k)}-b_j)}{(W^k)^{n-1}W^{(k+1)}}$$

Then

$$nm_0\left(r,\frac{1}{\prod_{i=1}^{p}(W-a_i)}\right) \le m_0\left(r,\frac{\prod_{j=1}^{q}(W^{(k)}-b_j)}{(W^k)^{n-1}W^{(k+1)}}\right) + S_0(r,W^{(k)}). \tag{4.5}$$

From (4.3), Lemma 3.1 and 3.3, we have

$$\begin{aligned}
m_0\left(r,\frac{\prod\limits_{j=1}^{q}(W^{(k)}-b_j)}{(W^{(k)})^{n-1}W^{(k+1)}}\right) &= N_0\left(r,\frac{(W^{(k)})^{n-1}W^{(k+1)}}{\prod\limits_{j=1}^{q}(W^{(k)}-b_j)}\right) - N_0\left(r,\frac{\prod\limits_{j=1}^{q}(W^{(k)}-b_j)}{(W^{(k)})^{n-1}W^{(k+1)}}\right) + S_0(r,W^{(k)})\\
&= \overline{N}_0(r,W) - (q-n)N_0(R,W^{(k)}) + \sum_{j=1}^{q} N_0\left(r,\frac{1}{W^{(k)}-b_j}\right)\\
&\quad -(n-1)N_0\left(r,\frac{1}{W^{(k)}}\right) - N_0\left(r,\frac{1}{W^{(k+1)}}\right) + S_0(r,W^{(k)}).
\end{aligned} \tag{4.6}$$

From Lemma 3.1 and (4.2), (4.3), the left of (4.6) can be replaced by

$$\begin{aligned}
nm_0\left(r,\frac{1}{\prod\limits_{i=1}^{p}(W-a_i)}\right) &= nT_0\left(r,\prod_{i=1}^{p}(W-a_i)\right) - nN_0\left(r,\frac{1}{\prod\limits_{i=1}^{p}(W-a_i)}\right) + O(1)\\
&= npT_0(r,W) - n\sum_{i=1} pN_0\left(r,\frac{1}{(W-a_i)}\right) + S_0(r,W^{(k)}).
\end{aligned} \tag{4.7}$$

Put (4.6) and (4.7) into (4.5), then we have

$$npT_0(r,W) \le \overline{N}_0(r,W) + n\sum_{i=1}^{p} N_0\left(r, \frac{1}{(W-a_i)}\right) + \sum_{j=1}^{q} N_0\left(r, \frac{1}{W^{(k)}-b_j}\right)$$
$$-(q-n)N_0(r,W^{(k)}) - (n-1)N_0\left(r, \frac{1}{W^{(k)}}\right) - N_0\left(r, \frac{1}{W^{(k+1)}}\right) + S_0(r,W).$$

Let $n = q$, $p = 1$, we can get the inequality (4.1). The proof of Theorem 4.1 is completed. □

It is natural question to ask if $W^{(n)}(z)$ and $\widehat{W}^{(n)}(z)$ be two ν-valued and μ-valued algebroid functions on annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$, partially share more than five values for a positive integer n, what the corresponding inequality becomes? Below we answer for this equations by proving the following theorem.

Theorem 4.2. *Let $W(z)$ and $\widehat{W}(z)$ be two ν-valued and μ-valued algebroid functions determined by (2.1) on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$, respectively and $\mu \le \nu$, let a_j $(j = 1,2,\ldots,k)$ be k distinct small functions, where $k \ge 4\nu+1$ and for a non negative integers n, if*

$$E(a_j, W^{(n)}) \subseteq E(a_j, \widehat{W}^{(n)}), \quad forall 1 \le j \le k, \tag{4.8}$$

$$E(0, W_1) \subseteq E(0, W^{(n)}) \ and \ E(0, \widehat{W}) \subseteq E(0, \widehat{W}^{(n)}), \tag{4.9}$$

and

$$\frac{\underline{\lim}_{r\to\infty} \sum_{j=1}^{k} N_0\left(r, \frac{1}{W^{(n)}-a_j}\right)}{\underline{\lim}_{r\to\infty} \sum_{j=1}^{k} N_0\left(r, \frac{1}{\widehat{W}^{(n)}-a_j}\right)} > \frac{n+1}{k-(n+2\nu+1)}, \tag{4.10}$$

then $W^{(n)}(z) \equiv \widehat{W}^{(n)}(z)$.

Proof. Given $\epsilon > 0$ and from Theorem 4.1, we have

$$(k-2\nu-\epsilon)T_0(r,W) \le \overline{N}_0(r,W) + (k-2\nu)N_0\left(r, \frac{1}{W}\right) + \sum_{j=1}^{k-2\nu} N_0\left(r, \frac{1}{W^{(n)}-a_j}\right)$$
$$-(k-(2\nu+1))N_0\left(r, \frac{1}{W^{(n)}}\right) + S_0(r,W) \tag{4.11}$$

and

$$(k-2\nu-\epsilon)T_0(r,\widehat{W}) \le \overline{N}_0(r,\widehat{W}) + (k-2\nu)N_0\left(r, \frac{1}{\widehat{W}}\right) + \sum_{j=1}^{k-2\nu} N_0\left(r, \frac{1}{\widehat{W}^{(n)}-a_j}\right)$$
$$-(k-(2\nu+1))N_0\left(r, \frac{1}{\widehat{W}^{(n)}}\right) + S_0(r,\widehat{W}). \tag{4.12}$$

Using (4.9), (4.11) and (4.12) reduces to

$$(k-2\nu-\epsilon)T_0(r,W) \le \overline{N}_0(r,W)+N_0\left(r,\frac{1}{W^{(n)}}\right) + \sum_{j=1}^{k-2\nu} N_0\left(r,\frac{1}{W^{(n)}-a_j}\right)+S_0(r,W) \tag{4.13}$$

and

$$(k-2\nu-\epsilon)T_0(r,\widehat{W}) \le \overline{N}_0(r,\widehat{W})+N_0\left(r,\frac{1}{\widehat{W}^{(n)}}\right) + \sum_{j=1}^{k-2\nu} N_0\left(r,\frac{1}{\widehat{W}^{(n)}-a_j}\right)+S_0(r,\widehat{W}). \tag{4.14}$$

Without loss of generality, we may assume $a_k=\infty$ and $a_{k-1}=0$.
First we may assume that all $a_j (1\le j\le k)$ in (4.8) are finite. Then by (4.13) and (4.14), we have

$$(k-(2\nu+1)-\epsilon)T_0(r,W) \le \sum_{j=1}^{k-1} N_0\left(r,\frac{1}{W^{(n)}-a_j}\right)+S_0(r,W) \tag{4.15}$$

and

$$(k-(2\nu+1)-\epsilon)T_0(r,\widehat{W}) \le \sum_{j=1}^{k-1} N_0\left(r,\frac{1}{\widehat{W}^{(n)}-a_j}\right)+S_0(r,\widehat{W}). \tag{4.16}$$

From (4.15), (4.16) and by Remark 3.2, we have

$$\begin{aligned}(q-(2\nu+1)-\epsilon)[T_0(r,W)+T_0(r,\widehat{W})] &\le \sum_{j=1}^{k-1} \overline{N}_0\left(r,\frac{1}{W^{(n)}-a_j}\right)+\sum_{j=1}^{q} \overline{N}_0\left(r,\frac{1}{\widehat{W}^{(n)}-a_j}\right)\\ &\quad +S_0(r,W)+S_0(r,\widehat{W}), \qquad (4.17)\\ &\le \sum_{j=1}^{k-1}\overline{N}_{12}(r,a_j)+2\sum_{j=1}^{k-1}\overline{N}_{\Delta}(r,a_j)\\ &\quad +S_0(r,W)+S_0(r,\widehat{W}). \qquad (4.18)\end{aligned}$$

If $W^{(n)}(z)\not\equiv \widehat{W}^{(n)}(z)$, then we have

$$\sum \overline{n}_{\Delta}(r,a) \le n_0\left(r,\frac{1}{R(\varphi,\psi)}\right),$$

$R(\varphi,\psi)$ denotes the resultant of $\varphi(z,W^{(n)})$ and $\psi(z,W^{(n)})$, it can be written as the following

$$R(\varphi,\psi)=[A_\nu(z)]^\mu[B_\mu(z)]^\nu \prod_{\substack{1\le j\le\nu\\1\le k\le\mu}}[w_j^{(n)}(z)-\widehat{w}_j^{(n)}(z)].$$

It can be written in the another form

$$R(\varphi,\psi)=\begin{vmatrix} A_\nu(z) & A_{\nu-1}(z) & \dots & \dots & A_0(z) & 0 & \dots & 0\\ 0 & A_\nu(z) & A_{\nu-1}(z) & \dots & A_1(z) & A_0(z) & \dots & 0\\ \vdots & \vdots & & & \vdots & & & \\ 0 & 0 & 0 & A_\nu(z) & A_{\nu-1}(z) & \dots & \dots & A_0(z)\\ B_\mu(z) & B_{\mu-1}(z) & \dots & \dots & B_0(z) & 0 & \dots & 0\\ 0 & B_\mu(z) & B_{\mu-1}(z) & \dots & B_1(z) & B_0(z) & \dots & 0\\ \vdots & \vdots & & & \vdots & & & \\ 0 & 0 & 0 & B_\mu(z) & B_{\mu-1}(z) & \dots & \dots & B_0(z)\end{vmatrix}$$

So we know that $R(\varphi,\psi)$ is a holomorphic function and using Jensen Theorem for meromorphic function on annuli, we have

$$
\begin{aligned}
&N_0\left(r,\frac{1}{R(\varphi,\psi)}\right)\\
&=\frac{1}{2\pi}\int_0^{2\pi}\log|R[\psi(re^{i\theta},W^{(n)}),\varphi(re^{i\theta},\widehat{W}^{(n)})]|d\theta\\
&\quad+\frac{1}{2\pi}\int_0^{2\pi}\log\left|R\left[\psi\left(\frac{1}{r}e^{i\theta},W^{(n)}\right),\varphi\left(\frac{1}{r}e^{i\theta},\widehat{W}^{(n)}\right)\right]\right|d\theta\\
&\quad+2\frac{1}{2\pi}\int_0^{2\pi}\log|R[\psi(e^{i\theta},W^{(n)}),\varphi(e^{i\theta},\widehat{W}^{(n)})]|d\theta\\
&=\frac{\mu}{2\pi}\int_0^{2\pi}\log|A_\nu(re^{i\theta})|d\theta+\frac{\nu}{2\pi}\int_0^{2\pi}\log|B_\mu(re^{i\theta})|d\theta\\
&\quad+\frac{1}{2\pi}\int_0^{2\pi}\log\left|\prod_{\substack{1\le j\le\nu\\1\le k\le\mu}}[w_j^{(n)}(re^{i\theta})-\widehat{w}_j^{(n)}(re^{i\theta})]\right|d\theta\\
&\quad+\frac{\mu}{2\pi}\int_0^{2\pi}\log\left|A_\nu\left(\frac{1}{r}e^{i\theta}\right)\right|d\theta+\frac{\nu}{2\pi}\int_0^{2\pi}\log\left|B_\mu\left(\frac{1}{r}e^{i\theta}\right)\right|d\theta\\
&\quad+\frac{1}{2\pi}\int_0^{2\pi}\log\left|\prod_{\substack{1\le j\le\nu\\1\le k\le\mu}}\left[w_j^{(n)}\left(\frac{1}{r}e^{i\theta}\right)-\widehat{w}_j^{(n)}\left(\frac{1}{r}e^{i\theta}\right)\right]\right|d\theta-2.\frac{\mu}{2\pi}\int_0^{2\pi}\log|A_\nu(e^{i\theta})|d\theta\\
&\quad-2\frac{\nu}{2\pi}\int_0^{2\pi}\log|B_\mu(e^{i\theta})|d\theta-2\frac{1}{2\pi}\int_0^{2\pi}\log\left|\prod_{\substack{1\le j\le\nu\\1\le k\le\mu}}[w_j^{(n)}(e^{i\theta})-\widehat{w}_j^{(n)}(e^{i\theta})]\right|d\theta\\
&=\frac{\mu}{2\pi}\int_0^{2\pi}\log|A_\nu(re^{i\theta})|d\theta+\frac{\mu}{2\pi}\int_0^{2\pi}\log\left|A_\nu\left(\frac{1}{r}e^{i\theta}\right)\right|d\theta-2\frac{\mu}{2\pi}\int_0^{2\pi}\log|A_\nu(e^{i\theta})|d\theta\\
&\quad+\frac{\nu}{2\pi}\int_0^{2\pi}\log|B_\mu(re^{i\theta})|d\theta+\frac{\nu}{2\pi}\int_0^{2\pi}\log\left|B_\mu\left(\frac{1}{r}e^{i\theta}\right)\right|d\theta-2\frac{\nu}{2\pi}\int_0^{2\pi}\log|B_\mu(e^{i\theta})|d\theta\\
&\quad+\frac{1}{2\pi}\int_0^{2\pi}\log\left|\prod_{\substack{1\le j\le\nu\\1\le k\le\mu}}[w_j^{(n)}(re^{i\theta})-\widehat{w}_j^{(n)}(re^{i\theta})]\right|d\theta\\
&\quad+\frac{1}{2\pi}\int_0^{2\pi}\log\left|\prod_{\substack{1\le j\le\nu\\1\le k\le\mu}}\left[w_j^{(n)}\left(\frac{1}{r}e^{i\theta}\right)-\widehat{w}_j^{(n)}\left(\frac{1}{r}e^{i\theta}\right)\right]\right|d\theta\\
&\quad-2.\frac{1}{2\pi}\int_0^{2\pi}\log\left|\prod_{\substack{1\le j\le\nu\\1\le k\le\mu}}[w_j^{(n)}(e^{i\theta})-\widehat{w}_j^{(n)}(e^{i\theta})]\right|d\theta
\end{aligned}
$$

$$\leq \mu\left[m_0(r,A_\nu)-m_0\left(r,\frac{1}{A_\nu}\right)\right]+\nu\left[m_0(r,B_\mu)-m_0\left(r,\frac{1}{B_\mu}\right)\right]$$
$$+\mu\nu[m_0(r,W^{(n)})+m_0(r,\widehat{W}^{(n)})]+O(1)$$
$$=\mu\nu[T_0(r,W^{(n)})+T_0(r,\widehat{W}^{(n)})]+O(1).$$

Then we get

$$\begin{aligned}\sum\overline{N}_\Delta(r,a_j) &\leq \frac{2\mu\nu}{\mu+\nu}[T_0(r,W^{(n)})+T_0(r,\widehat{W}^{(n)})]+O(1)\\ &\leq (n+1)\nu[T_0(r,W)+T_0(r,\widehat{W})]+O(1).\end{aligned}\tag{4.19}$$

By the condition of Theorem 4.2, we know that the set of zeros of $W(z)-a_j$ and $\widehat{W}(z)-a_j$ in which each point counts only once, at the same time we get $\overline{N}_{12}(r,a_j)=0$.

Therefore

$$\sum_{j=1}^{k-1}\overline{N}_0\left(r,\frac{1}{W^{(n)}-a_j}\right)\leq\sum_{j=1}^{k-1}\overline{N}_0\left(r,\frac{1}{W^{(n)}-\widehat{W}^{(n)}}\right)\leq(n+1)\nu[T_0(r,W)+T_0(r,\widehat{W})]+O(1).\tag{4.20}$$

From (4.15), (4.16) and (4.20), we have

$$\sum_{j=1}^{k-1}\overline{N}_0\left(r,\frac{1}{W^{(n)}-a_j}\right)\leq\left(\frac{n+1}{k-(2\nu+1)}+O(1)\right)\left[\sum_{j=1}^{k-1}\overline{N}_0\left(r,\frac{1}{W^{(n)}-a_j}\right)+\sum_{j=1}^{k-1}\overline{N}_0\left(r,\frac{1}{\widehat{W}^{(n)}-a_j}\right)\right]$$

for $r\not\in E$, which implies

$$\left(\frac{k-(n+4\nu)-\epsilon}{(k-2\nu+1)-\epsilon}+O(1)\right)\sum_{j=1}^{k-1}\overline{N}_0\left(r,\frac{1}{W^{(n)}-a_j}\right)\leq\left(\frac{n+1}{k-(2\nu+1)-\epsilon}+O(1)\right)\sum_{j=1}^{k-1}\overline{N}_0\left(r,\frac{1}{\widehat{W}^{(n)}-a_j}\right)$$

for $r\not\in E$.

Therefore, we obtain

$$\varliminf_{r\to\infty}\frac{\sum_{j=1}^{k-1}\overline{N}_0\left(r,\frac{1}{W-a_j}\right)}{\sum_{j=1}^{k-1}\overline{N}_0\left(r,\frac{1}{\widehat{W}-a_j}\right)}\leq\frac{n+1}{k-(n+4\nu)-\epsilon}$$

which is true for all $\epsilon>0$ and replace $k-1$ by k. Hence

$$\varliminf_{r\to\infty}\frac{\sum_{j=1}^{k-1}\overline{N}_0\left(r,\frac{1}{W^{(n)}-a_j}\right)}{\sum_{j=1}^{k-1}\overline{N}_0\left(r,\frac{1}{\widehat{W}^{(n)}-a_j}\right)}\leq\frac{n+1}{k-(n+4\nu)}.\tag{4.21}$$

Where a_k is finite (since all $a_j(1\leq j\leq k)$ are finite).

From (4.21) contradicts to (4.10) and hence $W^{(n)}(z) \equiv \widehat{W}^{(n)}(z)$. Now assume that one of the $a_j (1 \le j \le k)$ in (4.8) is infinity say $a_k = \infty$. Taking any finite value a such that $a \ne a_j (1 \le j \le k-1)$. Set

$$F^{(n)}(z) = \frac{1}{W^{(n)} - a}, \ G^{(n)}(z) = \frac{1}{\widehat{W}^{(n)} - a}.$$

Put $b_j = \frac{1}{a_j - a} (1 \le j \le k-1)$ and $b_k = 0$.

Since $F^{(n)}(z)$ and $G^{(n)}(z)$ partially share finite values $b_j (1 \le j \le k-1)$ IM. Thus by the above case $F^{(n)}(z) \equiv G^{(n)}(z)$. Which completes the proof of theorem. □

If $n = 0$ in Theorem 4.2, then the conditions $E(0, W) \subseteq E(0, W^{(n)})$ and $E(0, \widehat{W}) \subseteq E(0, \widehat{W}^{(n)})$ are obvious and hence in this case, Theorem 4.2 reduces as follows

Theorem 4.3. *Let $W(z)$ and $\widehat{W}(z)$ be two ν-valued and μ-valued algebroid functions on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$, respectively and $\mu \le \nu$, let a_j $(j = 1, 2, \ldots, k)$ be k distinct small functions, where $k \ge 4\nu + 1$. If $\overline{E}(a_j, W) \subseteq \overline{E}(a_j, \widehat{W})$ for all $1 \le j \le k$. If*

$$\frac{\underline{\lim}_{r\to\infty} \sum_{j=1}^{k} N_0\left(r, \frac{1}{W - a_j}\right)}{\underline{\lim}_{r\to\infty} \sum_{j=1}^{k} N_0\left(r, \frac{1}{\widehat{W} - a_j}\right)} > \frac{1}{k - (2\nu + 1)},$$

then $W(z) \equiv \widehat{W}(z)$.

If $n = 0$ and $k = 4\nu + 1$ in Theorem 4.2, then Theorem 4.2 reduces as follows

Corollary 4.1. *Let $W(z)$ and $\widehat{W}(z)$ be two ν-valued and μ-valued algebroid functions on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$, respectively and $\mu \le \nu$, let a_j $(j = 1, 2, \ldots, 4\nu + 1)$ be $4\nu + 1$ distinct small functions. If $\overline{E}(a_j, W) \subseteq \overline{E}(a_j, \widehat{W})$ for all $1 \le j \le 4\nu + 1$, and*

$$\underline{\lim}_{r\to\infty} \frac{\sum_{j=1}^{4\nu+1} \overline{N}_0\left(r, \frac{1}{W - a_j}\right)}{\sum_{j=1}^{4\nu+1} \overline{N}_0\left(r, \frac{1}{\widehat{W} - a_j}\right)} > \frac{1}{2\nu},$$

then $W(z) \equiv \widehat{W}(z)$.

Definition 4.2. Let $W(z)$ be an algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$ and $a(z)$ be a values of $W(z)$. We define

$$\overline{W}(a, W) = \{z | W(z) - a(z) = 0\}$$

in which each zero is counted only once.

We consider, two algebroid functions partially share five or more values on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$. Precisely speaking, if two algebroid functions $W(z)$ and $\widehat{W}(z)$ on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$ and k be distinct values $a_1, a_2, \ldots, a_k$, $k \geq 4\nu + 1$ such that $\overline{E}(a_j, W) \subseteq \overline{E}(a_j, \widehat{W})$, for all $1 \leq j \leq k$.

Now we can state and prove our theorem as follows

Theorem 4.4. *Let $W(z)$ and $\widehat{W}(z)$ be two ν-valued and μ-valued algebroid functions on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, respectively and $\mu \leq \nu$, let a_j $(j = 1, 2, \ldots, k)$ be k distinct values, where $k \geq 4\nu + 1$ and for a non negative integers n, if*

$$E(a_j, W^{(n)}) \subseteq E(a_j, \widehat{W}^{(n)}), \quad for all 1 \leq j \leq k,$$

$$E(0, W_1) \subseteq E(0, W^{(n)}) \ and \ E(0, \widehat{W}) \subseteq E(0, \widehat{W}^{(n)}),$$

and

$$\frac{\underline{\lim}_{r\to\infty} \sum_{j=1}^{k} N_0\left(r, \frac{1}{W^{(n)} - a_j}\right)}{\underline{\lim}_{r\to\infty} \sum_{j=1}^{k} N_0\left(r, \frac{1}{\widehat{W}^{(n)} - a_j}\right)} > \frac{n+1}{k - (n + 2\nu + 1)},$$

then $W^{(n)}(z) \equiv \widehat{W}^{(n)}(z)$.

Proof. Using a similar argument as Theorem 4.2, we can prove it. □

If $n = 0$ in Theorem 4.4, then the conditions $E(0, W) \subseteq E(0, W^{(n)})$ and $E(0, \widehat{W}) \subseteq E(0, \widehat{W}^{(n)})$ are obvious and hence in this case, Theorem 4.4 reduces as follows

Theorem 4.5. *Let $W(z)$ and $\widehat{W}(z)$ be two ν-valued and μ-valued algebroid functions on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \leq +\infty)$, respectively and $\mu \leq \nu$, let a_j $(j = 1, 2, \ldots, k)$ be k distinct values, where $k \geq 4\nu + 1$. If $\overline{E}(a_j, W) \subseteq \overline{E}(a_j, \widehat{W})$ for all $1 \leq j \leq k$. If*

$$\frac{\underline{\lim}_{r\to\infty} \sum_{j=1}^{k} N_0\left(r, \frac{1}{W - a_j}\right)}{\underline{\lim}_{r\to\infty} \sum_{j=1}^{k} N_0\left(r, \frac{1}{\widehat{W} - a_j}\right)} > \frac{1}{k - (2\nu + 1)},$$

then $W(z) \equiv \widehat{W}(z)$.

If $n = 0$ and $k = 4\nu + 1$ in Theorem 4.4, then Theorem 4.4 reduces as follows

Corollary 4.2. *Let $W(z)$ and $\widehat{W}(z)$ be two ν-valued and μ-valued algebroid functions on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$, respectively and $\mu \le \nu$, let a_j $(j = 1, 2, \dots, 4\nu+1)$ be $4\nu+1$ distinct values. If $\overline{E}(a_j, W) \subseteq \overline{E}(a_j, \widehat{W})$ for all $1 \le j \le 4\nu+1$, and*

$$\lim_{r\to\infty} \frac{\sum_{j=1}^{4\nu+1} \overline{N}_0\left(r, \frac{1}{W-a_j}\right)}{\sum_{j=1}^{4\nu+1} \overline{N}_0\left(r, \frac{1}{\widehat{W}-a_j}\right)} > \frac{1}{2\nu},$$

then $W(z) \equiv \widehat{W}(z)$.

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