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ON SOLUTIONS OF SOME NON-LINEAR DIFFERENTIAL EQUATIONS IN CONNECTION TO BRÜCK CONJECTURE

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Abstract. In this paper, we investigate on the non-constant entire solutions of some non-linear complex differential equations in connection to Brück conjecture and prove some results which improve and extend the results of Xu and Yang[**Xu HY, Yang LZ. On a conjecture of R. Brück and some linear differential equations. Springer Plus 2015; 4:748,:1-10, DOI 10.1186/s40064-015-1530-5.**]

1. Introduction

Let $f(z)$ be a nonconstant meromorphic function in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$ (e.g., see [2, 5, 12, 13]). By $S(r,f)$ we denote any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to +\infty$, possibly outside a set of r with finite linear measure. A meromorphic function $\alpha(z)$ is said to be small with respect to $f(z)$ if $T(r,\alpha) = S(r,f)$.

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions. For a small function $a(z)$ of f and g, if $f(z)-a(z)$ and $g(z)-a(z)$ have same zeros with same multiplicities, we say that $f(z)$ and $g(z)$ share the function $a(z)$ CM (counting multiplicities) and if $f(z)-a(z)$ and $g(z)-a(z)$ have same zeros with ignoring multiplicities, we say that $f(z)$ and $g(z)$ share $a(z)$ IM (ignoring multiplicities). Note that $a(z)$ can be a value in $\mathbb{C}\cup\{\infty\}$.

For an entire function $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ the central index $\nu(r,f)$ is the greatest exponent m such that $|a_m|r^m = \mu(r,f)$, where $\mu(r,f) = \max\limits_{n\geq 0}|a_n|r^n$ denote the maximum term of f on $|z| = r$. In this paper, we also need the following definitions.

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Definition 1.1. Let $f(z)$ be a nonconstant meromorphic function, then the order $\sigma(f)$ of $f(z)$ is defined by

$$\sigma(f)=\limsup_{r\to+\infty}\frac{\log T(r,f)}{\log r}=\limsup_{r\to+\infty}\frac{\log\log M(r,f)}{\log r}$$

and the lower order $\mu(f)$ of $f(z)$ is defined by

$$\mu(f)=\liminf_{r\to+\infty}\frac{\log T(r,f)}{\log r}=\liminf_{r\to+\infty}\frac{\log\log M(r,f)}{\log r}$$

where and in the sequel

$$M(r,f)=\max_{|z|=r}\left|f(z)\right|.$$

Definition 1.2. [[5]]The type $\tau(f)$ of an entire function $f(z)$ with $0<\sigma(f)=\sigma<+\infty$ is defined by

$$\tau(f)=\limsup_{r\to+\infty}\frac{\log M(r,f)}{r^{\sigma}}.$$

Following Yi an Yang [13] we define

Definition 1.3. Let f be a nonconstant meromorphic function, the hyper order $\sigma_2(f)$ of $f(z)$ is defined as follows

$$\sigma_2(f)=\limsup_{r\to+\infty}\frac{\log\log T(r,f)}{\log r}$$

and finally

Definition 1.4. Let $n_{0j},n_{1j},n_{2j},\dots,n_{kj}$ are non negative integers. The expression $M_j[f]=(f)^{n_{0j}}\left(f^{(1)}\right)^{n_{1j}}\left(f^{(2)}\right)^{n_{2j}}\cdots\left(f^{(k)}\right)^{n_{kj}}$ is called a differential monomial generated by f of degree $d(M_j)=\sum_{i=0}^{k}n_{ij}$ and weight $\Gamma_{M_j}=\sum_{i=0}^{k}(i+1)n_{ij}$. The sum $P[f]=\sum_{j=1}^{t}a_jM_j[f]$ is called a differential polynomial generated by f of degree $\overline{d}(P)=\max\{d(M_j):1\le j\le t\}$ and weight $\Gamma_P=\max\{\Gamma_{M_j}:1\le j\le t\}$, where $a_j\ne 0\,(j=1,2,\dots,t)$ and $T(r,a_j)=S(r,f)$ for $j=1,2,\dots,t$. The numbers $\underline{d}_P=\min\{d(M_j):1\le j\le t\}$ and k (the highest order of the derivative of f in $P[f]$) are called respectively the lower degree and the order of $P[f]$. $P[f]$ is said to be homogeneous if $\overline{d}_P=\underline{d}_P$. $P[f]$ is called a Linear Differential Polynomial generated by f if $\overline{d}_P=1$. Otherwise, $P[f]$ is called Non-linear Differential Polynomial. We denote by $Q=\max\{\Gamma_{M_j}-d(M_j):1\le j\le t\}=\max\left\{\sum_{i=1}^{k}i.n_{ij}:1\le j\le t\right\}$

Rubel and Yang [10] proved that if a nonconstant entire function f and its derivative f' share two distinct finite complex numbers CM, then $f=f'$. How is the relation between f and f', if an entire function f and its derivative f' share one finite complex number a CM? Brück [1] made the conjecture that if f is a nonconstant entire function satisfying $\sigma_2(f)<\infty$, where $\sigma_2(f)$ is not a positive integer and if f and f' share one finite complex number a CM, then $f' - a = c(f - a)$ for some nonzero finite complex number c. In 1998, Gundersen and Yang [4] proved that the conjecture is true for entire functions of finite order. Also, in 2008 Li and Cao[8] improved the Brück conjecture for entire function and its derivation sharing polynomials and proved the following theorem:

Theorem 1.1. *Let $\phi(z)$ be any polynomial. If f is a nonconstant entire solution of the equation $f^{(k)} - Q_1 = e^{\phi}(f - Q_2)$, where Q_1 and Q_2 are non-zero polynomials, then $\sigma_2(f)$ = degree of ϕ.*

Mao [9] improve the above theorem in which he replaced the k-th derivative $f^{(k)}$ by the linear differential polynomial $L(f) = a_k f^{(k)} + a_{k-1} f^{(k-1)} + \cdots + a_0 f$ and prove that

Theorem 1.2. *Let $a_k \equiv 0, a_{k-1}, \ldots, a_0, P(z)$ be polynomials, $k \geq 1$ and f be an entire function of order $\sigma(f) > \max\limits_{0 \leq j \leq k-1} \left\{\frac{\deg a_j - \deg a_k}{k-j}, 0\right\}$ and hyper-order $\sigma_2(f) < \frac{1}{2}$. If f and $L(f)$ share P CM, then $\frac{L(f)-P}{f-P} = c$ for a non zero constant c.*

Later in 2009 Chang and Zhu [3] proved that Brück conjecture is true if the constant a is replaced by a function $a(z)$, provided $\sigma(a) < \sigma(f)$.

Theorem 1.3. *Let f be an entire function of finite order and $a(z)$ be a function such that $\sigma(a) < \sigma(f) < +\infty$. If f and f' share $a(z)$ CM, then $\frac{f'-a}{f-a} = c$ for some constant $c \neq 0$.*

In the year 2015, Xu and Yang [11] prove the following theorems:

Theorem 1.4. *Let $f(z)$ and $\alpha(z)$ be two nonconstant entire functions and satisfy $0 < \sigma(\alpha) = \sigma(f) < +\infty, \tau(f) > \tau(\alpha)$ and let $\phi(z)$ be a polynomial such that*

$$\sigma(f) > \deg(\phi) + \max_{0 \leq j \leq k-1} \left\{\frac{\deg a_j - \deg a_k}{k-j}, 0\right\}.$$

If f is a nonconstant entire solution of the following differential equation $L(f) - \alpha(z) = (f(z) - \alpha(z))e^{\phi(z)}$, where $L(f)$ is defined as above. Then $\phi(z)$ is a constant.

Theorem 1.5. *Let $f(z)$ and $\alpha(z)$ be two nonconstant entire functions satisfying $0 < \sigma(\alpha) = \sigma(f) < +\infty, \tau(f) > \tau(\alpha)$ and let $\phi(z)$ be a polynomial such that*

$$\sigma(f) > \deg(\phi) + \max_{0 \leq j \leq k-1} \left\{\frac{\deg a_j - \deg a_k}{k-j}, 0\right\}.$$

If f is a nonconstant entire solution of the following differential equation $L_1(f) - \alpha(z) = (f(z) - \alpha(z))e^{\phi(z)}$, where $L_1(f) = L(f) + \beta(z)$ and β is an entire function satisfying $0 < \sigma(\beta) = \sigma(f) < +\infty, \tau(f) > \tau(\beta)$. Then $\phi(z)$ is a constant.

Theorem 1.6. *Let $f(z)$ and $\alpha(z)$ be two nonconstant entire functions satisfying $\sigma(\alpha) < \mu(f)$, and let $\phi(z)$ be a polynomial such that*

$$\sigma(f) > \deg(\phi) + \max_{0\le j\le k-1}\left\{\frac{\deg a_j - \deg a_k}{k-j}, 0\right\}.$$

If f is a nonconstant entire solution of the following differential equation $L_1(f) - \alpha(z) = (f(z) - \alpha(z))e^{\phi(z)}$, where $L_1(f) = L(f) + \beta(z)$ and β is an entire function satisfying $\sigma(\beta) < \mu(f)$. Then $\sigma_2(f) = \deg\phi(z)$.

In this paper, we improve and extend the results of Xu and Yang [11] in which we replaced the linear differential polynomial by the differential polynomial $P[f]$ and f by $f^{\overline{d}_P}$ and proved the following theorems:

Theorem 1.7. *Let $f(z)$ and $\alpha(z)$ be two nonconstant entire functions and satisfy $0 < \sigma(\alpha) = \sigma(f) < +\infty$ and $\tau(f) > \tau(\alpha)$. Also, let $\phi(z)$ be a polynomial. If f is a nonconstant entire solution of the following differential equation*

$$P[f] - \alpha(z) = \left(f^{\overline{d}_P} - \alpha(z)\right)e^{\phi(z)}, \tag{1.1}$$

then $\phi(z)$ is a constant.

Theorem 1.8. *Let $f(z)$ and $\alpha(z)$ be two nonconstant entire functions and satisfy $0 < \sigma(\alpha) = \sigma(f) < +\infty$ and $\tau(f) > \tau(\alpha)$. Also, let $\phi(z)$ be a polynomial. If f is a nonconstant entire solution of the following differential equation*

$$P[f] + \beta(z) - \alpha(z) = \left(f^{\overline{d}_P} - \alpha(z)\right)e^{\phi(z)}, \tag{1.2}$$

where $\beta(z)$ is an entire function satisfying $0 < \sigma(\beta) = \sigma(f) < +\infty$ and $\tau(f) > \tau(\beta)$. Then $\phi(z)$ is a constant.

Theorem 1.9. *Let $f(z)$ and $\alpha(z)$ be two nonconstant entire functions satisfying $\sigma(\alpha) < \mu(f)$ and $\phi(z)$ be a polynomial. If f is a nonconstant entire solution of the following differential equation*

$$P[f] + \beta(z) - \alpha(z) = \left(f^{\overline{d}_P} - \alpha(z)\right)e^{\phi(z)}, \tag{1.3}$$

where $\beta(z)$ is an entire function satisfying $\sigma(\beta) < \mu(f)$. Then $\sigma_2(f) = \deg\phi$.

2. Preparatory lemmas

In this section we state some lemmas needed in the sequel.

Lemma 2.1 ([7]). *Let $f(z)$ be a transcendental entire function, $\nu(r,f)$ be the central index of $f(z)$. Then there exists a set $E\subset(1,+\infty)$ with finite logarithmic measure, we choose z satisfying $|z|=r\notin[0,1]\cup E$ and $|f(z)|=M(r,f)$, we get*

$$\frac{f^{j}(z)}{f(z)}=\left\{\frac{\nu(r,f)}{z}\right\}^{j}(1+o(1)),\ \textit{for } j\in N.$$

Lemma 2.2 ([6]). *Let $f(z)$ be an entire function of finite order $\sigma(f)=\sigma<+\infty$ and let $\nu(r,f)$ be the central index of f. Then*

$$\limsup_{r\to+\infty}\frac{\log\nu(r,f)}{\log r}=\sigma(f)$$

and if f is a transcendental entire function of hyper order $\sigma_2(f)$, then

$$\limsup_{r\to+\infty}\frac{\log\log\nu(r,f)}{\log r}=\sigma_2(f).$$

Lemma 2.3 ([9]). *Let $f(z)$ be a transcendental entire function and let $E\subset[1,+\infty)$ be a set having finite logarithmic measure. Then there exists $\{z_n=r_ne^{i\theta_n}\}$ such that $|f(z_n)|=M(r_n,f)$, $\theta_n\in[0,2\pi)$, $\lim_{n\to+\infty}\theta_n=\theta_0\in[0,2\pi)$, $r_n\notin E$ and if $0<\sigma(f)<+\infty$, then for any given $\varepsilon>0$ and sufficiently large r_n,*

$$r_n^{\sigma(f)-\varepsilon}<\nu(r_n,f)<r_n^{\sigma(f)+\varepsilon}.$$

If $\sigma(f)=+\infty$, then for any given large $K>0$ and sufficiently large r_n,

$$\nu(r_n,f)>r_n^{K}.$$

Lemma 2.4 ([7]). *Let $\phi(z)=b_n\,z^n+b_{n-1}\,z^{n-1}+\cdots+b_0$ with $b_n\neq0$ be a polynomial. Then for every $\varepsilon>0$, there exists $r_0>0$ such that for all $r=|z|>r_0$ the inequalities*

$$(1-\varepsilon)\,|b_n|\,r^n\le|\phi(z)|\le(1+\varepsilon)\,|b_n|\,r^n$$

hold.

Lemma 2.5 ([11]). *Let $f(z)$ and $A(z)$ be two entire functions with $0<\sigma(f)=\sigma(A)=\sigma<+\infty$, $0<\tau(A)=\tau(f)<+\infty$, then there exists a set $E\subset[1,+\infty)$ that has infinite logarithmic measure such that for all $r\in E$ and a positive number κ, we have*

$$\frac{M(r,A)}{M(r,f)}<\exp\{-\kappa r^{\sigma}\}.$$

3. Proofs of the main results

Proof of Theorem 1.7. Suppose that $\deg\phi=m\ge1$. Let

$$\phi(z)=b_mz^m+b_{m-1}z^{m-1}+\cdots+b_0,$$

where $b_m, b_{m-1}, \ldots, b_0$ are constants and $b_m \neq 0$. Thus, it follows from (1.1) and Lemma 2.4 that

$$|b_m| r^m (1+o(1)) = |\phi(z)| = \left| \log \frac{\frac{P[f]}{f^{\overline{d}_P}} - \frac{\alpha}{f^{\overline{d}_P}}}{1 - \frac{\alpha}{f^{\overline{d}_P}}} \right| = \left| \log \frac{\frac{P[f]}{f^{\overline{d}_P}} - \frac{\alpha}{f} \cdot \frac{1}{f^{\overline{d}_P - 1}}}{1 - \frac{\alpha}{f} \cdot \frac{1}{f^{\overline{d}_P - 1}}} \right| \tag{3.1}$$

Since for each $j = 1, 2, \ldots, t$,

$$M_j[f] = (f)^{n_{0j}} \left(f^{(1)}\right)^{n_{1j}} \left(f^{(2)}\right)^{n_2 j} \cdots \left(f^{(k)}\right)^{n_{kj}} = f^{\left(\sum_{i=0}^{k} n_{ij}\right)} \prod_{i=1}^{k} \left(\frac{f^{(i)}}{f}\right)^{n_{ij}} = f^{d_{M_j}} \prod_{i=1}^{k} \left(\frac{f^{(i)}}{f}\right)^{n_{ij}}$$

and from Lemma 2.1, there exists a subset $E_1 \subset (1, +\infty)$ with finite logarithmic measure, such that for some point $|z| = re^{i\theta}$ $(\theta \in [0, 2\pi])$, $r \notin E_1$ and $M(r, f) = |f(z)|$, we have

$$\frac{f^i(z)}{f(z)} = \left\{\frac{\nu(r,f)}{z}\right\}^i (1+o(1)), 1 \leq i \leq k.$$

Thus, it follows that

$$\frac{M_j[f]}{f^{d_{M_j}}} = \prod_{i=1}^{k} \left\{\frac{\nu(r,f)}{z}\right\}^{i.n_{ij}} (1+o(1)) = \left\{\frac{\nu(r,f)}{z}\right\}^{\left(\sum_{i=1}^{k} i.n_{ij}\right)} (1+o(1))$$

and

$$\frac{P[f]}{f^{\overline{d}_P}} \leq \sum_{j=1}^{t} |a_j| \left|\frac{M_j[f]}{f^{\overline{d}_P}}\right| \leq \sum_{j=1}^{t} |a_j| \left|\frac{M_j[f]}{f^{d_{M_j}}}\right| \leq \sum_{j=1}^{t} |a_j| \left|\frac{\nu(r,f)}{z}\right|^{\left(\sum_{i=1}^{k} i.n_{ij}\right)} (1+o(1)) \leq \sum_{j=1}^{t} |a_j| \left|\frac{\nu(r,f)}{z}\right|^{Q} (1+o(1))$$

$$\leq \left|\frac{\nu(r,f)}{z}\right|^{Q} . \left(\sum_{j=1}^{t} |a_j|\right) . (1+o(1)) . \tag{3.2}$$

From Lemma 2.3 there exists $\{z_n = r_n e^{i\theta_n}\}$ such that $|f(z_n)| = M(r_n, f)$, $\theta_n \in [0, 2\pi]$, $\lim_{n\to\infty} \theta_n = \theta_0 \in [0, 2\pi]$, $r_n \notin E_1$, then for any given $\varepsilon > 0$ and sufficiently large r_n,

$$r_n^{\sigma(f)-\varepsilon} < \nu(r_n, f) < r_n^{\sigma(f)+\varepsilon}. \tag{3.3}$$

Then, from (3.2) and (3.3) we have

$$\begin{aligned}\frac{P[f]}{f^{\bar{d}_P}} &\leq \left\{\frac{\nu(r_n,f)}{r_n}\right\}^{Q} . \left(\sum_{j=1}^{t} |a_j|\right) . (1+o(1)) \\ &< r_n^{(\sigma(f)+\varepsilon-1)Q} . \left(\sum_{j=1}^{t} |a_j|\right) . (1+o(1)) .\end{aligned} \tag{3.4}$$

Since $0 < \sigma(\alpha) = \sigma(f) < +\infty$ and $\tau(f) > \tau(\alpha)$, using Lemma 2.5, there exists a set $E \subset [1, +\infty)$ that has infinite logarithmic measure such that for a sequence $\{r_n\}_{n=1}^{\infty} \in E_2 = E - E_1$, we have

$$\frac{M(r_n,\alpha)}{M(r_n,f)} < \exp\left\{-\kappa r_n^{\sigma(f)}\right\} \to 0 \text{ as } n \to +\infty. \tag{3.5}$$

From (3.1), (3.4) and (3.5) and Lemma 2.2, we get that

$$|b_m|\, r^m (1+o(1)) = |\phi(z)| = O(\log r_n),$$

which is impossible. Thus, $\phi(z)$ is not a polynomial, that is, $\phi(z)$ is a constant.

Proof of Theorem 1.8. Rewritting (1.2) as

$$\frac{\frac{P[f]}{f^{\bar{d}_P}} + \frac{\beta}{f}\cdot\frac{1}{f^{\bar{d}_P-1}} - \frac{\alpha}{f}\cdot\frac{1}{f^{\bar{d}_P-1}}}{1 - \frac{\alpha}{f}\cdot\frac{1}{f^{\bar{d}_P-1}}} = e^{\phi(z)}.$$

Our assumptions on τ and σ values give, using Lemma 2.5, that there exists a set $E \subset [1, +\infty)$ that has infinite logarithmic measure such that for a sequence $\{r_n\}_{n=1}^{\infty} \in E_3 = E - E_1$, we have

$$\frac{M(r_n,\alpha)}{M(r_n,f)} < \exp\left\{-\kappa r_n^{\sigma(f)}\right\} \to 0 \text{ as } n \to +\infty$$

and

$$\frac{M(r_n,\beta)}{M(r_n,f)} < \exp\left\{-\kappa r_n^{\sigma(f)}\right\} \to 0 \text{ as } n \to +\infty.$$

Proceeding as in the proof of Theorem 1.7 we can show that $\phi(z)$ is a constant.

Proof of Theorem 1.9. We will consider two cases (i) $\sigma(f) < +\infty$ and (ii) $\sigma(f) = +\infty$.

Case (i). Suppose that $\sigma(f) < +\infty$. Then $\sigma_2(f) = 0$. Since $\sigma(\alpha) < \mu(f)$ and $\sigma(\beta) < \mu(f)$, from definitions of the order and the lower order, there exists infinite sequence $\{z_n\}_{n=1}^{\infty}$, we have

$$\frac{|\alpha(z_n)|}{|f(z_n)|} \to 0 \text{ and } \frac{|\beta(z_n)|}{|f(z_n)|} \to 0 \text{ as } n \to \infty.$$

Thus, by using the same argument as in Theorem 1.7, we get that $\phi(z)$ is a constant, that is, $\deg\phi = 0$. Therefore, $\sigma_2(f) = \deg\phi$.

Case (ii). Suppose that $\sigma(f) = +\infty$.

Rewritting (1.3), we have

$$\frac{\frac{P[f]}{f^{\overline{d}_P}} + \frac{\beta}{f} \cdot \frac{1}{f^{\overline{d}_P - 1}} - \frac{\alpha}{f} \cdot \frac{1}{f^{\overline{d}_P - 1}}}{1 - \frac{\alpha}{f} \cdot \frac{1}{f^{\overline{d}_P - 1}}} = e^{\phi(z)}.$$

Since for each $j = 1, 2, \ldots, t$,

$$\begin{aligned} M_j[f] &= (f)^{n_{0j}} \left(f^{(1)}\right)^{n_{1j}} \left(f^{(2)}\right)^{n_{2j}} \cdots \left(f^{(k)}\right)^{n_{kj}} \\ &= f^{\left(\sum_{i=0}^{k} n_{ij}\right)} \prod_{i=1}^{k} \left(\frac{f^{(i)}}{f}\right)^{n_{ij}} \\ &= f^{d_{M_j}} \prod_{i=1}^{k} \left(\frac{f^{(i)}}{f}\right)^{n_{ij}} \end{aligned}$$

and from Lemma 2.1, there exists a subset $E_4 \subset [1, +\infty)$ with finite logarithmic measure, we choose z satisfying $|z| = r \notin [0,1] \cup E_4$ and $|f(z)| = M(r, f)$, we have

$$\frac{f^i(z)}{f(z)} = \left\{\frac{\nu(r,f)}{z}\right\}^i (1 + o(1)),\ 1 \le i \le k.$$

Thus, it follows that

$$\frac{M_j[f]}{f^{d_{M_j}}} = \prod_{i=1}^{k} \left\{\frac{\nu(r,f)}{z}\right\}^{i.n_{ij}} (1 + o(1)) = \left\{\frac{\nu(r,f)}{z}\right\}^{\left(\sum_{i=1}^{k} i.n_{ij}\right)} (1 + o(1))$$

and

$$\begin{aligned} \frac{P[f]}{f^{\overline{d}_P}} &\le \sum_{j=1}^{t} |a_j| \left|\frac{M_j[f]}{f^{\overline{d}_P}}\right| \\ &\le \sum_{j=1}^{t} |a_j| \left|\frac{M_j[f]}{f^{d_{M_j}}}\right| \end{aligned}$$

$$\leq \sum_{j=1}^{t} |a_j| \left|\frac{\nu(r,f)}{z}\right|^{\left(\sum_{i=1}^{k} i.n_{ij}\right)} (1+o(1))$$

$$\leq \sum_{j=1}^{t} |a_j| \left|\frac{\nu(r,f)}{z}\right|^{Q} (1+o(1))$$

$$\leq \left|\frac{\nu(r,f)}{z}\right|^{Q} . \left(\sum_{j=1}^{t} |a_j|\right) . (1+o(1)) . \tag{3.6}$$

Since $\sigma(f) = +\infty$, then it follows from Lemma 2.3 that there exists $\{z_n = r_n e^{i\theta_n}\}$ with $|f(z_n)| = M(r_n, f)$, $\theta_n \in [0, 2\pi]$, $\lim_{n\to\infty} \theta_n = \theta_0 \in [0, 2\pi]$, $r_n \notin E_5 \subset [1, +\infty)$, such that for any large constant K and for sufficiently large r_n, we have

$$\nu(r_n, f) \geq r_n^K. \tag{3.7}$$

Since $\sigma(\alpha) < \mu(f)$ and $\sigma(\beta) < \mu(f)$, from definitions of order and lower order, there exists infinite sequence $\{z_n\}_{n=1}^{\infty}$, we have

$$\frac{|\alpha(z_n)|}{|f(z_n)|} \to 0 \text{ and } \frac{|\beta(z_n)|}{|f(z_n)|} \to 0 \text{ as } n \to \infty. \tag{3.8}$$

Thus, it follows from (1.3), (3.6), (3.7) and (3.8) that

$$e^{\phi(z_n)} \leq \left\{\frac{\nu(r_n, f)}{r_n}\right\}^{Q} . \left(\sum_{j=1}^{t} |a_j|\right) . (1+o(1)) . \tag{3.9}$$

Let

$$\phi(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0,$$

where $b_m, b_{m-1}, \ldots, b_0$ are constants and $b_m \neq 0, m \geq 1$. From Lemma 2.4, there exists sufficiently large positive number r_0 and $n_0 \in N_+$, such that for sufficiently large positive integer $n > n_0$ satisfying $|z_n| = r_n > r_0$, we have for every $\varepsilon' > 0$

$$\log|b_m| + m\log|z_n| + \log\left|1-\varepsilon'\right| \leq \log\left|\phi(z_n)\right| \leq \left|\log\log e^{\phi(z_n)}\right|. \tag{3.10}$$

It follows from (3.9) that

$$\left|\log\log e^{\phi(z_n)}\right| \leq \log\log\left(\sum_{j=1}^{t} |a_j|\right) + \log\log \nu(r_n, f) + \log\log r_n + O(1)$$

$$\leq \log\log \nu(r_n, f) + O(\log\log r_n). \tag{3.11}$$

Thus, we have from (3.10) and (3.11) and Lemma 2.2 that

$$m = \deg\phi \leq \sigma_2(f). \tag{3.12}$$

Also, it follows from (3.9) and Lemma 2.4 that

$$M(r_n, e^{\phi(z_n)}) \geq \left\{\frac{\nu(r_n, f)}{r_n}\right\}^Q . \left(\sum_{j=1}^{t} |a_j|\right).$$

Then, we have

$$\{\nu(r_n, f)\}^Q \leq \left(\sum_{j=1}^{t} |a_j|\right)^{-1} . (r_n)^Q . M(r_n, e^{\phi(z_n)}). \tag{3.13}$$

Thus, it follows from (3.13) and Lemma 2.2 that

$$\begin{aligned}\sigma_2(f) &= \limsup_{r_n \to +\infty} \frac{\log\log \nu(r_n, f)}{\log r_n} \\ &= \limsup_{r_n \to +\infty} \frac{\log\log (\nu(r_n, f))^Q}{\log r_n} \\ &\leq \limsup_{r_n \to +\infty} \frac{\log\log \left(\left(\sum_{j=1}^{t} |a_j|\right)^{-1} . (r_n)^Q . M(r_n, e^{\phi(z_n)})\right)}{\log r_n} \\ &= \sigma\left(e^{\phi}\right). \end{aligned} \tag{3.14}$$

Since $\phi(z)$ is a polynomial, then $\sigma\left(e^{\phi}\right) = \deg\phi = m$. By combining (3.12) and (3.14), we have $\sigma_2(f) = \deg\phi$.

Corollary 3.1. *Let $f(z)$ and $\alpha(z)$ be two nonconstant entire functions and satisfy $\sigma(\alpha) < \mu(f)$. Also, let $\phi(z)$ be a polynomial. If f is a nonconstant entire solution of the following differential equation*

$$P[f] - \alpha(z) = (f^{\overline{d}_P} - \alpha(z))e^{\phi(z)},$$

then $\sigma_2(f) = \deg\phi$.

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