INVERSE PROBLEMS FOR DIFFERENTIAL OPERATORS WITH NONSEPARATED BOUNDARY CONDITIONS IN THE CENTRAL SYMMETRIC CASE

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Abstract. Inverse spectral problems for Sturm-Liouville operators on a finite interval with non-separated boundary conditions are studied in the central symmetric case, when the potential is symmetric with respect to the middle of the interval. We discuss statements of the problems, provide algorithms for their solutions along with necessary and sufficient conditions for the solvability of the inverse problems considered.

1. Introduction

We study inverse spectral problems for the Sturm-Liouville operator

$$\ell y := y'' + q(x)y, \; x \in (0, \pi),$$

on the finite interval $(0, \pi)$ with non-separated boundary conditions. Inverse problems consist in recovering coefficients of differential operators from their spectral characteristics. Such problems often appear in mathematics, mechanics, physics, geophysics, electronics and other branches of natural sciences and engineering. Inverse problems also play an important role in solving nonlinear evolution equations in mathematical physics. Inverse problems for differential operators with separated boundary conditions have been studied fairly completely by many authors (see the monographs [1]−[5] and the references therein). Inverse problems for Sturm-Liouville operators with non-separated boundary conditions, which are more difficult for the investigation, were treated in [6]−[17] and other works. In particular, the periodic boundary value problem was considered in [6, 7, 9, 14]. Stankevich [6] suggested a statement of the inverse problem and proved the corresponding uniqueness theorem. Marchenko and Ostrovskii [7] gave the characterization of the spectrum for the periodic boundary value problem in terms of a special conformal mapping. Conditions considered in [7] are difficult

to verify. Another method, used in [9], allowed to obtain necessary and sufficient conditions for the solvability of the inverse problem for the periodic case that are easier to verify. Similar results were obtained in [9] for another type of boundary conditions, namely

$$y'(0) - a y(0) + b y(\pi) = y'(\pi) + d y(\pi) - b y(0) = 0.$$

Later analogous results were established in [12, 13].

In this paper we study the case when the potential q is symmetric with respect to the middle of the interval, i.e. $q(x) = q(\pi - x)$ a.e. on $(0, \pi)$. The symmetric case requires non-trivial modifications in the method and allows us to specify less spectral information than in the general case. Some results for the symmetric case were obtained in [10] and [17]. In the present paper for the symmetric case we construct the solution of the inverse spectral problem and give the characterization of the spectrum for various non-separated boundary conditions. For convenience of readers in Section 2 we describe briefly the known results for the general (non-symmetric) case.

2. Periodic boundary value problem

Consider the differential equation

$$-y'' + q(x) y = \lambda y, \quad x \in (0, \pi), \tag{1}$$

where λ is the spectral parameter, and $q(x) \in L_2(0, T)$ is a real-valued function. The function $q(x)$ is called the potential. Let $C(x, \lambda), S(x, \lambda)$ and $\psi(x, \lambda)$ be solutions of Eq. (1) with the initial conditions $C(0, \lambda) = S'(0, \lambda) = -\psi'(\pi, \lambda) = 1, C'(0, \lambda) = S(0, \lambda) = \psi(\pi, \lambda) = 0$. For each fixed x, the functions $C^{(v)}(x, \lambda), S^{(v)}(x, \lambda)$ and $\psi^{(v)}(x, \lambda), v = 0, 1$, are entire in λ of order 1/2. Moreover,

$$\langle C(x, \lambda), S(x, \lambda) \rangle \equiv 1, \tag{2}$$

where $\langle y, z \rangle := y z' - y' z$ is the Wronskian of y and z. Denote

$$\Delta(\lambda) = (C(\pi, \lambda) + S'(\pi, \lambda))/2, \ \delta(\lambda) = (C(\pi, \lambda) - S'(\pi, \lambda))/2, \ p(\lambda) = 1 - \Delta(\lambda).$$

Zeros $\Lambda = \{\lambda_n\}_{n \geq 0}$ of the entire function $p(\lambda)$ coincide with the eigenvalues of the boundary value problem (BVP) $L = L(q)$ for Eq. (1) with periodic boundary conditions

$$y(0) - y(\pi) = y'(0) - y'(\pi) = 0.$$

The function $p(\lambda)$ is called the characteristic function for L. For convenience of readers we describe briefly the well-known results related to the BVP L (see [6, 7, 9] for details).

(1) All eigenvalues λ_n are real, and

$$\lambda_0 < \lambda_1 \le \lambda_2 < \lambda_3 \le \lambda_4 < \dots, \tag{3}$$

$$\lambda_{2n} = (2n)^2 + \alpha + \kappa_{2n}, \ \lambda_{2n-1} = (2n)^2 + \alpha + \kappa_{2n-1}, \quad \{\kappa_n\} \in l_2, \tag{4}$$

where $\alpha = \frac{1}{\pi} \int_0^\pi q(t) \, dt$. Here and everywhere below one and the same symbol $\{\kappa_n\}$ denotes various sequences from l_2. The specification of Λ uniquely determines the characteristic function $p(\lambda)$ by the formula

$$p(\lambda) = \frac{\pi^2}{2} (\lambda - \lambda_0) \prod_{n=1}^\infty \frac{\lambda_{2n} - \lambda}{(2n)^2} \prod_{n=1}^\infty \frac{\lambda_{2n-1} - \lambda}{(2n)^2}. \tag{5}$$

Moreover,

$$\max_{\lambda \in [\lambda_{2n}, \lambda_{2n+1}]} p(\lambda) \ge 2, \quad n \ge 0. \tag{6}$$

(2) Let $\Lambda^+ = \{\lambda_n^+\}_{n \ge 1}$ be zeros of the entire function $p^+(\lambda) := p(\lambda) - 2$. Then $\{\lambda_n^+\}_{n \ge 0}$ are real and

$$\lambda_0 < \lambda_1^+ \le \lambda_2^+ < \lambda_1 \le \lambda_2 < \lambda_3^+ \le \lambda_4^+ < \lambda_3 \le \lambda_4 \dots, \tag{7}$$

$$\lambda_{2n}^+ = (2n-1)^2 + \alpha + \kappa_{2n}, \ \lambda_{2n-1}^+ = (2n-1)^2 + \alpha + \kappa_{2n-1}, \quad \{\kappa_n\} \in l_2. \tag{8}$$

Denote $a_{2n} = [\lambda_{2n-1}, \lambda_{2n}]$, $a_{2n-1} = [\lambda_{2n-1}^+, \lambda_{2n}^+]$, $n \ge 1$. Segments a_n are called the gaps.

(3) Denote $d(\lambda) := \langle \psi(x, \lambda), S(x, \lambda) \rangle = S(\pi, \lambda) = \psi(0, \lambda)$. Then zeros $\gamma = \{\gamma_n\}_{n \ge 1}$ of the entire function $d(\lambda)$ coincide with the eigenvalues of the BVP $L_0 = L_0(q)$ for Eq. (1) with Dirichlet boundary conditions $y(0) = y(\pi) = 0$. The numbers γ_n are real, $\gamma_n \in a_n$, and

$$\gamma_1 < \gamma_2 < \gamma_3 < \cdots; \quad \gamma_n = n^2 + \alpha + \kappa_n, \ \{\kappa_n\} \in l_2. \tag{9}$$

The specification of γ uniquely determines the characteristic function $d(\lambda)$ of L_0 by the formula

$$d(\lambda) = \pi \prod_{n=1}^\infty \frac{\gamma_n - \lambda}{n^2}. \tag{10}$$

The numbers $\alpha_n := \int_0^\pi S^2(x, \gamma_n) \, dx$ are called the weight numbers, and numbers $\{\gamma_n, \alpha_n\}_{n \ge 1}$ are called the spectral data for the BVP L_0. One has

$$\alpha_n = \dot{d}(\gamma_n) S'(\pi, \gamma_n), \quad \dot{d}(\lambda) := \frac{d}{d\lambda} d(\lambda), \tag{11}$$

$$\alpha_n > 0; \quad \alpha_n = \frac{\pi}{2n^2} \left(1 + \frac{\kappa_n}{n}\right), \quad \{\kappa_n\} \in l_2, \tag{12}$$

$$\dot{d}(\gamma_n) = \frac{(-1)^n \pi}{2n^2} \left(1 + \frac{\kappa_n}{n}\right), \quad \{\kappa_n\} \in l_2, \qquad \text{sign} \, \dot{d}(\gamma_n) = (-1)^n. \tag{13}$$

The functions $S(x, \gamma_n)$ and $\psi(x, \gamma_n)$ are eigenfunctions for L_0, and

$$\psi(x, \gamma_n) = \beta_n S(x, \gamma_n), \quad \beta_n \ne 0. \tag{14}$$

Lemma 1. *The following relation holds*

$$\alpha_n \beta_n = -\dot{d}(\gamma_n). \tag{15}$$

Proof. Since

$$-\psi''(x,\lambda) + q(x)\psi(x,\lambda) = \lambda\psi(x,\lambda), \ -S''(x,\gamma_n) + q(x)S(x,\gamma_n) = \gamma_n S(x,\gamma_n),$$

we get

$$\frac{d}{dx}\langle\psi(x,\lambda), S(x,\gamma_n)\rangle = (\lambda - \gamma_n)\psi(x,\lambda)S(x,\gamma_n),$$

and hence,

$$(\lambda - \gamma_n)\int_0^\pi \psi(x,\lambda)S(x,\gamma_n)\,dx = \langle\psi(x,\lambda), S(x,\gamma_n)\rangle\Big|_0^\pi = -d(\lambda).$$

For $\lambda \to \lambda_n$, this yields

$$\int_0^\pi \psi(x,\gamma_n)S(x,\gamma_n)\,dx = -\dot{d}(\gamma_n).$$

Using (14) we arrive at (15). $\square$

The inverse problem for the BVP L_0 is formulated as follows.

Inverse problem 1. Given the spectral data $\{\gamma_n, \alpha_n\}_{n\geq 1}$, construct the potential q.

This inverse problem is related to the case of separated boundary conditions. It is known that the specification of the spectral data $\{\gamma_n, \alpha_n\}_{n\geq 1}$ uniquely determines the potential q. The global solution of Inverse problem 1 can be constructed by the transformation operator method or by the method of spectral mappings (see [1]−[5] for details). In particular, these methods allow one to describe necessary and sufficient conditions for the solvability of Inverse problem 1 which are presented in the next theorem.

Theorem 1. *For real numbers $\{\gamma_n, \alpha_n\}_{n\geq 1}$ to be the spectral data for a certain BVP L_0 with a real potential $q(x) \in L_2(0,\pi)$, it is necessary and sufficient that* (9) *and* (12) *hold.*

Let us now return to the periodic BVP L. It follows from (2) that

$$\Delta^2(\lambda) - \delta^2(\lambda) - d(\lambda)d_1(\lambda) \equiv 1, \tag{16}$$

where $d_1(\lambda) := C'(\pi,\lambda)$. In particular, (16) yields

$$\delta^2(\gamma_n) = \Delta^2(\gamma_n) - 1. \tag{17}$$

Denote $\Omega = \{\omega_n\}_{n\geq 1}$, $\omega_n = \text{sign}\,\delta(\gamma_n)$. The sequence Ω is called the Ω- sequence for q. In view of (17) one has

$$\delta(\gamma_n) = \omega_n(\Delta^2(\gamma_n) - 1)^{1/2}, \tag{18}$$

Since $S'(\pi, \gamma_n) = \Delta(\gamma_n) - \delta(\gamma_n)$, it follows from (11) and (18) that

$$\alpha_n = \dot{d}(\gamma_n)(\Delta(\gamma_n) - \omega_n(\Delta^2(\gamma_n) - 1)^{1/2}). \tag{19}$$

The inverse problem for the periodic case is formulated as follows [6].

Inverse problem 2. Given Λ, γ and Ω, construct the potential q.

This inverse problem was studied in [6, 7, 9, 14] and other works. It was proved in [6] that the specification of Λ, γ and Ω uniquely determines the potential q. In order to construct q one can calculate the functions $p(\lambda)$ and $d(\lambda)$ according to (5) and (10), and construct $\{\alpha_n\}_{n \geq 1}$ via (19), where $\Delta(\lambda) = 1 - p(\lambda)$. Then using data $\{\gamma_n, \alpha_n\}_{n \geq 1}$, we can construct the potential q by solving Inverse problem 1.

Lemma 2. *Fix $n \geq 1$. Relation $\delta(\gamma_n) = 0$ holds iff γ_n lies at one of the endpoints of the gap a_n.*

Indeed, in view of (17), $\delta(\gamma_n) = 0$, iff $\Delta(\gamma_n) = \pm 1$, i.e. γ_n lies at one of the endpoints of the gap a_n.

Denote by J the set of sequences $\Omega = \{\omega_n\}_{n \geq 1}$ such that $\omega_n = 0$ if γ_n lies at one of the endpoints of the gap a_n, and $\omega_n = \pm 1$, otherwise. Clearly, if Ω is the Ω- sequence for L, then $\Omega \in J$. The next theorem [9] establishes necessary and sufficient conditions for the solvability of Inverse problem 2.

Theorem 2 ([9]). *Let real numbers $\Lambda = \{\lambda_n\}_{n \geq 0}$ satisfying (3)−(4) be given. The sequence Λ is the spectrum for a certain BVP L with a real potential $q(x) \in L_2(0, \pi)$, iff relation (6) holds, where $p(\lambda)$ is constructed via (5). Moreover, if additionally we have a sequence $\gamma = \{\gamma_n\}_{n \geq 1}$, $\gamma_n \in a_n$, satisfying (9), where $\Lambda^+ = \{\lambda_n^+\}_{n \geq 1}$ are zeros of $p^+(\lambda) = p(\lambda) - 2$, and a sequence $\Omega = \{\omega_n\}_{n \geq 1} \in J$, then there exists a unique real function $q(x) \in L_2(0, \pi)$ such that Λ and γ are the spectra of L and L_0, respectively, and Ω is the Ω- sequence for L.*

The next theorem [9] shows that one of the endpoints of each gap can be chosen arbitrary taking only asymptotics into account.

Theorem 3 ([9]). *Let real numbers θ_n of the form $\theta_n = n^2 + \alpha + \kappa_n$, $\{\kappa_n\} \in l_2$, $\theta_n < \theta_{n+1}$, be given. Then there exists a real function $q(x) \in L_2(0, \pi)$ (not unique!) such that for this potential the number θ_n lies at one of the endpoints of the gap a_n for all $n \geq 1$.*

3. Central symmetric case.

In this section we consider the case when the potential q is symmetric with respect to the middle of the interval, i.e. with respect to the replacement $x \to \pi - x$. We will say that $q(x) \in L_2'(0, \pi)$ if $q(x) \in L_2(0, \pi)$ and $q(x) = q(\pi - x)$ a.e. on $(0, \pi)$.

Theorem 4. $q(x) \in L'_2(0, \pi)$ *iff* $\beta_n = (-1)^{n-1}$, $n \geq 1$.

Proof.

(1) Let $q(x) \in L'_2(0, \pi)$. Then $\psi(x, \lambda) \equiv S(\pi - x, \lambda)$. Using (14) we calculate

$$\psi(x, \gamma_n) = \beta_n S(x, \gamma_n) = \beta_n \psi(\pi - x, \gamma_n) = \beta_n^2 S(\pi - x, \gamma_n) = \beta_n^2 \psi(x, \gamma_n).$$

Hence, $\beta_n^2 = 1$. On the other hand, it follows from (14) that $\beta_n S'(\pi, \gamma_n) = -1$. Using Sturm's oscillation theorem we conclude that $\beta_n = (-1)^{n-1}$, $n \geq 1$.

(2) Let $\beta_n = (-1)^{n-1}$, $n \geq 1$. Denote $\tilde{q}(x) := q(\pi - x)$. We agree that here and below, if a certain symbol θ denotes an object related to q, then $\tilde{\theta}$ will denote the analogous object related to $\tilde{q}$.

Obviously, $\tilde{\psi}(x, \lambda) \equiv S(\pi - x, \lambda)$, $\tilde{S}(x, \lambda) \equiv \psi(\pi - x, \lambda)$, and consequently, $d(\lambda) \equiv \tilde{d}(\lambda)$ and $\gamma_n = \tilde{\gamma}_n$, $n \geq 1$. Since $\beta_n = (-1)^{n-1}$, it follows from (14) that $\psi(x, \gamma_n) = (-1)^{n-1} S(x, \gamma_n)$. Moreover, according to (14), $\tilde{\psi}(x, \gamma_n) = \tilde{\beta}_n \tilde{S}(x, \gamma_n)$, hence $S(\pi - x, \gamma_n) = \tilde{\beta}_n \psi(\pi - x, \gamma_n)$, i.e. $\tilde{\beta}_n = (\beta_n)^{-1} = (-1)^{n-1}$. Thus, $\beta_n = \tilde{\beta}_n$ for all $n \geq 1$. Taking (15) into account we conclude that $\alpha_n = \tilde{\alpha}_n$ for all $n \geq 1$. Since the specification of the spectral data $\{\gamma_n, \alpha_n\}_{n \geq 1}$ uniquely determines the potential, we obtain that $q(x) = \tilde{q}(x)$ a.e. on $(0, \pi)$, i.e. $q(x) \in L'_2(0, \pi)$.   □

Let us consider the inverse problem for the BVP L_0. In the central symmetric case $q(x) \in L'_2(0, \pi)$ we do not need to specify the weight numbers $\{\alpha_n\}_{n \geq 1}$; it is sufficient to specify only the spectrum γ.

Inverse problem 3. Given the spectrum $\gamma = \{\gamma_n\}_{n \geq 1}$, construct the potential q.

It is known [1]−[5] that for the central symmetric case the specification of the spectrum $\gamma = \{\gamma_n\}_{n \geq 1}$ of the BVP L_0 uniquely determines the potential q. In order to construct q, one can calculate $d(\lambda)$ via (10) and the weight numbers $\alpha_n = (-1)^n \dot{d}(\gamma_n)$, and then find q by solving Inverse problem 1. Moreover, the characterization of the spectrum of L_0 is given by the following assertion.

Theorem 5. *For real numbers $\{\gamma_n\}_{n \geq 1}$ to be the spectrum of a BVP L_0 with a real potential $q(x) \in L'_2(0, \pi)$, it is necessary and sufficient that (9) holds.*

Proof. The necessity is obvious. We will prove the sufficiency. Let real numbers $\{\gamma_n\}_{n \geq 1}$ satisfying (9) be given. We construct $d(\lambda)$ via (10) and the numbers $\{\alpha_n\}_{n \geq 1}$ by $\alpha_n = (-1)^n \dot{d}(\gamma_n)$. Our plan is to use Theorem 1. For this purpose we should obtain the asymptotics for the numbers α_n. This seems to be difficult because the function $d(\lambda)$ is by construction the infinite product. But fortunately, for calculating the asymptotics of α_n one can also use

Theorem 1, as an auxiliary assertion. Indeed, by virtue of Theorem 1 there exists a potential $\tilde{q}(x) \in L_2(0,\pi)$ (not unique) such that $\gamma = \{\gamma_n\}_{n\geq 1}$ is the spectrum of $\tilde{L}_0 := L_0(\tilde{q})$ with this potential. Then $d(\lambda)$ is the characteristic function of $\tilde{L}_0$, and consequently, (13) holds. Therefore, (12) is valid. Then, by Theorem 1 there exists a unique potential $q(x) \in L_2(0,\pi)$ such that $\{\gamma_n, \alpha_n\}_{n\geq 1}$ are the spectral data of $L_0(q)$. Since $\beta_n = (-1)^{n-1}$, $n \geq 1$, it follows from Theorem 4 that $q(x) \in L_2'(0,\pi)$. $\qquad\square$

Theorem 6 ([9]). $q(x) \in L_2'(0,\pi)$ iff γ_n lies at one of the endpoints of the gap a_n for all $n \geq 1$.

Proof.

(1) Let $q(x) = q(\pi - x)$ a.e. on $(0,\pi)$. Using Lemma 4 from [8] we get $C(\pi,\lambda) \equiv S'(\pi,\lambda)$, i.e. $\delta(\lambda) \equiv 0$. By Lemma 2 we conclude that γ_n lies at one of the endpoints of the gap a_n for all $n \geq 1$.

(2) Let γ_n lie at one of the endpoints of the gap a_n for all $n \geq 1$. By Lemma 1 one has $\delta(\gamma_n) = 0$ for all $n \geq 1$. Then the function $F(\lambda) := \delta(\lambda)/d(\lambda)$ is entire in λ, and it vanishes at infinity. This means that $F(\lambda) \equiv 0$, and consequently, $C(\pi,\lambda) \equiv S'(\pi,\lambda)$. Using Lemma 4 from [8] we get $q(x) = q(\pi - x)$ a.e. on $(0,\pi)$. $\qquad\square$

We will write $a_n \in I_0$, if the length of the gap a_n is equal to zero, and $a_n \in I_1$, otherwise.

Let us now consider the inverse problem for the periodic BVP L. In the general case in Inverse problem 2 we have to specify Λ, γ and Ω. In the central symmetric case we do not need γ. On the other hand, the sequence $\Omega = \{\omega_n\}_{n\geq 1}$ does not bring any information because in the central symmetric case $\omega_n = 0$ for all $n \geq 1$. Unfortunately, in contrast to the separated boundary conditions, for the periodic case the specification of the spectrum Λ does not uniquely determine the potential q, and we need additional information. For this purpose we introduce the sequence $E = \{\xi_n\}_{n\geq 1}$, where $\xi_n = 0$ if $a_n \in I_0$, $\xi_n = 1$, if $a_n \in I_1$ and γ_n lies at the right endpoint of a_n, $\xi_n = -1$, if $a_n \in I_1$ and γ_n lies at the left endpoint of a_n. The sequence $E = \{\xi_n\}_{n\geq 1}$ is called the E- sequence for the potential $q(x) \in L_2'(0,\pi)$. The inverse problem for the periodic BVP L in the central symmetric case is formulated as follows.

Inverse problem 4. Given Λ and E, construct q.

Theorem 7 ([9]). *Let $q(x) \in L_2'(0,\pi)$. Then the specification of Λ and E uniquely determines the potential q. The solution of Inverse problem* 1 *can be found by the following algorithm.*

Algorithm 1. *Given Λ and E.*

(1) *Construct $p(\lambda)$ by* (5).

(2) *Calculate the functions $\Delta(\lambda) = 1 - p(\lambda)$ and $p^+(\lambda) = p(\lambda) - 2$.*

(3) *Find zeros* $\Lambda^+ = \{\lambda_n^+\}_{n\geq 1}$ *of* $p^+(\lambda)$.

(4) *Construct* $\gamma = \{\gamma_n\}_{n\geq 1}$ *as follows:* γ_n *lies at the right endpoint of* a_n *if* $\xi_n = 1$; γ_n *lies at the left endpoint of* a_n *if* $\xi_n = -1$, *and* $\gamma_n = a_n$ *if* $\xi_n = 0$.

(5) *Using* $\{\gamma_n\}$ *construct the potential* $q(x) \in L_2'(0,\pi)$ *by solving Inverse problem* 3.

Denote by J_1 the set of sequences $E = \{\xi_n\}_{n\geq 1}$ such that $\xi_n = 0$ if $a_n \in I_0$, and $\xi_n = \pm 1$ if $a_n \in I_1$. Clearly, if E is the E- sequence for q, then $E \in J_1$. The next theorem [9] establishes necessary and sufficient conditions for the solvability of Inverse problem 4.

Theorem 8 ([9]). *Let real numbers* $\Lambda = \{\lambda_n\}_{n\geq 0}$ *satisfying* (3)$-$(4) *be given. The sequence* Λ *is the spectrum for a certain BVP L with a real potential* $q(x) \in L_2'(0,\pi)$, *iff relation* (6) *holds, where* $p(\lambda)$ *is constructed via* (5). *Moreover, if additionally we have a sequence* $E = \{\xi_n\}_{n\geq 1} \in J_1$, *then there exists a unique real function* $q(x) \in L_2'(0,\pi)$ *such that* Λ *is the spectrum of L, and E is the E- sequence for q.*

Proof. The necessity is obvious. We will prove the sufficiency. Let real numbers $\Lambda = \{\lambda_n\}_{n\geq 0}$ satisfying (3)$-$(4) be given. We construct the function $p(\lambda)$ by (5), and calculate the functions $\Delta(\lambda) = 1 - p(\lambda)$ and $p^+(\lambda) = p(\lambda) - 2$. Let (6) holds. Then there exist zeros $\Lambda^+ = \{\lambda_n^+\}_{n\geq 1}$ of the function $p^+(\lambda)$, and (7) holds. Using (5) by similar arguments as in the proof of Theorem 5 (see also [4]) one gets

$$p(\lambda) = 1 - \cos\rho\pi - a\frac{\sin\rho\pi}{\rho} - \frac{\kappa(\rho)}{\rho}, \tag{20}$$

where $\kappa(\rho) \in L_2(-\infty,\infty)$ for real ρ. Since $p^+(\lambda) = p(\lambda) - 2$, it follows from (20) that (8) is valid. Let a sequence $E = \{\xi_n\}_{n\geq 1} \in J_1$ be given. We introduce real numbers $\gamma = \{\gamma_n\}_{n\geq 1}$ as follows: γ_n lies at the right endpoint of a_n if $\xi_n = 1$; γ_n lies at the left endpoint of a_n if $\xi_n = -1$; $\gamma_n = a_n$ if $\xi_n = 0$. Clearly, (9) is valid. We construct the function $d(\lambda)$ by (10), and the sequence $\{\alpha_n\}_{n\geq 1}$ via

$$\alpha_n = \dot{d}(\gamma_n)\Delta(\gamma_n), \quad n \geq 1. \tag{21}$$

Since $\Delta(\lambda) = 1 - p(\lambda)$, it follows from (20) that

$$\Delta(\lambda) = \cos\rho\pi + a\frac{\sin\rho\pi}{\rho} + \frac{\kappa(\rho)}{\rho}.$$

Together with (9) this yields

$$\Delta(\gamma_n) = (-1)^n\left(1 + \frac{\kappa_n}{n}\right), \quad \{\kappa_n\} \in l_2. \tag{22}$$

Moreover, (13) is valid. It follows from (13), (21) and (22) that (12) holds. It is easy to check that

$$\operatorname{sign}\dot{d}(\gamma_n) = (-1)^n, \quad \operatorname{sign}\Delta(\gamma_n) = (-1)^n. \tag{23}$$

In view of (21) and (23) we conclude that $\alpha_n > 0$, $n \geq 1$. By Theorem 1 we infer that there exists a unique real potential $q(x) \in L_2(0, \pi)$ such that $\{\gamma_n, \alpha_n\}_{n \geq 1}$ are the spectral data for the BVP L_0 for this potential. We construct solutions $C(x, \lambda), S(x, \lambda)$ for Eq. (1) with this potential. Denote

$$\tilde{\Delta}(\lambda) = (C(\pi, \lambda) + S'(\pi, \lambda))/2, \quad \tilde{p}(\lambda) = 1 - \tilde{\Delta}(\lambda), \quad \tilde{p}^+(\lambda) = \tilde{p}(\lambda) - 2.$$

Using (10) and (21) we get

$$\Delta(\gamma_n) = \tilde{\Delta}(\gamma_n), \quad n \geq 1.$$

Then the function $F_0(\lambda) := (\Delta(\lambda) - \tilde{\Delta}(\lambda))/d(\lambda)$ is entire in λ, and it vanishes at infinity. This yields $F_0(\lambda) \equiv 0$, i.e. $\Delta(\lambda) \equiv \tilde{\Delta}(\lambda)$, and consequently, $p(\lambda) \equiv \tilde{p}(\lambda)$, $p^+(\lambda) \equiv \tilde{p}^+(\lambda)$. In particular, this means that the sequence $\Lambda = \{\lambda_n\}_{n \geq 0}$ coincides with the spectrum of the BVP L for the potential q. Since γ_n lies at one of the endpoints of the gap a_n for all $n \geq 1$, it follows from Theorem 6 that $q(x) \in L_2'(0, \pi)$. Now it is clear that E is the E- sequence for q. $\qquad \square$

Similar results are valid for other non-separated boundary conditions. For convenience of readers and for completeness of the presentation, we formulate here briefly the main results from [10] related to the boundary conditions

$$y'(0) - a y(0) + b y(\pi) = y'(\pi) + a y(\pi) - b y(0) = 0. \tag{24}$$

We consider the BVP B for the differential equation

$$-y'' + q(x) y = \lambda y, \ x \in (0, \pi), \ q(x) \in L_2'(0, \pi), \tag{25}$$

with the non-separated boundary conditions (24), where a and b are real numbers, $b \neq 0$. Let $\theta(x, \lambda)$ be the solution of Eq. (25) under the initial conditions $\theta(0, \lambda) = 1$, $\theta'(0, \lambda) = a$. Eigenvalues $\mu = \{\mu_n\}_{n \geq 0}$ of the BVP B coincide with zeros of the entire function

$$r(\lambda) = -\theta'(\pi, \lambda) - a\theta(\pi, \lambda) + b^2 S(\pi, \lambda) + 2b.$$

The eigenvalues μ_n are real, and

$$\mu_n < \mu_{n+2}, \quad \mu_n = n^2 + \pi^{-1}(h + (-1)^{n+1} 4b) + \kappa_n, \ \{\kappa_n\} \in l_2, \tag{26}$$

where $h = 4a + \int_0^\pi q(t)\, dt$. The specification of the spectrum μ uniquely determines the characteristic function $r(\lambda)$ via

$$r(\lambda) = \pi(\lambda - \mu_0) \prod_{n=1}^\infty \frac{\mu_n - \lambda}{n^2}. \tag{27}$$

Moreover,

$$\max_{\lambda \in Q_n} |r(\lambda)| \geq |4b|, \tag{28}$$

where $Q_n = [\mu_{2n}, \mu_{2n+1}]$ if $b > 0$, and $Q_n = [\mu_{2n-1}, \mu_{2n}]$ if $b < 0$. Let $\nu = \{\nu_n\}_{n \geq 0}$ be zeros of the entire function $\theta(\pi, \lambda)$. Denote $\eta_n = \text{sign}(|\theta'(\pi, \nu_n)| - |b|)$. The sequence $\eta = \{\eta_n\}_{n \geq 0}$ is called the η- sequence for B. The inverse problem is formulated as follows

Inverse problem 5. Given μ and η, construct q, a and b.

The next theorem gives us the characterization of the spectrum of the BVP B.

Theorem 9. *For real numbers* $\{\mu_n\}_{n \geq 0}$ $(\mu_n \leq \mu_{n+1})$ *to be the eigenvalues of a certain BVP B with real potential* $q(x) \in L_2'(0, \pi)$, *it is necessary and sufficient that* (26) *and* (28) *hold, where* $r(\lambda)$ *is constructed by* (27).

Denote by J' the set of sequences $\eta = \{\eta_n\}_{n \geq 0}$ such that

 (i) $\eta_n = \pm 1$ if the corresponding zeros of the functions $r(\lambda)$ and $r(\lambda) - 4b$ are simple, and $\eta_n = 0$ otherwise;

(ii) there exists N (depending on the sequence) such that $\eta_n = 1$ for all $n > N$.

Clearly, if η is the η- sequence for B, then $\eta \in J'$. The next theorem gives us necessary and sufficient conditions for the solvability of Inverse problem 5.

Theorem 10. *Let real numbers* $\{\mu_n\}_{n \geq 0}$ $(\mu_n \leq \mu_{n+1})$ *satisfying* (26) *and* (28) *be given, where* $r(\lambda)$ *is constructed by* (27). *Then for each sequence* $\eta \in J'$ *there exists a unique real function* $q(x) \in L_2'(0, \pi)$ *and real numbers* a *and* b *such that* $\mu = \{\mu_n\}_{n \geq 0}$ *is the spectrum of B, and* η *is the* η- *sequence for B.*

We note that in [17] stability of the solution of the inverse problem for the BVP B is established. Some other results related to the inverse problem theory for the Sturm-liouville operators can be found in [18] − [20] and other works.

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