A GENERALIZATION OF $H$-CLOSED SPACES

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Abstract. Whereas a space $X$ can be embedded in a compact space if and only if it is Tychonoff, every space $X$ can be embedded in an $H$-closed space (a generalization of compact space). In this paper, we further generalize, the concept of $H$-closedness into $gH$-closedness and have shown that every connected space is either a $gH$-closed space or can be embedded in a $gH$-closed space. Also, in a locally connected regular space the concept of $gH$-closedness is equivalent to the concepts of $J$-ness and strong $J$-ness due to E. Michael [7] and $\theta J$-ness due to C.K. Basu et. al [1]. Several characterizations and properties of $gH$-closed spaces with respect to subspaces, products and functional preservations (along with various examples) are given.

1. Introduction

The concept of an $H$-set (a generalization of an $H$-closed space) was initiated by N. Veličko [15]. Since then $H$-sets played a major role in the development of the theory of $H$-closed spaces, locally $H$-closed spaces [10] although the exact relationship between $H$-sets and $H$-closed subspaces is as yet unknown. Indeed, unlike compactness, $H$-closure is not an absolute property.

Attempts have been made to use such $H$-sets in place of compact sets as is in the case of strong $J$-spaces due to E. Michael [7], to initiate a new class of spaces called $gH$-closed spaces. In what follows, attention will be focused upon $gH$-closed spaces because of the fact that not only every $H$-closed space is $gH$-closed (shown in section 2) but also every connected space is either a $gH$-closed space or can be embedded in a $gH$-closed space (shown in section 4). This result may make a new insight in investigating connected non $H$-closed (non compact as well) spaces. Several characterizations and properties of $gH$-closed spaces analogous to strong $J$-spaces due to E. Michael [7] have been achieved.

All the spaces considered herein are assumed to be Hausdorff. We assume that the reader is familiar with the concepts of $H$-closedness, $H$-sets, $\theta$-closed sets and $\theta$-continuity; [12, 16] might very well serve as the necessary background. The $\theta$-closure of a subset $A$ of a space $X$ is the set $[A]_\theta \equiv \{ x \in X : U \cap A \neq \emptyset \text{ for all open sets } U \text{ containing } x \}$. A subset $A$ is $\theta$-closed if $A = [A]_\theta$ and the complement of a $\theta$-closed set is a $\theta$-open set; a subset which is both $\theta$-open as well as $\theta$-closed is called $\theta$-clopen. The $\theta$-boundary [1] of a subset $A$ of $X$ ($\theta$-bd $A$, for short) is defined as $[A]_\theta \cap (X - [A]_\theta)$.

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The other concepts needed for investigation of $gH$-closed spaces are: almost regularity, \( \theta \)-connectedness, \( \theta \)-rigidity and \( \theta \)-perfectness. A space \( (X, \tau) \) is said to be almost regular \([13, 8]\) iff \( \tau_\theta = \tau_0 \) (where \( \tau_\theta, \tau_0 \) are respectively the semi-regularization topology and the \( \theta \)-topology). Every regular space is almost regular but not conversely. A pair \( (P, Q) \) of non-empty subsets of \( X \) is called a \( \theta \)-separation relative to \( X \) iff \( (P \cap (Q_\theta) \cup (Q \cap (P_\theta)) = \emptyset \); a subset \( A \) is called \( \theta \)-connected \([2]\) iff \( A \) is not the union of \( P \) and \( Q \) where \( (P, Q) \) is a \( \theta \)-separation relative to \( X \). Clearly every connected set is \( \theta \)-connected but the converse is not true and in a regular space these two concepts coincide. A subset \( A \) of \( X \) is \( \theta \)-rigid \([3]\) iff for each open cover \( \mathcal{U} \) of \( A \), there is a finite subfamily \( \{U_1, U_2, \ldots, U_n\} \) of \( \mathcal{U} \) such that \( A \subseteq \text{int}(\bigcup_{i=1}^n U_i) \); in a Hausdorff space, every \( \theta \)-rigid set is an \( H \)-set \([3]\). A filter \( \mathcal{F} \) in \( X \) almost converges \([3]\) to a subset \( A \) (written \( \mathcal{F} \rightarrow A \)) if for each open cover \( \mathcal{A} \) of \( A \), there is a finite subfamily \( \mathcal{B} \subseteq \mathcal{A} \) such that \( \bigcup({\text{cl} } V : V \in \mathcal{B}) \in \mathcal{F} \). A function \( f : X \rightarrow Y \) is \( \theta \)-perfect \([3]\) iff for every filter base \( \mathcal{F} \) on \( f(X) \), \( \mathcal{F} \rightarrow y \) implies \( f^{-1}(\mathcal{F}) \rightarrow f^{-1}(y) \).

Some results from literature are cited below:

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\[\text{\#1.1. If } f : X \rightarrow Y \text{ is } \theta \text{-continuous surjective and } X \text{ is } H \text{-closed then } Y \text{ is } H \text{-closed} \ [16].\]

\[\text{\#1.2. If } f : X \rightarrow Y \text{ is } \theta \text{-continuous, when } A \subset X \text{ is an } H \text{-set, then } f(A) \text{ is an } H \text{-set in } Y \ [16].\]

\[\text{\#1.3. An almost regular } T_2 \text{ space is Urysohn} \ [13].\]

\[\text{\#1.4. An } H \text{-set in a Urysohn space is } \theta \text{-closed} \ [14].\]

\[\text{\#1.5. Let } f : X \rightarrow Y \text{ be a function, where } X \text{ and } Y \text{ are almost regular spaces. Then } f \text{ is } \theta \text{-continuous iff inverse image of every } \theta \text{-open (resp. } \theta \text{-closed) set of } Y \text{ is } \theta \text{-open (resp. } \theta \text{-closed) in } X \ [5].\]

\[\text{\#1.6. A } \theta \text{-closed subset of an } H \text{-closed space is an } H \text{-set} \ [3].\]

\[\text{\#1.7. A space } X \text{ is } H \text{-closed iff for every space } Y \text{, the projection map from } X \times Y \text{ onto } Y \text{ takes a } \theta \text{-closed subset onto a } \theta \text{-closed subset} \ [6].\]

\[\text{\#1.8. If } A \subset X \text{ is an } H \text{-set of } X \text{ and } X \subset Y \text{, then } A \text{ is an } H \text{-set in } Y \ [16].\]

\[\text{\#1.9. Let } B \text{ be a regular closed subset of a } T_2 \text{ space } X. \text{ If } A \subset X \text{ is an } H \text{-set and } B \subset A \text{ then } B \text{ is } H \text{-closed} \ [16].\]

\[\text{\#1.10. For any subset } A \subset X \text{, } |A|_\theta \text{ is } \theta \text{-closed if } X \text{ is almost regular} \ [5].\]

\[\text{\#1.11. In an almost regular space } X \text{, every regular closed (resp. every regular open) subset is } \theta \text{-closed (resp. } \theta \text{-open)} \ [5].\]

\[\text{\#1.12. If } f : X \rightarrow Y \text{ is } \theta \text{-continuous, the mapping } f : X \rightarrow f(X) \text{ need not be } \theta \text{-continuous (even if } f(X) \text{ is a regular subspace of } Y) \ [15].\]

\[\text{\#1.13. If } f : X \rightarrow Y \text{ is } \theta \text{-continuous and } f(X) \subset Z \subset Y \text{ and } Z \text{ is dense in } Y \text{, then } f : X \rightarrow Z \text{ is } \theta \text{-continuous} \ [16].\]

\[\text{\#1.14. If } f : X \rightarrow Y \text{ is } \theta \text{-continuous and } A \subset X \text{, then } f / A : A \rightarrow Y \text{ is } \theta \text{-continuous} \ [16].\]

\[\text{\#1.15. If } Y \text{ is an } H \text{-set in } X \text{ and } A \text{ is a } \theta \text{-closed subset of } X \text{ then } A \text{ is an } H \text{-set if } A \subset Y \ [14].\]

\[\text{\#1.16. If } A \subset Y \subset X \text{, } A \text{ is } \theta \text{-open in } Y \text{ and } Y \text{ is } \theta \text{-open in } X \text{, then } A \text{ is } \theta \text{-open in } X \ [14].\]

\[\text{\#1.17. If } A \subset Y \subset X \text{ and } Y \text{ is } \theta \text{-open in } X; \text{ } A \text{ is } \theta \text{-closed in } Y \text{, then } A = f \cap Y \text{, where } F \text{ is } \theta \text{-closed in } X \ [14].\]

\[\text{\#1.18. Let } Y \text{ be an open subset of } X \text{ and } X \text{ be almost regular; then } Y \text{ is almost regular} \ [5].\]

\[\text{\#1.19. In an } H \text{-closed Urysohn space, every } H \text{-set is } \theta \text{-closed and every } \theta \text{-closed set is an } H \text{-set} \ [3].\]
1.20. Let \( f : X \to Y \) be a \( \theta \)-continuous \( \theta \)-bd perfect map from an almost regular space \( X \) onto a Urysohn space \( Y \). If \( X \) is a \( \theta \)-J-space then so is \( Y \) [1].

2. \( gH \)-closed spaces.

**Definition 2.1.** A space \( X \) is called a generalized \( H \)-closed space (\( gH \)-closed, for short) if for every \( H \)-set \( H \) of \( X \), there is an \( H \)-set \( K \) in \( X \) such that \( H \subset K \) and \( X - K \) is \( \theta \)-connected relative to \( X \).

**Remark 2.2.** Every \( H \)-closed space is a \( gH \)-closed space. But the converse is not true.

**Examples of \( gH \)-closed spaces that are not \( H \)-closed.**

**Example 2.3.** On \( R^+ = [0, \infty) \), let us consider \( \tau \), the countable complement extension topology of the usual topology \( \mathcal{U} \) on \( R^+ = [0, \infty) \). Then as \( \tau_\theta = \tau_\emptyset = \mathcal{U} \), the space \( (R^+, \tau) \) is almost regular and Hausdorff. It can be checked that this space is non-regular, non \( H \)-closed and non locally connected. In this space \( (R^+, \tau) \), let \( H \) be an \( H \)-set and hence \( H \) is compact in \( (R^+, \mathcal{U}) \). Therefore, \( h = \sup H \) exists. Obviously, \( H \subset [0, h] \) and \( R^+ - [0, h] = (h, \infty) \) is connected in \( (R^+, \mathcal{U}) \). As \((h, \infty)\) is \( \theta \)-open in \( (R^+, \tau) \) and is connected in \( (R^+, \tau_\emptyset) \), by proposition 3 [8], \((h, \infty)\) is \( \theta \)-connected relative to \( (R^+, \tau) \). Clearly \([0, h]\) is an \( H \)-set in \( (R^+, \tau) \). Hence \((R^+, \tau)\) is a \( gH \)-closed space.

**Example 2.4.** Let \( \tau \) be the countable complement extension topology of the real line \((R, \mathcal{U})\). Then as \( \tau_\emptyset = \tau_\emptyset = \mathcal{U} \), the space \((R, \tau)\) is almost regular. Easy verification shows that this space is connected but not \( H \)-closed; also \((R, \tau)\) is not \( gH \)-closed because of the Theorem 2.16 (given latter) and it is not \( \theta \)J [Example 2.4, 1]. Therefore, by Theorem 4.1 (given latter), \((R^n, \tau^n)\), \( n > 1 \) is a \( gH \)-closed space. Obviously \((R^n, \tau^n)\) is not \( H \)-closed; otherwise, \((R, \tau)\) would be \( H \)-closed — a contradiction.

**Lemma 2.5.** If \( Y \subset X \) and \( B \) is \( \theta \)-closed \((\theta \text{-open})\) in \((X, \tau)\) then \( B \cap Y \) is \( \theta \)-closed \((\theta \text{-open})\) in \((Y, \tau_Y)\), where \( \tau_Y \) is the subspace topology on \( Y \).

**Lemma 2.6.** If \( A \subset Y \subset X \), and \( A \) is \( \theta \)-open in \( Y \) and \( Y \) is \( \theta \)-open in \( X \) then \( A \) is \( \theta \)-open in \( X \).

But for \( \theta \)-closed sets, the lemma 2.6 does not hold.

**Example 2.7.** ([1]) Let \( \tau \) be the countable complement extension topology of the real line \((R, \mathcal{U})\). Now, let \( Y = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots \} \), obviously \( \{0\} \) is open and closed in \((Y, \tau_Y)\) and \( Y \) is \( \theta \)-closed in \((R, \tau)\) as \( Y \) is closed in \((R, \mathcal{U})\). Hence \( \{0\} \cap \{1, \frac{1}{2}, \frac{1}{3}, \cdots \} = \emptyset \), implying that \( A = \{1, \frac{1}{2}, \frac{1}{3}, \cdots \} \) is \( \theta \)-closed in \((Y, \tau_Y)\) (where the closure is taken in the subspace topology) but \( A \) is not \( \theta \)-closed in \((R, \tau)\), since \( A \) is not closed in \((R, \mathcal{U})\).

**Lemma 2.8.** ([1]) A subset \( H \) of an almost regular space \( X \) is an \( H \)-set, iff every \( \theta \)-open cover of \( A \) has a finite subcover.
It is well known that every \( H \)-closed subspace is an \( H \)-set but example exists in [15], which shows that the converse is not true in general. Now we give the following characterization theorem for \( gH \)-closed spaces.

**Theorem 2.9.** Let \( \{H_1, H_2\} \) be a \( \theta \)-clopen cover of an almost regular space \( X \), with \( H_1 \cap H_2 \) an \( H \)-set. Then the following are equivalent:

(i) \( X \) is \( gH \)-closed.
(ii) \( H_1 \) and \( H_2 \) are \( gH \)-closed, and \( H_1 \) or \( H_2 \) is an \( H \)-set.

**Proof.** (i)⇒(ii) Let \( H_1 \) be an \( H \)-set. Then obviously \( H_1 \) is regular closed and hence \( H_1 \) is an \( H \)-closed subspace of \( X \) (by result 1.9). Therefore, by Remark 2.2, \( H_1 \) is a \( gH \)-closed subspace of \( X \). To show \( H_2 \) is a \( gH \)-closed subspace of \( X \), let \( K_2 \subset H_2 \) be an \( H \)-set in the subspace \( H_2 \). Then, by result 1.8, \( K_2 \) is an \( H \)-set in \( X \). So, \( H = K_2 \cup H_1 \) is an \( H \)-set in \( X \). Since \( X \) is \( gH \)-closed, there exists some \( H \)-set \( L \) in \( X \) with \( H \subset L \) and \( X - L \) is \( \theta \)-connected. Let \( L_2 = L \cap H_2 \). Then \( K_2 \subset L_2 \) and \( L_2 \) is not only a \( \theta \)-closed set in \( H_2 \) but also in \( X \) (by result 1.3 and 1.4). In addition, \( L_2 \subset H_2 \) is an \( H \)-set in \( X \) (by result 1.9). Therefore, \( L_2 \) is a \( gH \)-closed space.

(ii)⇒(i) Suppose (ii) holds and also suppose \( H_1 \) be an \( H \)-set. Let \( H \subset X \) be an \( H \)-set. Then \( K = (H \cup H_1) \cap H_2 \) is an \( H \)-set in \( X \) such that \( K \subset H_2 \). Since \( H_2 \) is \( \theta \)-open, by lemma 2.8, we can easily prove that \( K \) is an \( H \)-set in \( H_2 \). The \( gH \)-closedness of the subspace \( H_2 \) implies that there exists some \( H \)-set \( L \) in the subspace \( H_2 \) such that \( K \subset L \) and \( X - L \) is \( \theta \)-connected in \( X \). Therefore, \( L \) is a \( gH \)-closed subspace.

**Corollary 2.10.** If \( A \) is a \( \theta \)-clopen subset of an almost regular \( gH \)-closed space \( X \) then \( A \) is a \( gH \)-closed space.

**Remark 2.11.** (a) In [1] we considered a kind of spaces termed \( \theta \)-J spaces (see Definition 2.15) which satisfy a condition weaker than that given in (ii) of Theorem 2.9. Indeed, we shall show shortly (see Theorem 2.16) that every \( gH \)-closed space is a \( \theta \)-J space for which we need not assume the condition of almost regularity on the underlying space.

(b) The condition in the above corollary is not necessary. In the space \( (R^+, \tau) \) in Example 2.3, \( Y = [0, 1] \) with the subspace topology \( \tau_Y \) is \( H \)-closed and hence is a \( gH \)-closed subspace. Although \( Y \) is \( \theta \)-closed in \( (R^+, \tau) \) but because of the fact that \( Y \) is not open in \( (R^+, \emptyset) \), it is not \( \theta \)-open in \( (R^+, \tau) \).

On the other hand, union of even two \( \theta \)-closed \( gH \)-closed subspaces may not be \( gH \)-closed. In Example 2.4, \( R^+ = \{x \in R : x \geq 0\} \) and \( R^- = \{x \in X : x \leq 0\} \) are both \( \theta \)-closed \( gH \)-closed subspaces but their union \( (R, \tau) \) is not so. But we have the following corollary.

**Corollary 2.12.** If an almost regular space \( X \) be such that \( X = H_1 \cup H_2 \) with \( H_1 \) an open \( gH \)-closed subspace, \( H_2 \) a \( \theta \)-clopen \( H \)-set, then \( X \) is a \( gH \)-closed space.
Proof. Let $A = X - H_2$. Then $A$ is $\theta$-clopen subset in the subspace $H_1$. By result 1.18 $H_1$ is almost regular (being an open subspace of an almost regular space $X$) and as $H_1$ is a $gH$-closed space, the Corollary 2.10 implies $A$ is a $gH$-closed subspace. Again $H_2$ is being a regular closed $H$-set in $X$, by result 1.9, $H_2$ is an $H$-closed subspace of $X$ and hence is a $gH$-closed subspace in $X$ (by Remark 2.2). Since $\{A, H_2\}$ is a $\theta$-clopen of $X$, $A$ and $H_2$ are $gH$-closed subspaces and $H_2$ is an $H$-set, by Theorem 2.19, $X$ is a $gH$-closed space.

Theorem 2.13. Let $X$ be an almost regular $gH$-closed space. Then every regular closed set $B$ which is a union of $\theta$-components of $X$ [2] is $gH$-closed.

Proof. Let $A \subset B$ be an $H$-set in the subspace $B$. Then by result 1.8, $A$ is an $H$-set in $X$ and hence the $gH$-closedness of $X$ implies the existence of an $H$-set $H((in X)$ such that $A \subset H$ and $X - H$ is $\theta$-connected relative to $X$. Now, if $B \subset H$, then because of $X$ is almost regular, $B$ is a $\theta$-closed set (by result 1.11) contained in an $H$-set $H$ and hence (by result 1.15) $B$ is an $H$-set. In fact, $B$ is a $H$-closed subspace of $X$ (by result 1.9). So, by Remark 2.2, $B$ is $gH$-closed. If $B \notin H$, then the $\theta$-connected set $X - H$ intersects $B$; but as $B$ is a union of $\theta$-components of $X$, the only possibility is $X - H \subset B$. Let $H_1 = H \cap B$. Using results 1.3, 1.4 and 1.15, it can be shown that $H_1$ is an $H$-set in $X$. Obviously $A \subset H_1$ and also $B - H_1 = X - H$ is $\theta$-connected relative to $X$. Since $X - H$ is a $\theta$-open set (in $X$) contained in $B$, then by lemma 2.5, $X - H$ is $\theta$-open in the subspace $B$. If $(P, Q)$ is any $\theta$-separation of the $\theta$-open set $X - H$ in the subspace $B$, such that $X - H = P \cup Q$ then by [8, Proposition 1], $P, Q$ are $\theta$-open sets in $B$. Since the $\theta$-closure and the closure of an open set are equal and $B$ is closed in $X$, $(P, Q)$ is therefore a separation of the connected set $X - H$ (because an open subset of $\theta$-set) is connected if it is $\theta$-connected relative to $X$ [8, Proposition 2]) — a contradiction. So, $B - H_1 = X - H$ is $\theta$-connected in the subspace $B$ and hence $B$ is a $gH$-closed subspace of $X$.

Corollary 2.14. Every regular closed subset which is a union of $\theta$-components of an almost regular $H$-closed space is $gH$-closed.

Definition 2.15. A space $X$ is a $\theta J$-space [1] if whenever $\{H_1, H_2\}$ is a $\theta$-closed cover of $X$ with $H_1 \cap H_2$ an $H$-set, $H_1$ or $H_2$ is an $H$-set.

Theorem 2.16. Every $gH$-closed space is a $\theta J$-space.

Proof. Suppose $(X, \tau)$ is a $gH$-closed space. Let $\{H_1, H_2\}$ be a $\theta$-closed cover of $X$ with $H_1 \cap H_2$ an $H$-set. As $(X, \tau)$ is $gH$-closed and $H_1 \cap H_2 \subset X$ is an $H$-set, so $H_1 \cap H_2 \subset K$ for some $H$-set $K$ of $(X, \tau)$ with $X - K$ $\theta$-connected. By lemma 2.5, $(H_1 \cap (X - K), H_2 \cap (X - K))$ is a disjoint $\theta$-closed cover of the $\theta$-connected set $X - K$. So, either $H_1 \cap (X - K) = \emptyset$ or $H_2 \cap (X - K) = \emptyset$ implying either $H_1 \subset K$ or $H_2 \subset K$. But $H_1$ or $H_2$ is a $\theta$-closed subset contained in an $H$-set $K$. So, by result 1.15, either $H_1$ or $H_2$ is an $H$-set. Therefore $(X, \tau)$ is a $\theta J$-space.

We shall show that in a locally connected almost regular space, these two concepts are equivalent. For this we first state a theorem.

Theorem 2.17. ([1]) An almost regular space $X$ is a $\theta J$-space iff whenever $H \subset X$ is an $H$-set and $U \in \mathcal{U}$ is a disjoint $\theta$-open cover of $X - H$, then $X - U$ is an $H$-set for some $U \in \mathcal{U}$. 

In view of the following proposition due to Mrševic et al.

**Proposition 2.18.** ([8]) In \((X, τ)\), the conditions (i) — (iii) below are equivalent:

Every point has a neighbourhood basis in \((X, τ)\) consisting of:

(i) connected neighbourhoods.
(ii) \(δ\)-connected neighbourhoods.
(iii) neighbourhoods \(θ\)-connected relative to \(X\).

We have the following theorem:

**Theorem 2.19.** A locally connected almost regular space is a \(θ\)-space iff it is a \(gH\)-closed space.

**Proof.** Every \(gH\)-closed space is a \(θ\)-space (by Theorem 2.15). Let \(X\) be a locally connected \(θ\)-space and \(H \subset X\) be an \(H\)-set. As \(X\) is almost regular \(T_2\), \(X\) is not only a Urysohn space but also every \(H\)-set of \(X\) is \(θ\)-closed; further for any open neighbourhood \(U\) of \(x\) there exists a \(θ\)-open set \(V\) of \(x\) such that \(V < T\). Indeed, by definition of almost regularity, there exists, an open set \(W\) such that \(x \in W \subset \overline{W} \subset \text{int} \overline{U} \subset \overline{U}\); but in the almost regular space \(X\), the regular open set \(V = \text{int} \overline{U}\) is \(θ\)-open. Because of \(X\) is locally connected Urysohn and \(H\) is \(θ\) closed in \(X\), it can be easily shown that, there is a disjoint \(θ\)-open cover \(\mathcal{U}\) of \(X - H\) with each \(U \in \mathcal{U}\) \(θ\)-connected. Since \(X\) is \(θ\)-space by Theorem 2.17, there exists a \(U^* \in \mathcal{U}\) such that \(X - U^*\) is an \(H\)-set. If we take \(K = X - U^*\), then \(H \subset K\) and \(X - K = U^*\) is \(θ\)-connected. So, \(X\) is a \(gH\)-closed space.

3. Preservation of \(gH\)-closedness in terms of \(θ\)-perfect (\(θ\)-bd perfect) functions.

**Definition 3.1.** ([1]) A map \(f : X \to Y\) is called \(θ\)-boundary perfect (\(θ\)-bd perfect, for short) if \(f\) is almost closed [3] (i.e. \(f(\text{int} A) = \text{int} f(A), \forall A \subset X\)) and \(θ\)-bd \(f^{-1}(y)\) is \(θ\)-rigid [3] for every \(y \in Y\).

**Theorem 3.2.** For an almost regular space \(X\) if

(i) \(X\) is \(gH\)-closed space then

(ii) every \(θ\)-continuous \(θ\)-bd perfect map \(f : X \to Y\) onto a non-\(H\)-closed Urysohn space \(Y\) is \(θ\)-perfect.

**Proof.** Let \(f : X \to Y\) be a \(θ\)-continuous \(θ\)-bd perfect map and let \(y \in Y\). Since \(θ\)-bd \(f^{-1}(y)\) is a \(θ\)-rigid set and hence is an \(H\)-set (by remark after Corollary 6.3, [3]), then as \(X\) is a \(gH\)-closed space there exist an \(H\)-set \(K\) such that \([f^{-1}(y)]_θ \cap [X - f^{-1}(y)]_θ = θ\)-bd \(f^{-1}(y)\) \(\subset K\) and \(X - K\) is \(θ\)-connected. But \([f^{-1}(y)]_θ \cap (X - K), [X - f^{-1}(y)]_θ \cap (X - K)\) is a \(θ\)-separation of \((X - K)\). So, either \([f^{-1}(y)]_θ \subset K\) or \([X - f^{-1}(y)]_θ \subset K\). But, by results 1.10 and 1.15, either \([f^{-1}(y)]_θ\) or \([X - f^{-1}(y)]_θ\) is an \(H\)-set. Now as \(f\) is \(θ\)-continuous and \((y)\) is a \(θ\)-closed set being an \(H\)-set in the Urysohn space \(Y\) (by result 1.4), \(f^{-1}(y)\) is \(θ\)-closed i.e. \(f^{-1}(y) = [f^{-1}(y)]_θ\). Since \([X - f^{-1}(y)]_θ\) is an non-\(H\)-set otherwise \(Y\) would be \(H\)-closed, so by Theorem 3.4 [3], \(f\) is \(θ\)-perfect.
Remark 3.3. If $Y$ is $H$-closed then the above theorem fails. In fact, if $f : X \to Y$, where $Y$ is a singleton and $X$ is $gH$-closed but not $H$-closed then $f$ is a $\theta$-continuous $\theta$-bd perfect map onto a Urysohn space $Y$. But $f$ is not $\theta$-perfect as $f^{-1}(y)$ is not a $\theta$-rigid set.

Theorem 3.4. Let $f : X \to Y$ be a $\theta$-continuous, $\theta$-bd perfect $\theta$-open map (i.e. maps $\theta$-open sets into $\theta$-open sets) from an almost regular space $X$ onto a Urysohn space $Y$. Then $Y$ is a $gH$-closed space if $X$ is so.

Proof. If $Y$ is $H$-closed then by Remark 2.2, $Y$ is $gH$-closed. Suppose $Y$ is not an $H$-closed space. Then by above Theorem 3.2, $f$ is $\theta$-perfect. Let $H \subseteq Y$ be an $H$-set then $H_1 = f^{-1}(H) \subset X$ is an $H$-set, by Corollary 3.1.1.(c) [3]; but the $gH$-closedness of $X$ implies the existence of an $H$-set $K_1 \subset X$ such that $H_1 \subset K_1$ and $X - K_1$ is $\theta$-connected. Since $f$ is $\theta$-continuous, by Theorem 2.7 [2], $f(X - K_1)$ is $\theta$-connected. Obviously, $H_2 = Y - f(X - K_1)$ is $\theta$-closed and $f$ being $\theta$-continuous, the $\theta$-closed set $f^{-1}(H_2) \subset K_1$ is an $H$-set in $X$ by result 1.15. Since $f$ is $\theta$-continuous, by result 1.2, $H_2$ is an $H$-set in $Y$ such that $H \subset H_2$. But $Y - H_2 = f(X - K_1)$ is $\theta$-connected. So, $Y$ is a $gH$-closed space.

Theorem 3.5. Let $f : X \to Y$ (where $X$ and $Y$ are almost regular spaces) be a $\theta$-continuous, $\theta$-perfect map onto $Y$. Then, if $Y$ is a $gH$-closed space, so is $X$.

Proof. Let $Y$ be $gH$-closed and let $H \subseteq X$ be an $H$-set in $X$. Since $f$ is $\theta$-continuous, by result 1.2, $f(H) \subset Y$ is an $H$-set. Because of $gH$-closedness of $Y$, there is an $H$-set $K$ in $Y$ such that $f(H) \subset K$ with $Y - K$ $\theta$-connected. Hence $H \subset f^{-1}(K)$ and since $f$ is $\theta$-perfect, by Corollary 3.1.1.(c) [3], $f^{-1}(K)$ is an $H$-set in $X$. We shall show that $X - f^{-1}(K) = f^{-1}(Y - K)$ is $\theta$-connected. If, $\{B_1, B_2\}$ is a $\theta$-separation relative to $X$ of the $\theta$-open set $f^{-1}(Y - K)$ (since $f$ is $\theta$-continuous and $Y - K$ is $\theta$-open such that $f^{-1}(Y - K) = B_1 \cup B_2$). Then by proposition 1 [8], $B_1$ and $B_2$ are disjoint $\theta$-open sets in $X$. Since $f$ is $\theta$-perfect, by Corollary 3.1.1.(b) [3], for each $\theta$-closed set $A$ of $X$, $f(A)$ is $\theta$-closed. From this, one can easily verify that the sets $V_i = \{y \in Y - K: f^{-1}(y) \subset B_i\}$ for $i = 1, 2$ are disjoint $\theta$-open sets. Therefore, $Y - K$, is not $\theta$-connected — a contradiction. So, $f^{-1}(Y - K)$ is $\theta$-connected i.e. $X - f^{-1}(K)$ is $\theta$-connected. Hence $X$ is a $gH$-closed space.

Corollary 3.6. An almost regular space $(X, \tau)$ is $gH$-closed iff $(X, \tau_s)$ is $gH$-closed.

Proof. Since the $\theta$-closure of a subset $A$ in $(X, \tau)$ is the same as the $\theta$-closure of $A$ in $(X, \tau_s)$, the identity map $i_1 : (X, \tau_s) \to (X, \tau)$ is $\theta$-continuous and almost closed; also point inverses are $\theta$-rigid sets. Hence by Theorem 3.4 [4], the identity map $i_1 : (X, \tau_s) \to (X, \tau)$ is $\theta$-perfect. So, if $(X, \tau)$ is $gH$-closed then by above Theorem 3.5, $(X, \tau_s)$ is $gH$-closed. On the other hand, by the same reason, the identity map $i_2 : (X, \tau) \to (X, \tau_s)$ is $\theta$-perfect and $\theta$-continuous. So, if $(X, \tau_s)$ is $gH$-closed then by Theorem 3.5, $(X, \tau)$ is $gH$-closed.

Theorem 3.7. For an almost regular locally connected space $(X, \tau)$, the following are equivalent:

(i) $(X, \tau)$ is $gH$-closed.

(ii) $(X, \tau_s)$ is $gH$-closed.
(iii) \((X, \tau_\theta)\) is \(gH\)-closed.
(iv) \((X, \tau)\) is \(\theta J\).
(v) \((X, \tau_s)\) is \(\theta J\).
(vi) \((X, \tau_\theta)\) is \(\theta J\).

**Proof.** The proof follows from Corollary 3.6, Theorem 2.19 and from the fact that a space \((X, \tau)\) is almost regular iff \(\tau_s = \tau_\theta\).

**Definition 3.8.** A space \((X, \tau)\) is a \(J\)-space (resp. strong \(J\)-space) [7] if, whenever \(\{A, B\}\) is a closed cover of \(X\) with \(A \cap B\) compact, then \(A\) or \(B\) is compact (resp. if every compact set \(K \subset X\) is contained in a compact set \(L \subset X\) with \(X - L\) connected).

**Theorem 3.9.** For a regular locally connected space \((X, \tau)\) the following are equivalent:

(i) \((X, \tau)\) is \(gH\)-closed.
(ii) \((X, \tau)\) is \(J\).
(iii) \((X, \tau)\) is \(\theta J\).
(iv) \((X, \tau)\) strong \(J\).

4. \(gH\)-closedness in products.

**Theorem 4.1.** If \(X_1, X_2\) are connected and non-\(H\)-closed spaces then \(X_1 \times X_2\) is a \(gH\)-closed space.

**Proof.** Let \(H\) be an \(H\)-set in the product space \(X = X_1 \times X_2\) and let \(\pi_i : X \to X_i, i = 1, 2\) are projection maps. Since \(\pi_i\)'s are continuous and hence are \(\theta\)-continuous then by result 1.2, \(\pi_i(H) = H_i, i = 1, 2\) are \(H\)-sets in \(X_i, i = 1, 2\) respectively. Obviously \(H \subset H_1 \times H_2\), where \(H_1 \times H_2\) is an \(H\)-set, by result [12, Theorem 4.8].

But \(X_i - H_i \neq \emptyset, i = 1, 2\) otherwise \(X_i\)'s, \(i = 1, 2\) would have been \(H\)-closed. Let \(x_i \in X_i - H_i, i = 1, 2\). Now the set \(B = \{(x_1) \times X_2\} \cup (X_1 \times \{x_2\})\) being a union of two intersecting connected and hence \(\theta\)-connected sets is \(\theta\)-connected and also \(B \subset X - (H_1 \times H_2)\). But \(X - (H_1 \times H_2)\) is the union of \(\theta\)-connected sets of the form \(\{x_1\} \times X_2\) with \(x_1 \in X_1\) or \(X_1 \times \{x_2\}\) with \(x_2 \in X_2\) and all of which intersects \(B\) is thus \(\theta\)-connected. Therefore, \(X_1 \times X_2\) is a \(gH\)-closed space.

**Corollary 4.2.** Any connected space is either a \(gH\)-closed space or can be embedded in a \(gH\)-closed space.

**Proof.** Let \(X\) be a connected space. If it is \(H\)-closed then by Remark 2.2, \(X\) is \(gH\)-closed. If \(X\) is not an \(H\)-closed space, then by Theorem 4.1, \(X \times X\) is a \(gH\)-closed space. But as \(X\) is homeomorphic to some subspace of \(X \times X\), the proof follows immediately.

**Corollary 4.3.** \(\mathbb{R}^n, n > 1,\) where \(\mathbb{R}\) is the real line with the usual topology is a \(gH\)-closed space.

**Example 4.4.** Example of a \(gH\)-closed space which is not an \(H\)-closed space.
Let \((R, \mathcal{U})\) be the real line with usual topology. Then by above Corollary 4.3, \(R^n, n > 1\) is a \(gH\)-closed space. But \(R^n\) is not an \(H\)-closed space.

**Theorem 4.5.** The following are equivalent for any locally connected almost regular space \(X\).

(a) \(X\) is a \(gH\)-closed space.
(b) \(X \times Y\) is a \(gH\)-closed space for every connected \(H\)-closed almost regular space \(Y\).
(c) \(X \times Y\) is a \(gH\)-closed space for some \(H\)-closed almost regular space \(Y\).

**Proof.** For proving (a) \(\Rightarrow\) (b) and (b) \(\Rightarrow\) (c), the assumption of local connectedness of \(X\) is not needed. Let us prove:

(a) \(\Rightarrow\) (b). Let \(p_X : X \times Y \to X\) be the projection map. Now, \(p_X^{-1}(x) = \{x\} \times Y\) is an \(H\)-set in the almost regular space \(X \times Y\) (as the product of any family of almost regular spaces is almost regular [3, Theorem 5.1]) and therefore is \(\theta\)-rigid set in \(X \times Y\), by Theorem 6.4 [3].

We shall next show that for each \(A \subset X \times Y\), \(\{x\} \times Y\) \(\subset\) \(p_X([A]\theta)\). For this, we first show for each \(\theta\)-closed set \(B\) of \(X \times Y\), \(p_X(B)\) is \(\theta\)-closed. Indeed, if \(x \in X - p_X(B)\); then \((x, y) \cap B = \emptyset\). Therefore, for each point \((x', y)\), has a open neighbourhood \(V_{x'}(x') \times V(y)\) such that \((V_{y'}(x') \times V(y)) \cap B = \emptyset\). Since \(\{x\} \times Y\) is an \(H\)-set we can select \(V_{y_i}(x') \times V(y)_i\), \(i = 1, 2, \ldots, n\) such that the union of the closures of such subfamily covers \(\{x\} \times Y\). Then \(\cap_{i=1}^n V_{y_i}(x')\) is a open neighbourhood of \(x'\) in \(X\) such that \(\cap_{i=1}^n V_{y_i}(x') \cap p_X(B) = \emptyset\). Therefore, \(p_X(B)\) is \(\theta\)-closed.

Now, \(A \subset X \times Y\), \([A]\theta\) is \(\theta\)-closed in \(X \times Y\) (by result 1.10) and hence \(p_X([A]\theta)\) is \(\theta\)-closed. But we always have \(p_X(A) \subset p_X([A]\theta)\). So \([p_X(A)]\theta \subset [p_X([A]\theta)]\theta\). Therefore by Corollary 3.4.1 [3], \(p_X\) is \(\theta\)-perfect. Since \(p_X\) is continuous and hence is \(\theta\)-continuous, by Theorem 3.5, \(X \times Y\) is \(gH\)-closed.

(b) \(\Rightarrow\) (c). If \(X\) is \(H\)-closed then by Remark 2.2, \(X\) is \(gH\)-closed. Suppose \(X\) is non \(H\)-closed then for the projection map \(p_X : X \times Y \to X\) (where \(Y\) is some \(H\)-closed space), we have \([p_X(A)]\theta \subset [p_X([A]\theta)]\theta\) for every \(A \subset X \times Y\). Since \(p_X\) is \(\theta\)-continuous, by Corollary 2.10.1 [3], \([p_X(A)]\theta \subset [p_X([A]\theta)]\theta\). Therefore, \(p_X\) is an almost closed map. Again, \(p_X^{-1}(x)\) is an \(H\)-set in the almost regular space \(X \times Y\). So, \(p_X^{-1}(x)\) is \(\theta\)-closed. Now, as \(\theta\)-bd \(p_X^{-1}(x)\) is a \(\theta\)-closed subset of the \(H\)-set \(p_X^{-1}(x)\), then by result 1.15, \(\theta\)-bd \(p_X^{-1}(x)\) is an \(H\)-set in the almost regular space \(X \times Y\) and hence is \(\theta\)-rigid so \(p_X\) is \(\theta\)-bd perfect mapping. Since \(X \times Y\) is \(\theta\) (by Theorem 2.16) then by result 1.20, \(X\) is \(\theta\). Because of \(X\) is locally connected, \(X\) is a \(gH\)-closed space by Theorem 3.7.

**Remark 4.6.** In Example 2.4, we have seen the product space \((R^n, \tau^n = \tau \times \tau \times \ldots \times \tau)\) where \(\tau\) is the countable complement extension topology of the real line with usual topology \((R, \mathcal{U})\), is \(gH\)-closed but \((R, \tau)\) is not so. We now give some sufficient conditions so that the product of two spaces is \(gH\)-closed.

**Theorem 4.7.** If the spaces \(X\) and \(Y\) satisfy any one of the following conditions then \(X \times Y\) is \(gH\)-closed.
(i) $X$ and $Y$ are connected non $H$-closed.
(ii) $X$ and $Y$ are $H$-closed.
(iii) $X$ is $gH$-closed and $Y$ is connected $H$-closed (both $X$ and $Y$ are almost regular).
(iv) $X$ and $Y$ are connected $gH$-closed spaces.
(v) $X$ is connected, non $H$-closed $gH$-closed space and $Y$ is connected (both $X$ and $Y$ are almost regular).

**Proof.**

(i) Follows from Theorem 4.1.
(ii) By proposition 4.8L [12], $X \times Y$ is $H$-closed and hence is $gH$-closed, by Remark 2.2.
(iii) Follows from Theorem 4.5, (a) $\Rightarrow$ (b).
(iv) If $X$ or $Y$ is $H$-closed, this follows (iii). If $X$ and $Y$ are both non $H$-closed, then by Theorem 4.1, $X \times Y$ is $gH$-closed.
(v) If $Y$ is $H$-closed, then proof follows from (iii). If $Y$ is not $H$-closed, then the proof follows from Theorem 4.1.

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