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A GENERALIZATION OF H-CLOSED SPACES

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Abstract. Whereas a space X can be embedded in a compact space if and only if it is Tychonoff, every space X can be embedded in an H-closed space(a generalization of compact space). In this paper, we further generalize, the concept of H-closedness into gH-closedness and have shown that every connected space is either a gH-closed space or can be embedded in a gH-closed space. Also, in a locally connected regular space the concept of gH-closedness is equivalent to the concepts of J-ness and strong J-ness due to E. Michael [7] and θJ-ness due to C.K. Basu et. al [1]. Several characterizations and properties of gH-closed spaces with respect to subspaces, products and functional preservations (along with various examples) are given.

1. Introduction

The concept of an H-set (a generalization of an H-closed space) was initiated by N. Veličko [15]. Since then H-sets played a major role in the development of the theory of H-closed spaces, locally H-closed spaces [10] although the exact relationship between H-sets and H-closed subspaces is as yet unknown. Indeed, unlike compactness, H-closure is not an absolute property.

Attempts have been made to use such H-sets in place of compact sets as is in the case of strong J-spaces due to E. Michael [7], to initiate a new class of spaces called gH-closed spaces. In what follows, attention will be focused upon gH-closed spaces because of the fact that not only every H-closed space is gH-closed (shown in section 2) but also every connected space is either a gH-closed space or can be embedded in a gH-closed space (shown in section 4). This result may make a new insight in investigating connected non H-closed (non compact as well) spaces. Several characterizations and properties of gH-closed spaces analogous to strong J-spaces due to E. Michael [7] have been achieved.

All the spaces considered herein are assumed to be Hausdorff. We assume that the reader is familiar with the concepts of H-closedness, H-sets, θ-closed sets and θ-continuity; [12, 16] might very well serve as the necessary background. The θ-closure of a subset A of a space X is the set $[A]_\theta \equiv \{x \in X : \overline{U} \cap A \neq \emptyset$ for all open sets U containing $x\}$. A subset A is θ-closed if $A = [A]_\theta$ and the complement of a θ-closed set is a θ-open set; a subset which is both θ-open as well as θ-closed is called θ-clopen. The θ-boundary [1] of a subset A of X (θ-bd A, for short) is defined as $[A]_\theta \cap [X - A]_\theta$.

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The other concepts needed for investigation of gH-closed spaces are: almost regularity, θ-connectedness, θ-rigidity and θ-perfectness. A space (X,τ) is said to be almost regular [13, 8] iff $\tau_s = \tau_\theta$ (where τ_s, τ_θ are respectively the semi-regularization topology and the θ-topology). Every regular space is almost regular but not conversely. A pair (P,Q) of non-empty subsets of X is called a θ-separation relative to X iff $(P \cap [Q]_\theta) \cup (Q \cap [P]_\theta) = \emptyset$; a subset A is called θ-connected [2] iff A is not the union of P and Q where (P,Q) is a θ-separation relative to X. Clearly every connected set is θ-connected but the converse is not true and in a regular space these two concepts coincide. A subset A of X is θ-rigid [3] iff for each open cover $\mathscr{U}$ of A, there is a finite subfamily $\{U_1, U_2, \ldots, U_n\}$ of $\mathscr{U}$ such that $A \subseteq int(\overline{\bigcup_{i=1}^{n} U_i})$; in a Hausdorff space, every θ-rigid set is an H-set [3]. A filter $\mathscr{F}$ in X almost converges [3] to a subset A (written $\mathscr{F} \hookrightarrow A$) if for each open cover $\mathscr{A}$ of A, there is a finite subfamily $\mathscr{B} \subseteq \mathscr{A}$ such that $\cup\{clV : V \in \mathscr{B}\} \in \mathscr{F}$. A function $f : X \to Y$ is θ-perfect [3] iff for every filter base $\mathscr{F}$ on $f(X)$, $\mathscr{F} \hookrightarrow y$ implies $f^{-1}(\mathscr{F}) \hookrightarrow f^{-1}(y)$.

Some results from literature are cited below:
♯**1.1**. If $f : X \to Y$ is θ-continuous surjective and X is H-closed then Y is H-closed [16].
♯**1.2**. If $f : X \to Y$ is θ-continuous, when $A \subset X$ is an H-set, then $f(A)$ is an H-set in Y [16].
♯**1.3**. An almost regular T_2 space is Urysohn [13].
♯**1.4**. An H-set in a Urysohn space is θ-closed [14].
♯**1.5**. Let $f : X \to Y$ be a function, where X and Y are almost regular spaces. Then f is θ-continuous iff inverse image of every θ-open (resp. θ-closed) set of Y is θ-open (resp. θ-closed) in X [5].
♯**1.6**. A θ-closed subset of an H-closed space is an H-set [3].
♯**1.7**. A space X is H-closed iff for every space Y, the projection map from $X \times Y$ onto Y takes a θ-closed subset onto a θ-closed subset [6].
♯**1.8**. If $A \subset X$ is an H-set of X and $X \subset Y$, then A is an H-set in Y [16].
♯**1.9**. Let B be a regular closed subset of a T_2 space X. If $A \subset X$ is an H-set and $B \subset A$ then B is H-closed [16].
♯**1.10**. For any subset $A \subset X$, $[A]_\theta$ is θ-closed if X is almost regular [5].
♯**1.11**. In an almost regular space X, every regular closed (resp. every regular open) subset is θ-closed (resp. θ-open) [5].
♯**1.12**. If $f : X \to Y$ is θ-continuous, the mapping $f : X \to f(X)$ need not be θ-continuous (even if $f(X)$ is a regular subspace of Y)[15].
♯**1.13**. If $f : X \to Y$ is θ-continuous and $f(X) \subset Z \subset Y$ and Z is dense in Y, then $f : X \to Z$ is θ-continuous [16].
♯**1.14**. If $f : X \to Y$ is θ-continuous and $A \subset X$, then $f/A : A \to Y$ is θ-continuous [16].
♯**1.15**. If Y is an H-set in X and A is a θ-closed subset of X then A is an H-set if $A \subset Y$ [14].
♯**1.16**. If $A \subset Y \subset X$, A is θ-open in Y and Y is θ-open in X, then A is θ-open in X [14].
♯**1.17**. If $A \subset Y \subset X$ and Y is θ-open in X; A is θ-closed in Y, then $A = F \cap Y$, where F is θ-closed in X [14].
♯**1.18**. Let Y be an open subset of X and X be almost regular; then Y is almost regular [5].
♯**1.19**. In an H-closed Urysohn space, every H-set is θ-closed and every θ-closed set is an H-set [3].

♯**1.20**. Let $f : X \to Y$ be a θ-continuous θ-bd perfect map from an almost regular space X onto a Urysohn space Y. If X is a θJ-space then so is Y [1].

2. gH-closed spaces.

Definition 2.1. A space X is called a generalized H-closed space (gH-closed, for short) if for every H-set H of X, there is an H-set K in X such that $H \subset K$ and $X - K$ is θ-connected relative to X.

Remark 2.2. Every H-closed space is a gH-closed space. But the converse is not true.

Examples of gH-closed spaces that are not H-closed.

Example 2.3. On $R^+ = [0,\infty)$, let us consider τ, the countable complement extension topology of the usual topology $\mathscr{U}$ on $R^+ = [0,\infty)$. Then as $\tau_s = \tau_\theta = \mathscr{U}$, the space (R^+,τ) is almost regular and Hausdorff. It can be checked that this space is non-regular, non H-closed and non locally connected. In this space (R^+,τ), let H be an H-set and hence H is compact in $(R^+,\mathscr{U})$. Therefore, $h = supH$ exists. Obviously, $H \subset [0,h]$ and $R^+ - [0,h] = (h,\infty)$ is connected in $(R^+,\mathscr{U})$. As (h,∞) is θ-open in (R^+,τ) and is connected in (R^+,τ_θ), by proposition 3 [8], (h,∞) is θ-connected relative to (R^+,τ). Clearly $[0,h]$ is an H-set in (R^+,τ). Hence (R^+,τ) is a gH-closed space.

Example 2.4. Let τ be the countable complement extension topology of the real line $(R,\mathscr{U})$. Then as $\tau_s = \tau_\theta = \mathscr{U}$, the space (R,τ) is almost regular. Easy verification shows that this space is connected but not H-closed; also (R,τ) is not gH-closed because of the Theorem 2.16 (given latter) and it is not θJ [Example 2.4, 1]. Therefore, by Theorem 4.1(given latter), (R^n,τ^n), $n > 1$ is a gH-closed space. Obviously (R^n,τ^n) is not H-closed; otherwise, (R,τ) would be H-closed — a contradiction.

Lemma 2.5 [1]. If $Y \subset X$ and B is θ-closed (θ-open) in (X,τ) then $B \cap Y$ is θ-closed (θ-open) in (Y,τ_Y), where τ_Y is the subspace topology on Y.

Lemma 2.6 [14]. If $A \subset Y \subset X$, and A is θ-open in Y and Y is a θ-open in X then A is θ-open in X.

But for θ-closed sets, the lemma 2.6 does not hold.

Example 2.7.([1]) Let τ be the countable complement extension topology of the real line $(R,\mathscr{U})$. Now, let $Y = \{0,1,\frac{1}{2},\frac{1}{3},\cdots\}$, obviously $\{0\}$ is open and closed in (Y,τ_Y) and Y is θ-closed in (R,τ) as Y is closed in $(R,\mathscr{U})$. Hence $\overline{\{0\}} \cap \{1,\frac{1}{2},\frac{1}{3},\cdots\} = \emptyset$, implying that $A = \{1,\frac{1}{2},\frac{1}{3},\cdots\}$ is θ-closed in (Y,τ_Y) (where the closure is taken in the subspace topology) but A is not θ-closed in (R,τ), since A is not closed in $(R,\mathscr{U})$.

Lemma 2.8.([1]) *A subset H of an almost regular space X is an H-set, iff every θ-open cover of A has a finite subcover.*

It is well known that every H-closed subspace is an H-set but example exists in [15], which shows that the converse is not true in general. Now we give the following characterization theorem for gH-closed spaces.

Theorem 2.9. *Let $\{H_1, H_2\}$ be a θ-clopen cover of an almost regular space X, with $H_1 \cap H_2$ an H-set. Then the following are equivalent:*

(i) *X is gH-closed.*
(ii) *H_1 and H_2 are gH-closed, and H_1 or H_2 is an H-set.*

Proof. (i)⇒(ii) Let H_1 be an H-set. Then obviously H_1 is regular closed and hence H_1 is an H-closed subspace of X (by result 1.9). Therefore, by Remark 2.2, H_1 is a gH-closed subspace of X. To show H_2 is a gH-closed subspace of X, let $K_2 \subset H_2$ be an H-set in the subspace H_2. Then, by result 1.8, K_2 is an H-set in X. So, $H = K_2 \cup H_1$ is an H-set in X. Since X is gH-closed, there exists some H-set L in X with $H \subset L$ and $X - L$ is θ-connected. Let $L_2 = L \cap H_2$. Then $K_2 \subset L_2$ and L_2 is not only a θ-closed set in H_2 but also in X (by result 1.3 and 1.4). In addition, $L_2 \subset H_2$ is an H-set in X (by result 1.15). Since H_2 is θ-open and hence is almost regular (by result 1.18), then by lemma 2.8, it can be easily shown that L_2 is an H-set in subspace H_2. Since $H_2 - L_2 = X - L$ then $H_2 - L_2$ is θ-connected in X. Again as H_2 is θ-open, $H_2 - L_2$ is θ-connected in the subspace H_2. Therefore, H_2 is a gH-closed space.

(ii)⇒ (i) Suppose (ii) holds and also suppose H_1 be an H-set. Let $H \subset X$ be an H-set. Then $K = (H \cup H_1) \cap H_2$ is an H-set in X such that $K \subset H_2$. Since H_2 is θ-open, by lemma 2.8 and results 1.16, 1.18, we can easily prove that K is an H-set in H_2. The gH-closedness of the subspace H_2 implies that there exists some H-set L in the subspace H_2 such that $K \subset L$ and $H_2 - L$ is θ-connected in H_2. The set $L^\star = L \cup H_1$ is an H-set containing H and one can check that $X - L^\star = H_2 - L$ is θ-connected in X. Therefore X is a gH-closed space.

Corollary 2.10. *If A is a θ-clopen subset of an almost regular gH-closed space X then A is a gH-closed space.*

Remark 2.11. (a) In [1] we considered a kind of spaces termed θ-J spaces (see Definition 2.15) which satisfy a condition weaker than that given in (ii) of Theorem 2.9. Indeed, we shall show shortly (see Theorem 2.16) that every gH-closed space is a θ-J space for which we need not assume the condition of almost regularity on the underlying space.

(b) The condition in the above corollary is not necessary. In the space (R^+, τ) in Example 2.3, $Y = [0, 1]$ with the subspace topology τ_Y is H-closed and hence is a gH-closed subspace. Although Y is θ-closed in (R^+, τ) but because of the fact that Y is not open in $(R^+, \mathscr{U})$, it is not θ-open in (R^+, τ).

On the otherhand, union of even two θ-closed gH-closed subspaces may not be gH-closed. In Example 2.4, $R^+ = \{x \in R : x \geq 0\}$ and $R^- = \{x \in X : x \leq 0\}$ are both θ-closed gH-closed subspaces but their union (R, τ) is not so. But we have the following corollary.

Corollary 2.12. If an almost regular space X be such that $X = H_1 \cup H_2$ with H_1 an open gH-closed subspace, H_2 a θ-clopen H-set, then X is a gH-closed space.

Proof. Let $A = X - H_2$. Then A is θ-clopen subset in the subspace H_1. By result 1.18 H_1 is almost regular (being an open subspace of an almost regular space X) and as H_1 is a gH-closed space, the Corollary 2.10 implies A is a gH-closed subspace. Again H_2 is being a regular closed H-set in X, by result 1.9, H_2 is an H-closed subspace of X and hence is a gH-closed subspace in X (by Remark 2.2). Since $\{A, H_2\}$ is a θ-clopen cover of X, A and H_2 are gH-closed subspaces and H_2 is an H-set, by Theorem 2.9, X is a gH-closed space.

Theorem 2.13. Let X be an almost regular gH-closed space. Then every regular closed set B which is a union of θ-components of X [2] is gH-closed.

Proof. Let $A \subset B$ be an H-set in the subspace B. Then by result 1.8, A is an H-set in X and hence the gH-closedness of X implies the existence of an H-set H(in X) such that $A \subset H$ and $X - H$ is θ-connected relative to X. Now, if $B \subset H$, then because of X is almost regular, B is a θ-closed set (by result 1.11) contained in an H-set H and hence (by result 1.15) B is an H-set. In fact, B is a H-closed subspace of X (by result 1.9). So, by Remark 2.2, B is gH-closed. If $B \not\subset H$, then the θ-connected set $X - H$ intersects B; but as B is a union of θ-components of X, the only possibility is $X - H \subset B$. Let $H_1 = H \cap B$. Using results 1.3, 1.4 and 1.15, it can be shown that H_1 is an H-set in X. Obviously $A \subset H_1$ and also $B - H_1 = X - H$ is θ-connected relative to X. Since $X - H$ is a θ-open set (in X) contained in B, then by lemma 2.5, $X - H$ is θ-open in the subspace B. If (P, Q) is any θ-separation of the θ-open set $X - H$ in the subspace B, such that $X - H = P \cup Q$ then by [8, Proposition 1], P, Q are θ-open sets in B. Since the θ-closure and the closure of an open set are equal and B is closed in X, (P, Q) is therefore a separation of the connected set $X - H$ (because an open subset of (X, τ) is connected iff it is θ-connected relative to X [8, Proposition 2]) — a contradiction. So, $B - H_1 = X - H$ is θ-connected in the subspace B and hence B is a gH-closed subspace of X.

Corollary 2.14. *Every regular closed subset which is a union of θ-components of an almost regular H-closed space is gH-closed.*

Definition 2.15. A space X is a θJ-space [1] if whenever $\{H_1, H_2\}$ is a θ-closed cover of X with $H_1 \cap H_2$ an H-set, H_1 or H_2 is an H-set.

Theorem 2.16. *Every gH-closed space is a θJ-space.*

Proof. Suppose (X, τ) is a gH-closed space. Let $\{H_1, H_2\}$ be a θ-closed cover of X with $H_1 \cap H_2$ an H-set. As (X, τ) is gH-closed and $H_1 \cap H_2 \subset X$ is an H-set, so $H_1 \cap H_2 \subset K$ for some H-set K of (X, τ) with $X - K$ θ-connected. By lemma 2.5, $(H_1 \cap (X - K), H_2 \cap (X - K))$ is a disjoint θ-closed cover of the θ-connected set $X - K$. So, either $H_1 \cap (X - K) = \emptyset$ or $H_2 \cap (X - K) = \emptyset$ implying either $H_1 \subset K$ or $H_2 \subset K$. But H_1 or H_2 is a θ-closed subset contained in an H-set K. So, by result 1.15, either H_1 or H_2 is an H-set. Therefore (X, τ) is a θJ-space.

We shall show that in a locally connected almost regular space, these two concepts are equivalent. For this we first state a theorem.

Theorem 2.17.([1]) *An almost regular space X is a θJ-space iff whenever $H \subset X$ is an H-set and $\mathcal{U}$ is a disjoint θ-open cover of $X - H$, then $X - U$ is an H-set for some $U \in \mathcal{U}$.*

In view of the following proposition due to Mrševic et al i.e.

Proposition 2.18.([8]) *In* (X,τ), *the conditions* (i) — (iii) *below are equivalent: every point has a neighbourhood basis in* (X,τ) *consisting of:*

(i) *connected neighbourhoods.*
(ii) δ*-connected neighbourhoods.*
(iii) *neighbourhoods* θ*-connected relative to* X.

We have the following theorem:

Theorem 2.19. A locally connected almost regular space is a θJ-space iff it is a gH-closed space.

Proof. Every gH-closed space is a θJ-space (by Theorem 2.16).

Let X be a locally connected θJ-space and $H \subset X$ be an H-set. As X is almost regular T_2, X is not only a Urysohn space but also every H-set of X is θ-closed; further for any open neighbourhood U of x there exists a θ-open set V of x such that $V \subset \overline{U}$. Indeed, by definition of almost regularity, there exists, an open set W such that $x \in W \subset \overline{W} \subset int\overline{U} \subset \overline{U}$; but in the almost regular space X, the regular open set $V = int\overline{U}$ is θ-open. Because of X is locally connected Urysohn and H is θ closed in X, it can be easily shown that, there is a disjoint θ-open cover $\mathscr{U}$ of $X - H$ with each $U \in \mathscr{U}$ θ-connected. Since X is θJ-space by Theorem 2.17, there exists a $U^\star \in \mathscr{U}$ such that $X - U^\star$ is an H-set. If we take $K = X - U^\star$, then $H \subset K$ and $X - K = U^\star$ is θ-connected. So, X is a gH-closed space.

3. Preservation of gH-closedness in terms of θ-perfect (θ-bd perfect) functions.

Definition 3.1.([1]) A map $f : X \to Y$ is called θ-boundary perfect(θ-bd perfect, for short) if f is almost closed [3] (i.e. $f([A]_\theta) = [f(A)]_\theta, \forall A \subset X$) and θ-bd$f^{-1}(y)$ is θ-rigid [3] for every $y \in Y$.

Theorem 3.2. *For an almost regular space* X *if*

(i) X *is* gH*-closed space then*
(ii) *every* θ*-continuous* θ*-bd perfect map* $f : X \to Y$ *onto a non* H*-closed Urysohn space* Y *is* θ*-perfect.*

Proof. Let $f : X \to Y$ be a θ-continuous θ-bd perfect map and let $y \in Y$. Since θ-bd$f^{-1}(y)$ is a θ-rigid set and hence is an H-set (by remark after Corollary 6.3, [3]), then as X is a gH-closed space there exist an H-set K such that $[f^{-1}(y)]_\theta \cap [X - f^{-1}(y)]_\theta =$ θ-bd$f^{-1}(y) \subset K$ and $X - K$ is θ-connected. But $\{[f^{-1}(y)]_\theta \cap (X-K), [X - f^{-1}(y)]_\theta \cap (X-K)\}$ is a θ-separation of $(X - K)$. So, either $[f^{-1}(y)]_\theta \subset K$ or $[X - f^{-1}(y)]_\theta \subset K$. But, by results 1.10 and 1.15, either $[f^{-1}(y)]_\theta$ or $[X - f^{-1}(y)]_\theta$ is an H-set. Now as f is θ-continuous and $\{y\}$ is a θ-closed set being an H-set in the Urysohn space Y (by result 1.4), $f^{-1}(y)$ is θ-closed i.e. $f^{-1}(y) = [f^{-1}(y)]_\theta$. Since $[X - f^{-1}(y)]_\theta$ is an non-H-set otherwise Y would be H-closed, so by Theorem 3.4 [3], f is θ-perfect.

Remark 3.3. If Y is H-closed then the above theorem fails. In fact, if $f : X \to Y$, where Y is a singleton and X is gH-closed but not H-closed then f is a θ-continuous θ-bd perfect map onto a Urysohn space Y. But f is not θ-perfect as $f^{-1}(y)$ is not a θ-rigid set.

Theorem 3.4. Let $f : X \to Y$ be a θ-continuous, θ-bd perfect θ-open map (i.e. maps θ-open sets into θ-open sets) from an almost regular space X onto a Urysohn space Y. Then Y is a gH-closed space if X is so.

Proof. If Y is H-closed then by Remark 2.2, Y is gH-closed. Suppose Y is not an H-closed space. Then by above Theorem 3.2, f is θ-perfect. Let $H \subset Y$ be an H-set then $H_1 = f^{-1}(H) \subset X$ is an H-set, by Corollary 3.1.1.(c) [3]; but the gH-closedness of X implies the existence of an H-set $K_1 \subset X$ such that $H_1 \subset K_1$ and $X - K_1$ is θ-connected. Since f is θ-continuous, by Theorem 2.7 [2], $f(X - K_1)$ is θ-connected. Obviously, $H_2 = Y - f(X - K_1)$ is θ-closed and f being θ-continuous, the θ-closed set $f^{-1}(H_2) \subset K_1$ is an H-set in X by result 1.15. Since f is θ-continuous, by result 1.2, H_2 is an H-set in Y such that $H \subset H_2$. But $Y - H_2 = f(X - K_1)$ is θ-connected. So, Y is a gH-closed space.

Theorem 3.5. Let $f : X \to Y$ (where X and Y are almost regular spaces) be a θ-continuous, θ-perfect map onto Y. Then, if Y is a gH-closed space, so is X.

Proof. Let Y be gH-closed and let $H \subset X$ be an H-set in X. Since f is θ-continuous, by result 1.2, $f(H) \subset Y$ is an H-set. Because of gH-closedness of Y, there is an H-set K in Y such that $f(H) \subset K$ with $Y - K$ θ-connected. Hence $H \subset f^{-1}(K)$ and since f is θ-perfect, by Corollary 3.1.1(c) [3], $f^{-1}(K)$ is an H-set in X. We shall show that $X - f^{-1}(K) = f^{-1}(Y - K)$ is θ-connected. If, $\{B_1, B_2\}$ is a θ-separation relative to X of the θ-open set $f^{-1}(Y - K)$ (since f is θ-continuous and $Y - K$ is θ-open such that $f^{-1}(Y - K) = B_1 \cup B_2$). Then by proposition 1 [8], B_1 and B_2 are disjoint θ-open sets in X. Since f is θ-perfect, by Corollary 3.1.1(b) [3], for each θ-closed set A of X, $f(A)$ is θ-closed. From this, one can easily verify that the sets $V_i = \{y \in Y - K : f^{-1}(y) \subset B_i\}$ for $i = 1, 2$ are disjoint θ-open sets. Therefore, $Y - K$, is not θ-connected — a contradiction. So, $f^{-1}(Y - K)$ is θ-connected i.e. $X - f^{-1}(K)$ is θ-connected. Hence X is a gH-closed space.

Corollary 3.6. *An almost regular space (X, τ) is gH-closed iff (X, τ_s) is gH-closed.*

Proof. Since the θ-closure of a subset A in (X, τ) is the same as the θ-closure of A in (X, τ_s), the identity map $i_1 : (X, \tau_s) \to (X, \tau)$ is θ-continuous and almost closed; also point inverses are θ-rigid sets. Hence by Theorem 3.4 [4], the identity map $i_1 : (X, \tau_s) \to (X, \tau)$ is θ-perfect. So, if (X, τ) is gH-closed then by above Theorem 3.5, (X, τ_s) is gH-closed. On the otherhand, by the same reason, the identity map $i_2 : (X, \tau) \to (X, \tau_s)$ is θ-perfect and θ-continuous. So, if (X, τ_s) is gH-closed then by Theorem 3.5, (X, τ) is gH-closed.

Theorem 3.7. *For an almost regular locally connected space (X, τ), the following are equivalent:*

(i) (X, τ) *is gH-closed.*
(ii) (X, τ_s) *is gH-closed.*

(iii) (X, τ_θ) *is* gH*-closed.*
(iv) (X, τ) *is* θJ*.*
(v) (X, τ_s) *is* θJ*.*
(vi) (X, τ_θ) *is* θJ*.*

Proof. The proof follows from Corollary 3.6, Theorem 2.19 and from the fact that a space (X, τ) is almost regular iff $\tau_s = \tau_\theta$.

Definition 3.8. A space (X, τ) is a J-space (resp. strong J-space) [7] if, whenever $\{A, B\}$ is a closed cover of X with $A \cap B$ compact, then A or B is compact (resp. if every compact set $K \subset X$ is contained in a compact set $L \subset X$ with $X - L$ connected).

Theorem 3.9. *For a regular locally connected space* (X, τ) *the following are equivalent:*

(i) (X, τ) *is* gH*-closed.*
(ii) (X, τ) *is* J*.*
(iii) (X, τ) *is* θJ*.*
(iv) (X, τ) *strong* J*.*

4. gH-closedness in products.

Theorem 4.1. *If* X_1*,* X_2 *are connected and non-*H*-closed spaces then* $X_1 \times X_2$ *is a* gH*-closed space.*

Proof. Let H be an H-set in the product space $X = X_1 \times X_2$ and let $\pi_i : X \to X_i$, $i = 1, 2$ are projection maps. Since π_i's are continuous and hence are θ-continuous then by result 1.2, $\pi_i(H) = H_i$, $i = 1, 2$ are H-sets in X_i, $i = 1, 2$ respectively. Obviously $H \subset H_1 \times H_2$, where $H_1 \times H_2$ is an H-set, by result [12, Theorem 4.8L].

But $X_i - H_i \neq \emptyset$, $i = 1, 2$ otherwise X_i's ,$i = 1, 2$ would have been H-closed. Let $x_i \in X_i - H_i$, $i = 1, 2$. Now the set $B = (\{x_1\} \times X_2) \cup (X_1 \times \{x_2\})$ being a union of two intersecting connected and hence θ-connected sets is θ-connected and also $B \subset X - (H_1 \times H_2)$. But $X - (H_1 \times H_2)$ is the union of θ-connected sets of the form $\{x_1\} \times X_2$ with $x_1 \in X_1$ or $X_1 \times \{x_2\}$ with $x_2 \in X_2$ and all of which intersects B is thus θ-connected. Therefore, $X_1 \times X_2$ is a gH-closed space.

Corollary 4.2. *Any connected space is either a* gH*-closed space or can be embedded in a* gH*-closed space.*

Proof. Let X be a connected space. If it is H-closed then by Remark 2.2, X is gH-closed. If X is not an H-closed space, then by Theorem 4.1, $X \times X$ is a gH-closed space. But as X is homeomorphic to some subspace of $X \times X$, the proof follows immediately.

Corollary 4.3. R^n*,* $n > 1$*, where* R *is the real line with the usual topology is a* gH*-closed space.*

Example 4.4. Example of a gH-closed space which is not an H-closed space.

Let $(R,\mathscr{U})$ be the real line with usual topology . Then by above Corollary 4.3, R^n, $n>1$ is a gH-closed space. But R^n is not an H-closed space.

Theorem 4.5. *The following are equivalent for any locally connected almost regular space X.*

(a) *X is a gH-closed space.*
(b) *$X \times Y$ is a gH-closed space for every connected H-closed almost regular space Y.*
(c) *$X \times Y$ is a gH-closed space for some H-closed almost regular space Y.*

Proof. For proving (a)⇒ (b) and (b)⇒ (c), the assumption of local connectedness of X is not needed. Let us prove:

(a)⇒ (b). Let $p_X : X \times Y \to X$ be the projection map. Now, $p_X^{-1}(x) = \{x\} \times Y$ is an H-set in the almost regular space $X \times Y$ (as the product of any family of almost regular space is almost regular [3, Theorem 5.1]) and therefore is a θ-rigid set in $X \times Y$, by Theorem 6.4 [3]. We shall next show that for each $A \subset X \times Y$, $[p_X(A)]_\theta \subset p_X([A]_\theta)$. For this, we first show for each θ-closed set B of $X \times Y$, $p_X(B)$ is θ-closed. Indeed, if $x' \in X - p_X(B)$; then $(\{x'\} \times Y) \cap B = \emptyset$. Therefore, for each point (x', y), has a open neighbourhood $V_y(x') \times V(y)$ such that $(\overline{V_y(x')} \times \overline{V(y)}) \cap B = \emptyset$. Since $\{x'\} \times Y$ is an H-set we can select $V_{y_i}(x') \times V(y_i)$, $i = 1,2,\ldots,n$ such that the union of the closures of such subfamily covers $\{x'\} \times Y$. Then $\bigcap_{i=1}^{n} V_{y_i}(x')$ is a open neighbourhood of x' in X such that $\bigcap_{i=1}^{n} V_{y_i}(x') \bigcap p_X(B) = \emptyset$. Therefore, $p_X(B)$ is θ-closed.

Now, $A \subset X \times Y$, $[A]_\theta$ is θ-closed in $X \times Y$ (by result 1.10) and hence $p_X([A]_\theta)$ is θ-closed. But we always have $p_X(A) \subset p_X([A]_\theta)$. So $[p_X(A)]_\theta \subset p_X([A]_\theta)$. Therefore by Corollary 3.4.1 [3], p_X is θ-perfect. Since p_X is continuous and hence is θ-continuous, by Theorem 3.5, $X \times Y$ is gH-closed.

(b)⇒(c) is obvious.

(c)⇒ (a) If X is H-closed then by Remark 2.2, X is gH-closed. Suppose X is non H-closed then for the projection map $p_X : X \times Y \to X$ (where Y is some H-closed space), we have $[p_X(A)]_\theta \subset p_X([A]_\theta)$ for every $A \subset X \times Y$. Since p_X is θ-continuous, by Corollary 2.10.1 [3], $p_X([A]_\theta) \subset [p_X(A)]_\theta$. Therefore, p_X is a almost closed map. Again, $p_X^{-1}(x)$ is an H-set in the almost regular space $X \times Y$. So, $p_X^{-1}(x)$ is θ-closed. Now, as θ-bd$p_X^{-1}(x)$ is a θ-closed subset of the H-set $p_X^{-1}(x)$, then by result 1.15, θ-bd$p_X^{-1}(x)$ is an H-set in the almost regular space $X \times Y$ and hence is θ-rigid so p_X is a θ-bd perfect mapping. Since $X \times Y$ is θJ (by Theorem 2.16) then by result 1.20, X is θJ. Because of X is locally connected, X is a gH-closed space by Theorem 3.7.

Remark 4.6. In Example 2.4, we have seen the product space $(R^n, \tau^n = \tau \times \tau \times \ldots \times \tau)$ where τ is the countable complement extension topology of the real line with usual topology $(R,\mathscr{U})$, is gH-closed but (R,τ) is not so. We now give some sufficient conditions so that the product of two spaces is gH-closed.

Theorem 4.7. *If the spaces X and Y satisfy any one of the following conditions then $X \times Y$ is gH-closed.*

(i) *X and Y are connected non H-closed.*
(ii) *X and Y are H-closed.*
(iii) *X is gH-closed and Y is connected H-closed (both X and Y are almost regular).*
(iv) *X and Y are connected gH-closed spaces.*
(v) *X is connected, non H-closed gH-closed space and Y is connected (both X and Y are almost regular).*

Proof.
(i) Follows from Theorem 4.1.
(ii) By proposition 4.8L [12], $X \times Y$ is H-closed and hence is gH-closed, by Remark 2.2.
(iii) Follows from Theorem 4.5, (a)$\Rightarrow$ (b).
(iv) If X or Y is H-closed, this follows (iii). If X and Y are both non H-closed, then by Theorem 4.1, $X \times Y$ is gH-closed.
(v) If Y is H-closed, then proof follows from (iii). If Y is not H-closed, then the proof follows from Theorem 4.1.

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