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A COUPLED HYBRID FIXED POINT THEOREM FOR SUM OF TWO MIXED MONOTONE COUPLED OPERATORS IN A PARTIALLY ORDERED BANACH SPACE WITH APPLICATIONS

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Abstract. In this paper we prove a coupled hybrid fixed point theorem involving the sum of two coupled operators in a partially ordered Banach space and apply to a pair of nonlinear second order coupled linearly perturbed hybrid differential equations with the periodic boundary conditions for proving the existence and approximation of coupled solutions under certain mixed hybrid conditions. The abstract existence result of the coupled periodic boundary value problems is also illustrated by furnishing a numerical example.

1. Introduction

Let $(X, \preceq, d)$ denote a partially ordered metric space with the partial order relation $\preceq$ and the metric d defined on X and let $(E, \preceq, \|\cdot\|)$ denote a partially ordered Banach space with order relation $\preceq$ and the norm $\|\cdot\|$ defined in it. Given a mapping $\mathscr{T} : X \times X \to X$, consider a pair of mapping equations

$$x = \mathscr{T}(x, y) \tag{1.1}$$

and

$$y = \mathscr{T}(y, x) \tag{1.2}$$

which are called the coupled mapping equations and the mapping $\mathscr{F}$ involved in them is called a *coupled mapping* on $X \times X$.

A pair (x^*, y^*) of elements in $X \times X$ is called a *coupled fixed point* of the coupled mapping $\mathscr{T}$ or a *coupled solution* of the coupled mapping equations (1.1) and (1.2) if

$$x^* = \mathscr{T}(x^*, y^*) \quad \text{and} \quad y^* = \mathscr{T}(y^*, x^*).$$

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A coupled fixed point (x^*, y^*) is called *unique comparable* if there does not exist another coupled fixed point (u^*, v^*) which is comparable to it. A coupled fixed point (x^*, y^*) is called *unique* if it is the only coupled solution of the coupled mapping equations (1.1) and (1.2) in $X \times X$. Finally, a point (x^*, y^*) is called a *fixed point* if $x^* = y^*$, i.e., $x^* = \mathscr{T}(x^*, x^*)$.

The coupled fixed point theorems for mixed monotone operators using the properties of the cones in an ordered Banach space have been proved by Chang and Ma [7], Sun [33] and Nistri *et.al* [31]. But the coupled hybrid fixed point theorems for mixed monotone partially condensing coupled mappings on a partially ordered metric space guaranteeing the existence of coupled fixed points have been proved in Dhage [18] which includes the coupled fixed point theorems of Bhaskar and Laksmikatham [5], Berinde [4], Dhage and Dhage [25] and Dhage [16] as special cases. Bhaskar and Lakshmikathan [5] used a partial contraction type condition on the mixed monotone coupled mapping $\mathscr{T}$ which is further generalized by Berinde [4] by generalizing the partial contraction condition to symmetric partial contraction type condition to get the same conclusion via constructive method. See also Petrusel *et. al* [30] and references threin. However, Dhage [16] used a compactness type topological arguments on the mixed monotonic coupled mapping $\mathscr{T}$ and obtained an algorithm for the coupled solutions for the coupled mapping equations (1.1) and (1.2). Sometimes it may happen that the mixed monotone maping $\mathscr{T}$ on a patially ordered Banach space $E \times E$ neither satisfies condition of partial contraction condition nor the compactness type condition, but the splitting of the coupled operator $\mathscr{T}$ into two coupled operators $\mathscr{F}$ and $\mathscr{G}$ into the form $\mathscr{T} = \mathscr{F} + \mathscr{G}$ satisfy the above criteria. See Dhage [9, 10, 11, 12] and the references therein. So in this case it is interesting to establish the coupled hybrid fixed point theorems involving the sum of two operators in a partially ordered Banach space (cf. Dhage [13, 14, 15, 16]).

The rest of the paper is organized as follows: Section 2 deals with preliminaries and auxiliary results that will be used in the subsequent part of the paper. Section 3 consists of some basic results concerning regularity and Janhavi sets in a partially ordered metric space and the partial measure noncompactness along with a key coupled hybrid fixed point theorem is presented in Section 4. Section 5 contains our main coupled hybrid fixed point theorem for sum of two coupled operators and Section 5 contains PBVPs of second order differential equations. Finally, the application of our abstract hybrid fixed point theorem to coupled hybrid PBVPs is given in Section 7. We claim that the results of this paper are new to the literature on nonlinear analysis and applications.

2. Preliminaries and auxiliary results

In this section, we give some basic concepts and terminologies in a partially ordered metric space X with partial order $\preceq$ and the metric d on X. These concepts are also applicable to a partially ordered normed linear space and in an ordered Banach space because they are the examples of a partially ordered metric space. Two elements x and y in X are said to be *comparable* if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset C of X is called a *chain* or *totally ordered* if all the elements of C are comparable. It is known that X is *regular* if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in X and $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of X may be found in Guo and Lakshmikantham [28] and the references therein. Similarly a few details of a partially ordered normed linear space are given in Dhage [9] while orderings defined by different order cones are given in Deimling [8], Guo and Lakshmikantham [28], Heikkilä and Lakshmikantham [29], Carl and Heikkilä [6] and references therein.

We need the following definitions (see Dhage [12, 13, 14, 15, 16] and the references therein) in what follows.

A mapping $\mathcal{T} : X \to X$ is called *isotone* or *monotone nondecreasing* if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in X$. Similarly, $\mathcal{T}$ is called *monotone nonincreasing* if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in X$. Finally, $\mathcal{T}$ is called *monotonic* or simply *monotone* if it is either monotone nondecreasing or monotone nonincreasing on X. A mapping $\mathcal{T} : X \to X$ is called *partially continuous* at a point $a \in X$ if for given $\epsilon > 0$ there exists a $\delta > 0$ such that $d(\mathcal{T}x, \mathcal{T}a) < \epsilon$ whenever x is comparable to a and $d(x, a) < \delta$. $\mathcal{T}$ is called partially continuous on X if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on X, then it is continuous on every chain C contained in X and vice-versa. A non-empty subset S of the partially ordered metric space X is called *partially bounded* if every chain C in S is bounded. A mapping $\mathcal{T}$ on a partially ordered metric space X into itself is called *partially bounded* if $\mathcal{T}(X)$ is a partially bounded subset of X. $\mathcal{T}$ is called *uniformly partially bounded* if all chains C in $\mathcal{T}(X)$ are bounded by a unique constant. A non-empty subset S of the partially ordered metric space X is called *partially compact* if every chain C in S is a compact subset of X. A mapping $\mathcal{T} : X \to X$ is called *partially compact* if every chain C in $\mathcal{T}(X)$ is a compact subset of X. $\mathcal{T}$ is called *uniformly partially compact* if $\mathcal{T}$ is a uniformly partially bounded and partially compact operator on X. $\mathcal{T}$ is called *partially totally bounded* if for any bounded subset S of X, $\mathcal{T}(S)$ is a partially totally bounded subset of X. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called *partially completely continuous* on X.

Remark 2.1. Suppose that $\mathcal{T}$ is a nondecreasing operator on X into itself. Then $\mathcal{T}$ is a partially bounded or partially compact on X if $\mathcal{T}(C)$ is a bounded or relatively compact subset of X for each chain C in X.

Definition 2.1 (Dhage [13, 14], Dhage and Dhage [25])**.** The order relation $\leq$ and the metric d on a non-empty set X are said to be *$\mathscr{D}$-compatible* if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in X and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^* implies that the original sequence $\{x_n\}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \leq, \|\cdot\|)$, the order relation $\leq$ and the norm $\|\cdot\|$ are said to be $\mathscr{D}$-compatible if $\leq$ and the metric d defined through the norm $\|\cdot\|$ are $\mathscr{D}$-compatible. A subset S of X or E is called *Janhavi* if the order relation $\leq$ and the metric d or the norm $\|\cdot\|$ are $\mathscr{D}$-compatible in it. In particular, if $S = X$ or $S = E$, then X or E is called a *Janhavi metric* or *Janhavi Banach space.*

There do exist several examples of the regular and Janhavi metric spaces in the literature. In fact, every finite dimensional Euclidean space $\mathbb{R}^n$ is regular as well as Janhavi with respect to the usual componentwise order relation and the standard norm in $\mathbb{R}^n$. The following results are of fundamental importance concerning the regularity of a partially ordered Banach space and the Janhavi sets whereby it is possible to extend the utility or applicability of the abstract coupled hybrid fixed point theorems of this paper to the variety of nonlinear problems in a natural way.

We often need the concepts of regularity and janhavi sets in a partially ordered metric space X or Banach space E in the development of coupled hybrid fixed point theory and applications. In the following we obtain some basic results in this direction.

We recall that a non-empty closed and convex subset K of the Banach space E is called a *cone* if i) $K + K \subseteq K$, ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}$, $\lambda \geq 0$, and iii) $\{-K\} \bigcap K = \{\theta\}$, where θ is a zero element of E. The details of cones and their properties may be found in Guo and Lakshmikantham [28], Heikkilä and Lakshmikantham [29] and references therein. We define an order relation $\leq$ in the Banach space E by

$$x \leq y \iff y - x \in K \tag{2.1}$$

for all $x, y \in E$, where K is a cone in E. The Banach space E together with the order relation $\leq$ becomes a partially ordered or simply ordered Banach space and it is denoted by (E, K). We observe that every ordered Banach space (E, K) is not necessarily a Janhavi Banach space. The following two useful lemmas are recently proved in Dhage [19, 20] which play a crucial role in this connection. Since the proofs of these lemmas are not well-known, we give the details of them for completeness and ready reference.

Lemma 2.1 (Dhage [21, 22])**.** *Every ordered Banach space (E, K) is regular.*

Proof. Let $\{x_n\}$ be a monotone nondecreasing sequence of points in a partially ordered Banach space (E,K). Then,

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots . \tag{$*$}$$

Suppose that the sequence $\{x_n\}$ converges to a point x^*, that is, $x_n \to x^*$ as $n \to \infty$. Then, every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ also converges to the same limit point x^*, that is, $x_{n_k} \to x^*$ as $k \to \infty$. Since $\{x_n\}$ is nondecreasing, for any given positive integer n, we have $x_n \leq x_{n_k}$ for each $k \geq n \in \mathbb{N}$. This further by definition of the order relation $\leq$ implies that $x_{n_k} - x_n \in K$. As the cone K is closed and convex set in E, one has

$$\lim_{k\to\infty} \left(x_{n_k} - x_n \right) = x^* - x_n \in K$$

for each $n \in \mathbb{N}$. Therefore, $x_n \leq x^*$ for all $n \in \mathbb{N}$. Similarly, if $\{x_n\}$ is monotone noincreasing sequence of points in E, then using the similar arguments, it can be proved that $x^* \leq x_n$ for all $n \in \mathbb{N}$. As a result, (E,K) is a regular ordered Banach space and the proof of the lemma is complete. □

Lemma 2.2 (Dhage [21, 22])**.** *Every partially compact subset S of an ordered Banach space (E,K) is Janhavi.*

Proof. Let C be an arbitrary chain in a partially compact subset S of an ordered Banach space E. Then $C = \overline{C}$ is a compact set in E. Let $\{x_n\}$ be a monotone nondecreasing sequence of points in the chain C, that is,

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots . \tag{2.2}$$

Then $\{x_n\}$ is a relatively compact set in E. Therefore, $\{x_n\}$ has a convergent subsequence, say $\{x_{n_k}\}$ converging to a point x^*. We show that $\{x_n\}$ also converges to x^*. Suppose not. Then for $\epsilon > 0$ there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\|x_{n_i} - x^*\| \geq \epsilon \quad \text{for each } \ i = 1,2,\ldots. \tag{2.3}$$

Now, by relative compactness of $\{x_{n_i}\}$, there is a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_j}} \to x'$ as $j \to \infty$. Hence for any given positive integer k, by nondecreasing nature of $\{x_n\}$ it follows that when j is large enough $(j \geq k)$, we have that $x_{n_k} \leq x_{n_{i_j}}$. Then $x_{n_{i_j}} - x_{n_k} \in K$. As K is closed and convex, taking the limit first as $j \to \infty$ and then as $k \to \infty$, we obtain

$$x' - x^* \in K \quad \Longrightarrow \quad x^* \leq x'.$$

Similarly, it can be shown that $x' \leq x^*$. As a result, we have $x' = x^*$ and that $x_{n_{i_j}} \to x^*$ as $j \to \infty$. Therefore, we get

$$\|x_{n_{i_j}} - x^*\| < \epsilon \tag{2.4}$$

for large j. This is a contradiction to (2.3) and the proof of the lemma is complete. □

3. Regularity and Janhavi sets in product metric spaces

Next, we discuss some more information of the regularity and Janhavi sets in a partially ordered product metric space. We consider the following definitions in what follows.

Definition 3.1. A mapping $f:\mathbb{R}_+^n \to \mathbb{R}_+$ is called *sublinear* if

(i) $f(x+y) \le f(x)+f(y)$ (subadditivity), and
(ii) $f(\lambda x) = \lambda f(x)$ (homogeneity)

for all $x, y \in \mathbb{R}_+$ and $\lambda \in \mathbb{R}$, $\lambda \ge 0$.

Definition 3.2. A continuous mapping $f:\mathbb{R}_+^n \to \mathbb{R}_+$ is called *Kasu* if

(i) f is sublinear,
(ii) $f(r_1,\dots,r_n) = 0$ if and only if $x_i = 0$ for all i, $i = 1,2,\dots,n$, and
(iii) $f(r_1,\dots,r_n)$ is nondecreasing in each coordinate variable.

The class of Kasu functions on $\mathbb{R}_+^n$ is denoted by $\mathfrak{K}$.

Example 3.1. Define a mapping $f_s:\mathbb{R}_+^n \to \mathbb{R}_+$ by

$$f_s(r_1,\dots,r_n) = \sum_{i=1}^{n} a_i\, r_i,\ a_i > 0. \tag{3.1}$$

Then f_s is a Kasu function on $\mathbb{R}_+^n$.

Example 3.2. Let the mapping $f_m:\mathbb{R}_+^n \to \mathbb{R}_+$ be defined by

$$f_m(r_1,\dots,r_n) = a\max\{r_i,\dots,r_n\}, \tag{3.2}$$

where $a > 0$. Then f_m is a Kasu function on $\mathbb{R}_+^n$.

Proposition 3.1. *Let $d_1,\dots,d_n$ be the metrics on n metric spaces $X_1,\dots,X_n$ respectively and let $X = X_1\times\cdots\times X_n$. Then the function $d : X\times X \to \mathbb{R}_+$ defined by*

$$d = f\big(d_1,\dots,d_n\big) \tag{3.3}$$

is a metric on X and so (X,d) is a metric space, where $f \in \mathfrak{K}$.

Proof. We shall show that the function d satisfies all the properties of a metric on X.

(i) *Nonnegativity*: Let $x = \big(x_1,\dots,x_n\big)$ and $y = \big(y_1,\dots,y_n\big)$ be any two points in $X_1\times\cdots\times X_n = X$ Then, by definition of the Kasu function, we obtain

$$d(x,y) = d\Big(\big(x_1,\dots,x_n\big),\big(y_1,\dots,y_n\big)\Big)$$

$$= f\Big(d_1(x_1,y_1),\dots,d_n(x_n,y_n)\Big)$$
$$\geq 0$$

and so d is nonnegative real function on $X \times X$.

(ii) *Coincidence*: Now,

$$\begin{aligned}
d(x,y) &= d\Big(\big(x_1,\dots,x_n\big),\big(y_1,\dots,y_n\big)\Big)\\
&= f\Big(d_1(x_1,y_1),\dots,d_n(x_n,y_n)\Big)\\
&= 0
\end{aligned}$$

if and only if $d_1 = \cdots = d_n = 0$ which implies that $d(x,y) = 0 \iff x = y$ in view of the property (ii) of the Kasu function f.

(iii)*Symmetry*: Now,

$$\begin{aligned}
d(x,y) &= d\Big(\big(x_1,\dots,x_n\big),\big(y_1,\dots,y_n\big)\Big)\\
&= f\Big(d_1(y_1,x_1),\dots,d_n(y_n,x_n)\Big)\\
&= d(y,x)
\end{aligned}$$

and so d is symmetric function.

(iv) *Triangle inequality*: Let $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n)$ and $z = (z_1,\dots,z_n)$ be any three points in $X_1 \times \cdots \times X_n = X$. Then,

$$\begin{aligned}
d(x,y) &= d\Big(\big(x_1,\dots,x_n\big),\big(y_1,\dots,y_n\big)\Big)\\
&= f\Big(d_1(y_1,x_1),\dots,d_n(y_n,x_n)\Big)\\
&\leq f\Big(d_1(x_1,y_1)+d_1(y_1,z_1),\dots,d_n(x_n,y_n)+d_n(y_n,z_n)\Big)\\
&\leq f\Big(d_1(x_1,y_1),\dots,d_n(x_n,y_n)\Big)+f\Big(d_1(y_1,z_1),\dots,d_n(y_n,z_n)\Big)\\
&= d(x,y)+d(y,z)
\end{aligned}$$

for all $x,y,z \in X$ and so, d satisfies the triangle inequality. Thus, d is a metric on X and so (X,d) is a metric space. □

Proposition 3.2. *Let $\|\cdot\|_1,\dots,\|\cdot\|_n$ be the norms on n vector spaces $E_1,\dots,E_n$ respectively and let $E = E_1 \times \cdots \times E_n$. Then the function $\|\cdot\| : E \to \mathbb{R}_+$ defined by*

$$\|x\| = f\big(\|x_1\|_1,\dots,\|x_n\|_n\big) \tag{3.4}$$

is a norm on E, where $x = (x_1,\dots,x_n) \in E_1 \times \cdots \times E_n$ and $f \in \mathfrak{K}$.

Remark 3.1. The metric d defined by (3.3) is called a Kasu metric on the product metric space $X_1 \times \cdots \times X_n$. Similarly, the norm $\|\cdot\|$ defined by (3.4) is called a Kasu norm on a product normed linear space $E = E_1 \times \cdots \times E_n$.

Proposition 3.3. *Let $(X_1, d_1), \ldots, (X_n, d_n)$ be n metric spaces and let $X = X_1 \times \cdots \times X_n$. Suppose that the Kasu metric d is defined by (3.3). If each metric space $X_1, \ldots, X_n$ is complete, then so is also (X, d).*

Proof. We show that every Cauchy sequence of points in X converges to a point in X. Let $\{x^m\} = \big\{(x_1^m, \ldots, x_n^m)\big\}$ be a Cauchy sequence in X. Then, we have

$$\lim_{m,p\to\infty} d(x^m, x^p) = 0.$$

Now, by definition of the norm d, we have that

$$\lim_{m,p\to\infty} f\Big(d_1(x_1^m, x_1^p), \ldots, d_n(x_n^m, x_n^p)\Big) = 0$$

which further yields

$$\lim_{m,p\to\infty} d_i(x_i^m, x_i^p) = 0$$

for each i, $i = 1, 2, \ldots, n$. This shows that $\{x_i^m\}$ is a Cauchy sequence in X_i for $i = 1, 2, \ldots, n$. Since each X_i is complete , $\{x_i^m\}$ converges to a point, say $x_i^* \in X_i$ for $i = 1, 2, \ldots, n$. As a result, we have

$$\lim_{m\to\infty} d_i(x_i^m, x_i^*) = 0, \ \ i = 1, 2, \ldots, n.$$

Now, by definition of the norm d we obtain

$$\begin{aligned}
\lim_{m\to\infty} d(x^m, x^*) &= \lim_{m\to\infty} \Big(d\big(x_1^m, \ldots, x_n^m\big), \big(x_1^*, \ldots, x_n^*\big) \Big) \\
&= \lim_{m\to\infty} f\Big(d_1(x_1^m, x_1^*), \ldots, d_n(x_n^m, x_n^*)\Big) \\
&= f\Big(\lim_{m\to\infty} d_1(x_1^m, x_1^*), \ldots, \lim_{m\to\infty} d_n(x_n^m, x_n^*)\Big) \\
&= 0.
\end{aligned}$$

As a result every Cauchy sequence sequence in X is convergent and converges to a point in X. Hence, X is a complete metric space. □

Example 3.3. Let d be a metric in a metric space X. Then the functions d^* defined by

$$d_s\big(((x, y), (u, v)\big) = d(x, u) + d(y, v) \tag{3.5}$$

and

$$d_m\big(((x, y), (u, v)\big) = \max\big\{d(x, u), d(y, v)\big\} \tag{3.6}$$

are the Kasu metrics on $X^2 = X \times X$ in view of the relations (3.1) and (3.2), where $(x, y), (u, v) \in X^2$.

Now, we introduce an order relation α in the product metric space $X = X_1 \times \cdots \times X_n$. An order relation $\preceq$ is a binary relation which is reflexive, antisymmetric and transitive. Note that a metric space X together with the order relation $\preceq$ is called partially ordered metric space. A few details of a partially ordered metric space appear in Dhage [9] and references therein. If a partial order $\preceq$ is introduced in a metric space X which is also complete with respect to the metric d, then $(X, \preceq, d)$ is called a partially ordered complete metric space.

Let $\alpha_1, \ldots, \alpha_n$ be the partial order relations in the partially ordered metric spaces $X_1, \ldots, X_n$ respectively. Denote $X = X_1 \times \cdots \times X_n$ and $\alpha = \alpha_1 \times \cdots \times \alpha_n$. We define a partial order α in the product metric space X as follows. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two elements of X. Then,

$$x \,\alpha\, y \iff x_i \,\alpha_i\, y_i \text{ for } i = 1, 2, \ldots, n. \tag{3.7}$$

The order relation , α defined by the relation (3.7) is called the *Kasu partial order* on the product metric space $X = X_1 \times \cdots \times X_n$. The product metric space X together with the above partial order α becomes a partially ordered product metric space. Now, we are equipped with all the necessary details to state significant results concerning the Janhavi sets and regularity of the partially ordered product metric space (X, α, d).

Theorem 3.1. *Assume that each of the partially ordered metric spaces $(X_1, \alpha_1, d_1), \ldots, (X_n, \alpha_n, d_n)$ is regular. Suppose that $X = X_1 \times \cdots \times X_n$ and $\alpha = \alpha_1 \times \cdots \times \alpha_n$. If the metric d in X is defined by the Kasu function (3.3), then the partially ordered product metric space (X, α, d) is regular.*

Proof. Suppose first that $\{x^m\}$ is a monotone nondecreasing sequence of points in X. Then $x^m \,\alpha\, x^{m+1}$ for each $m \in \mathbb{N}$. By definition of the partial order α, we obtain $x_i^m \,\alpha_i\, x_i^{m+1}$ for each i, $i = 1, 2, \ldots, n$. Next, we assume that $x^m \to x^*$. Then,

$$\lim_{m\to\infty} d(x^m, x^*) = 0.$$

Now, by definition of the Kasu metric (3.3), we obtain

$$\lim_{m\to\infty} d_i(x_i^m, x_i^*) = 0$$

for each $i = 1, \ldots, n$. Thus the sequence $\{x_i^m\}$ is monotone nondecreasing and converges to a point x_i^* for $i = 1, \ldots, n$. Since each $\big(X_i, \alpha_i, d_i\big)$ is a regular partially ordered metric space, one has $x_i^m \,\alpha_i\, x_i^*$ for all $m \in \mathbb{N}$ and for each i, $i = 1, \ldots, n$. Hence, by definition of α, we get $x^m \,\alpha\, x^*$ for all $m \in \mathbb{N}$. Similarly, if $\{x^m\}$ is monotone nonincreasing sequence of points in X, that is, $x^{m+1} \,\alpha\, x^m$ for all $m \in \mathbb{N}$ and if $\{x^m\}$ converges to a point x^*, then it can be shown that $x^* \,\alpha\, x^m$ for all $m \in \mathbb{N}$. As a result (X, α, d) is a partially ordered regular metric space. This completes the proof. □

Corollary 3.1. *Let $(E_1,K_1,\|\cdot\|_1),\dots,(E_n,K_n,\|\cdot\|_n)$ be n ordered Banach spaces. Suppose that $E = E_1\times\cdots\times E_n$ and $K = K_1\times\cdots\times K_n$. If the norm $\|\cdot\|$ in E is defined by Kasu function* (3.4)*, then the ordered Banach space $(E,K,\|\cdot\|)$ is regular.*

Proof. By Lemma 2.1, each of the ordered Banach spaces $(E_1,K_1,\|\cdot\|_1),\dots,(E_n,K_n,\|\cdot\|_n)$ is regular. Now the desired conclusion follows by an application of Theorem 3.1. □

Theorem 3.2. *Assume that each of the partially partially ordered metric spaces $(X_1,\alpha_1,d_1),\dots,$ (X_n,α_n,d_n) is Janhavi. Suppose that $X = X_1\times\cdots\times X_n$ and $\alpha = \alpha_1\times\cdots\times\alpha_n$. If the metric Kasu d in X is defined by the relation* (3.3) *and the Kasu partial order α is defined by the relation* (3.7)*, then partially ordered metric space (X,α,d) is also Janhavi.*

Proof. Let $\{x^m\}$ be a monotone sequence of points in X and let a subsequence $\{x^{m_k}\}$ of $\{x^m\}$ be convergent converging to the point x^*. Then, from the nature of the sequence $\{x^m\}$, it follows that the sequence $\{x_i^m\}$ is monotone and has a convergent subsequence $\{x_i^{m_k}\}$ converging to a point x_i^* in X_i for $i = 1,\dots,n$. As each partially ordered metric space (X_i,α_i,d_i) is Janhavi, we have that $x_i^m \to x_i^*$ as $m\to\infty$ for each $i = 1,2,\dots,n$. Finally, from the definition of the Kasu function it follows that $x^m \to x^*$ as $n\to\infty$. As a result the partially ordered Banach space (X,α,d) is Janhavi. □

Corollary 3.2. *Let $(E_1,K_1,\|\cdot\|_1),\dots,(E_n,K_n,\|\cdot\|_n)$ be n partially ordered Banach spaces. Suppose that $E = E_1\times\cdots\times E_n$ and $K = K_1\times\cdots\times K_n$. If the norm $\|\cdot\|$ in E is defined by the Kasu function* (3.4)*, then the ordered Banach space $(E,K,\|\cdot\|)$ is also Janhavi.*

Proof. By Lemma 2.2, each of the ordered Banach spaces $(E_1,K_1,\|\cdot\|_1),\dots,(E_n,K_n,\|\cdot\|_n)$ is Janhavi. Now the desired conclusion follows by an application of Theorem 3.2. □

Theorem 3.3. *Assume that every partially compact subset of each of the partially ordered metric spaces $(X_1,\alpha_1,d_1),\dots,(X_n,\alpha_n,d_n)$ is Janhavi. Suppose that $X = X_1\times\cdots\times X_n$ and $\alpha = \alpha_1\times\cdots\times\alpha_n$. If the metric d in X is defined by the Kasu function* (3.3)*, then every partially compact subset of the partially ordered metric space (X,α,d) is also Janhavi.*

Proof. Suppose that S is a partially compact subset of the partially ordered metric space (X,α,d). Then $S = S_1\times\cdots\times S_n$, where $S_1,\dots,S_n$ are partially compact natural projections of S on $X_1,\dots,X_n$ respectively. Let $\mathscr{C}$ be a chain in S which is compact by virtue of partial compactness of S. Then $\mathscr{C} = C_1\times\cdots\times C_n$, where $C_1,\dots,C_n$ are compact chains and natural projections of $\mathscr{C}$ on $S_1,\dots,S_n$ respectively. Let $\{x^m\}$ be any monotone sequence of points in $\mathscr{C}$. Then, by compactness of $\mathscr{C}$, it has a convergent subsequence $\{x^{m_k}\}$ converging to a point, say $x^*\in\mathscr{C}$. Now, $x^m = \big(x_1^m,\dots,x_n^m\big)$, so that there are monotone sequences $\{x_i^m\}$ in C_i for $i = 1,\dots,n$ and the subsequences $\{x_i^{m_k}\}$ converging to the point x_i^* in view of the definition of the Kasu metric in X. Since every partially compact subset of the partially ordered metric spaces (X_i, α_i, d_i) is Janhavi, the sequence $\{x_i^m\}$ converges to x_i^* for each i, $i = 1, \ldots, n$. From definition of the metric d it follows that the original sequence $\{x^m\}$ converges to x^*. This shows that the partially compact subset S of the partially ordered metric space X is Janhavi. This completes the proof. $\square$

Corollary 3.3. *Let $(E_1, K_1, \|\cdot\|_1), \ldots, (E_n, K_n, \|\cdot\|_n)$ be n ordered Banach spaces. Suppose that $E = E_1 \times \cdots \times E_n$ and $K = K_1 \times \cdots \times K_n$. If the norm $\|\cdot\|$ in E is defined by the Kasu function (3.4), then every partially compact subset of the partially ordered Banach space $(E, K, \|\cdot\|)$ is also Janhavi.*

Proof. Suppose that S is a partially compact subset of the ordered Banach space E and suppose that $S_1, \ldots, S_n$ be the natural projections of S on the ordered Banach spaces $E_1, \ldots, E_n$ respectively. Then the sets $S_1, \ldots, S_n$ are also partially compact subset of $E_1, \ldots, E_n$ respectively. By Lemma 2.2, each of the sets $S_1, \ldots, S_n$ is Janhavi. Now the desired conclusion follows by an application of Theorem 3.3. This completes the proof. $\square$

Definition 3.3. An element u of the partially ordered set $(E, \preceq)$ is called a lower bound for a pair $\{x, y\}$ of elements in E if $u \preceq x$ and $u \preceq y$. Similarly, an upper bound for a pair of elements in the partially set E is defined. If every pair of elements in E have a lower as well as an upper bound, then the partially ordered set $(E, \preceq)$ is called a lattice. Moreover, if E is a Banach space, then it is called a Banach lattice.

The following results are sometimes useful for proving the uniqueness of fixed point for nonlinear operators and coupled operators in a partially ordered product Banach space satisfying partial contraction condition along with the applications to nonlinear simultaneous equations.

Lemma 3.1. *Let $(X_1, \alpha_1, d_1), \ldots, (X_n, \alpha_n, d_n)$ be n partially ordered metric spaces and let $X = X_1 \times \cdots \times X_n$. Suppose that d and α are respectively the Kasu metric and Kasu partial order in X defined by (3.3) and (3.7) respectively. If every pair of elements in each of $X_1, \ldots, X_n$ have a lower bound or an upper bound, then every pair of elements in X have a lower bound or an upper bound. In particular, the above conclusion holds if each of $X_1, \ldots, X_n$ is a lattice.*

Proof. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be any two elements of X. Then $x_i, y_i \in X_i$ for each $i = 1, \ldots, n$. Suppose that each pair of elements in each X_i have a lower bound, say $z_i \in X_i$. Then, we have $z_i \; \alpha_i \; x_i$ and $z_i \; \alpha_i \; y_i$ for each $i = 1, \ldots, n$. Therefore, the elements $z = (z_1, \ldots, z_n)$ serves as a lower bound for the pair of elements $\{x, y\}$ in X. Similarly, if each pair of elements in each X_i have an upper bound for each i, $i = 1, \ldots, n$, then it can be proved that every pair of elements of X have an upper bound in X. Again, if each of $X_1, \ldots, X_n$ is a lattice, then the partially ordered product metric space X is also a lattice and a fortiori, the above conclusion holds for all elements in X. This completes the proof. $\square$

Lemma 3.2. *Let $\big(E_1,\alpha_1,\|\cdot\|_1\big),\dots,\big(E_n,\alpha_n,\|\cdot\|_n\big)$ be n partially ordered Banach spaces and let $E=E_1\times\cdots\times E_n$. Suppose that $\|\cdot\|$ and α are respectively the Kasu norm and Kasu partial order in E defined by (3.4) and (3.7) respectively. If every pair of elements in each of $E_1,\dots,E_n$ have a lower bound or an upper bound, then every pair of elements in E have a lower bound or an upper bound. In particular, the above conclusion holds if each of $E_1,\dots,E_n$ is a Banach lattice.*

Proof. Since each partially ordered Banach space $\big(E_i,\alpha_i,\|\cdot\|_i\big)$, $i=1,\dots,n$, is a partially ordered metric space, where the metric d_i on E_i is defined through the norm $\|\cdot\|_i$ by $d_i(x,y) = \|x-y\|_i$, $x,y\in E_i$. Therefore, the desired conclusion follows by an application of Lemma 3.1. □

Lemma 3.3. *If every pair of elements in a partially ordered Banach space $(E,\preceq,\|\cdot\|_E)$ have a lower bound or an upper bound, then every pair of elements in the partially ordered product Banach space $(E^2,\alpha,\|\cdot\|)$ have a lower bound or an upper bound in E^2, where α and $\|\cdot\|$ are respectively the Kasu partial order and Kasu norm defined E^2. Moreover, the conclusion holds if E is a Banach lattice.*

Proof. Here, $E_1=E_2$. Hence, the proof of the lemma follows by an application of Lemma 3.2. We omit the details. □

Remark 3.2. The assertions of Lemma 3.1 remains true if we replace the partially ordered Banach spaces $(E_i,\alpha_i,\|\cdot\|_{E_i})$ with the ordered Banach spaces (E_i,K_i), $i=1,\dots,n$. Similarly the assertion of Lemma 3.2 also remains true if we replce the partially ordered Banach space E with the ordered Banach space (E,K).

4. Partial measure of noncompactness

The second most important concept that will be used in the development of coupled hybrid fixed point theory and applications is the partial measure of noncompactness in the partially ordered metric spaces. A few details concerning the partial measures of noncompactness along with their applications to nonlinear differential and integral equations appear in Dhage [12, 13, 14, 15, 16] and the references therein. For ready reference, we describe in the following some basic facts about the partial measures of noncompactness in a metric space X.

If C is a chain in X, then C' denotes the set of all limit points of C in X. The symbol $\overline{C}$ stands for the closure of C in X defined by $\overline{C}=C\cup C'$. The set $\overline{C}$ is also a closed chain in X. Thus, $\overline{C}$ is the intersection of all closed chains containing C. Clearly, $\inf C, \sup C\in\overline{C}$ provided $\inf C$ and $\sup C$ exist. The $\sup C$ is an element $z\in X$ such that for every $\epsilon>0$ there exists a

$c \in C$ such that $d(c,z) < \epsilon$ and $x \le z$ for all $x \in C$. Similarly, $\inf C$ is defined essentially in an analogous way.

In what follows, let $\mathscr{P}_p(X)$ denote the class of all subsets of X with property p. In particular, we denote by $\mathscr{P}_{cl}(X)$, $\mathscr{P}_{bd}(X)$, $\mathscr{P}_{rcp}(X)$, $\mathscr{P}_{cn}(X)$, $\mathscr{P}_{bd,cn}(X)$, $\mathscr{P}_{rcp,cn}(X)$ the family of all nonempty and closed, bounded, relatively compact, chains, bounded chains and relatively compact chains of X respectively. Now we introduce the concept of a partial measure of noncompactness of the chains in X on the lines of Dhage [13, 14, 15, 16]. The related idea of classical measure of noncompactness may be found in Appell [1], Banas and Goebel [3], Dhage [9], Arab [2] and references therein.

Definition 4.1. A mapping $\mu_p : \mathscr{P}_{bd,cn}(X) \to \mathbb{R}_+ = [0,\infty)$ is said to be a partial measure of noncompactness in X if it satisfies the following properties:

(P_1) $\emptyset \neq (\mu_p)^{-1}(\{0\}) \subset \mathscr{P}_{rcp,cn}(X)$. (kernel compactivity)

(P_2) $\mu_p(\overline{C}) = \mu_p(C)$. (closure invariance)

(P_3) μ_p is nondecreasing, i.e., if $C \subset D \Rightarrow \mu_p(C) \le \mu_p(D)$. (monotonicity)

(P_4) If $\{C_n\}$ is a sequence of closed chains from $\mathscr{P}_{bd,cn}(X)$ such that $C_{n+1} \subset C_n$, $n \in \mathbb{N}$ and if $\lim\limits_{n\to\infty} \mu_p(C_n) = 0$, then $\overline{C}_\infty = \bigcap_{n=1}^{\infty} C_n$ is nonempty. (limit intersection property)

The family of sets described in (P_1) is said to be the *kernel of the partial measure of noncompactness* μ_p and is defined as

$$\ker \mu_p = \big\{C \in \mathscr{P}_{bd,cn}(X) \,|\, \mu_p(C) = 0\big\}. \tag{4.1}$$

Clearly, $\ker \mu_p \subset \mathscr{P}_{rcp,cn}(X)$. Observe that the intersection set C_∞, from condition (P_3) is a member of the family $\ker \mu_p$. In fact, since $\mu_p(C_\infty) \le \mu_p(C_n)$ for any n, we infer that $\mu_p(C_\infty) = 0$. This yields that $C_\infty \in \ker \mu_p$. This simple observation will be essential in our further investigations.

The partial measure μ_p of noncompactness is called *full* or *complete* if it satisfies

(P_5) $\ker \mu_p = \mathscr{P}_{rcp,cn}(X)$.

Finally, μ_p is said to satisfy *maximum property* if

(P_6) $\mu_p\Big(C_1 \cup C_2\Big) = \max\big\{\mu_p(C_1), \mu_p(C_2)\big\}$.

Example 4.1. Define three functions $\alpha_p, \beta_p, \delta_p : \mathscr{P}_{bd,cn}(X) \to \mathbb{R}_+$ by

$$\alpha_p(C) = \inf\Big\{r > 0 \,\Big|\, C = \bigcup_{i=1}^{n} C_i,\ \mathrm{diam}\,(C_i) \le r\ \forall\, i\Big\}, \tag{4.2}$$

where $C \in \mathscr{P}_{bd,cn}(X)$ and $\operatorname{diam}(C_i) = \sup\{d(x,y) : x, y \in C_i\}$,

$$\beta_p(C) = \inf\Big\{ r > 0 \,\Big|\, C \subset \bigcup_{i=1}^{n} \mathscr{B}(x_i, r) \text{ for some } x_i \in X \Big\}, \tag{4.3}$$

where $\mathscr{B}(x_i, r) = \{x \in X : d(x_i, x) < r\}$, and

$$\delta_p(C) = \operatorname{diam}(C) = \sup\Big\{ d(x,y) : x, y \in C \Big\}. \tag{4.4}$$

It is easy to prove that α_p, β_p and δ_p are partial measures of noncompactness and are called the partial Kuratowskii, partial ball and partial diametric measures of noncompactness in X respectively. Note that partial measures α_p and β_p are full and enjoy the maximum property in X but the partial measure δ_p is not full as well as does not satisfy the maximum property.

When E is a Banach space, the partial Kuratowskii and partial ball measures of noncompatness satisfy the following two sublinear properties:

(P_7) $\mu_p(\lambda C) = |\lambda|\, \mu_p(C)$. (scalar multiplicativity)

(P_8) $\mu_p(C + D) \leq \mu_p(C) + \mu_p(D)$. (subadditivity)

The following proposition is very much useful for obtaining the partial measure of noncompactness in the partially ordered product metric spaces provided the partial measures of components in the partially ordered metric spaces are known to us.

Proposition 4.1. *Let $\mu_p^1, \ldots, \mu_p^n$ be the partial measures of noncompactness in the n partially ordered metric spaces $X_1, \ldots, X_n$ respectively and let $X = X_1 \times \cdots \times X_n$. Suppose that $f : \mathbb{R}_+^n \to \mathbb{R}_+$ is a Kasu function. Then the function $\mu_p : \mathscr{P}_{bd,cn}(X) \to \mathbb{R}_+$ defined by*

$$\mu_p(\mathscr{C}) = f\Big(\mu_p^1(C_1), \ldots, \mu_p^n(C_n)\Big) \tag{4.5}$$

is a partial measure of noncompactness in X, where $C_1, \ldots, C_n$ denote the natural projections of $\mathscr{C}$ on $X_1, \ldots, X_n$ respectively.

Proof. We shall show that the function μ_p satisfies all the conditions (P_1) through (P_4) of a partial measure of noncompactness on the partially ordered metric space (X, α, d).

(i) *Kernel compactivity :*

Let $C_1, \ldots, C_n$ be the natural projections of the chain $\mathscr{C}$ in X on $X_1, \ldots, X_n$ respectively. Then, $\mu_p(\mathscr{C}) = f\Big(\mu_p^1(C_1), \ldots, \mu_p^n(C_n)\Big) = 0 \Rightarrow \mu_p^i(C_i) = 0$ for $i = 1, \ldots, n$. Therefore, $C_1, \ldots, C_n$ are relatively compact chains in $X_1, \ldots, X_n$ respectively. As a result $\mathscr{C} = C_1 \times \cdots \times C_n$ is a relatively compact chain in the product metric space X.

(ii) *Closure invariance :*

Now, for any $\mathscr{C} = C_1 \times \cdots \times C_n$, we have that $\overline{C_1 \times \cdots \times C_n} = \overline{C_1} \times \cdots \times \overline{C_n}$. Therefore, we obtain

$$\mu_p(\overline{\mathscr{C}}) = f\Big(\mu_p^1(\overline{C_1}),\dots,\mu_p^n(\overline{C_n})\Big) = f\Big(\mu_p^1(C_1),\dots,\mu_p^n(C_n)\Big) = \mu_p(\mathscr{C}).$$

(iii) *Monotonicity :*

Let $\mathscr{C}$ and $\mathscr{D}$ be two chains in X with natural projections $C_1,\dots,C_n$ and $D_1,\dots,D_n$ on $X_1,\dots,X_n$ respectively. Suppose that $\mathscr{C} \subset \mathscr{D}$. Then, it follows that $C_i \subset D_i$ for each i, $i = 1,\dots,n$. Now, by nondecreasing nature of the Kasu function f in each co-ordinate variable, we obtain

$$\mu_p(\mathscr{C}) = f\Big(\mu_p^1(C_1),\dots,\mu_p^n(C_n)\Big) \le f\Big(\mu_p^1(D_1),\dots,\mu_p^n(D_n)\Big) = \mu_p(\mathscr{D}).$$

This shows that μ_p is nondecreasing on X.

(vi) *Limit intersection property :*

Let $\{\mathscr{C}^m\}$ be a decreasing sequence of closed and bounded chains in the partially ordered metric space X, that is, $\mathscr{C}^1 \supset \cdots \supset \mathscr{C}^m \cdots$; and let us assume that $\lim_{m\to\infty} \mu_p(\mathscr{C}^m) = 0$. Suppose that $C_1,\dots,C_n$ be the natural projections of the chain $\mathscr{C}$ on $X_1,\dots,X_n$ respectively. For the sake of convenience we write this as $\mathscr{C} = C_1 \times \cdots \times C_n$. Then $\{C_i^m\}$ is also a decreasing sequence of closed and bounded chains in the partially ordered metric space X_i for $i = 1,\dots,n$. Now by definition of μ_p,

$$\mu_p\big(\mathscr{C}^m\big) = f\Big(\mu_p^1(C_1^m),\dots,\mu_p^n(C_n^m)\Big).$$

Therefore,

$$\begin{aligned}
\lim_{m\to\infty} \mu_p\big(\mathscr{C}^m\big) &= \lim_{m\to\infty} f\Big(\mu_p^1\big(C_1^m\big),\dots,\mu_p^n\big(C_n^m\big)\Big) \\
&= f\Big(\lim_{m\to\infty} \mu_p^1\big(C_1^m\big),\dots,\lim_{m\to\infty} \mu_p^n\big(C_n^m\big)\Big) \\
&= 0
\end{aligned}$$

if and only if $\lim\limits_{m\to\infty} \mu_p^i(C_i^m) = 0$ for $i = 1,\dots,n$. As μ_p^i are the partial measures of noncompactness in the partially ordered metric spaces X_i, we have that $\bigcap_{m=1}^{\infty} C_i^m = C_i^{\infty} \neq \emptyset$ for each i, $i = 1,\dots,n$. Therefore, we obtain

$$\bigcap_{m=1}^{\infty} \mathscr{C}^m = \mathscr{C}^{\infty} = C_1^{\infty} \times \cdots \times C_n^{\infty} \neq \emptyset.$$

Thus, the function μ_p satisfies the properties (P_1) through (P_4) of the partial measure of noncompactness and hence it is a partial measure of noncompactness in X. This completes the proof. $\square$

Example 4.2. Let $\mu_p^1,\dots,\mu_p^n$ be the partial measures of noncompactness on the n partially ordered Banach spaces $X_1,\dots,X_n$ respectively and let $X = X_1\times\cdots\times X_n$. Define a two functions μ_p^s and μ_p^m on $\mathscr{P}_{bd,cn}(X)$ by

$$\mu_p^s(\mathscr{C}) = \sum_{i=1}^{n} a_i\,\mu_p^i(C_i),\; a_i > 0\;\forall\, i, \tag{4.6}$$

and

$$\mu_p^m(\mathscr{C}) = a\max\big\{\mu_p^1(C_1),\dots,\mu_p^n(C_n)\big\},\; a>0, \tag{4.7}$$

where $C_1,\dots,C_n$ are the natural projections of the chain $\mathscr{C}$ on $X_1,\dots,X_n$ respectively. Then the functions μ_p^s and μ_p^m are partial measures of noncompactness in X, because here the Kasu functions f_s abd f_m are defined by (3.1) and (3.2) respectively.

Example 4.3. Let μ_p be the partial measure in the partially ordered Banach space E. Then the partial measures μ_p^s and μ_p^m of noncompactness of a chain $\mathscr{C} = C\times D$ in $E^2 = E\times E$ may be defined as

$$\mu_p^s(\mathscr{C}) = \mu_p(C)+\mu_p(D) \tag{4.8}$$

and

$$\mu_p^m(\mathscr{C}) = \max\{\mu_p(C),\mu_p(D)\} \tag{4.9}$$

where C and D are the natural projections or components of the chain $\mathscr{C}$ in E.

In the following we prove a coupled hybrid fixed point theorem for partially condensing mappings in a partially ordered product metric space. We need the following useful definitions concerning the coupled operators are introduced in Dhage [17, 18].

Definition 4.2. Let (X^2,α,d) be a partially ordered metric space, where α and d are Kasu partial order and Kasu metric on X^2 respectively. A mapping $\mathscr{T}: X^2\to X^2$ is called monotone nondecreasing if it preserves the order relation α in X^2, that is, $\mathscr{T}z\,\alpha\,\mathscr{T}w$ for all $z,w\in X^2$, $z\,\alpha\,w$. Similarly, a mapping $\mathscr{T}$ on X^2 into itself is called monotone nonincreasing if $\mathscr{T}z\,\alpha'\,\mathscr{T}w$ for all $z,w\in X^2$, $z\,\alpha\,w$, where α' is the reverse of the order relation α in X. Finally, the mapping $\mathscr{T}$ is called monotone if it is either monotone nondecreasing or monotone nonincreasing on X.

Definition 4.3. Let $(X^2,\alpha,\tilde{d})$ be a partially ordered metric space, where α and $\tilde{d}$ are Kasu partial order and Kasu metric defined in $X^2 = X\times X$ respectively. A monotone mapping $\mathscr{T}: X^2\to X^2$ is called partially condensing if

$$\widetilde{\mu_p}\big(\mathscr{T}(\mathscr{C})\big) < \widetilde{\mu_p}(\mathscr{C}) \tag{4.10}$$

for all bounded chains $\mathscr{C}$ in X^2 for which $\widetilde{\mu_p}(\mathscr{C}) > 0$, where $\widetilde{\mu_p}$ is a Kasu partial measure of noncompactness in X^2.

Note that monotone partially compact and monotone partially contractions mappings on $X \times X$ are partially condensing, however the converse may not be true. Now we state a basic hybrid fixed point theorem for partially condensing monotone mappings in a partially ordered product metric space which is useful in the development of coupled hybrid fixed point theory and applications.

Theorem 4.1. *Let $(X, \preceq, d)$ be a regular partially ordered metric space and let every compact chain C in X be Janhavi. Suppose that α and $\tilde{d}$ are the Kasu partial order and Kasu metric defined in $X \times X$ respectively and suppose that $\mathcal{Q} : X \times X \to X \times X$ is a monotone nondecreasing, partially continuous, partially bounded and partially condensing mapping. If there exists an element $(x_0, y_0) \in X \times X$ such that $(x_0, y_0)\, \alpha\, \mathcal{Q}(x_0, y_0)$ or $\mathcal{Q}(x_0, y_0)\, \alpha\, (x_0, y_0)$, then $\mathcal{Q}$ has a fixed point $(x^*, y^*) \in X \times X$ and the sequence $\{\mathcal{Q}^n(x_0, y_0)\}$ of successive iterations converges monotonically to (x^*, y^*).*

Proof. Set $X^2 = X \times X$. As α and $\tilde{d}$ are respectively the Kasu order and Kasu metric defined in X^2, the triplet $(X^2, \alpha, \tilde{d})$ is a regular partially ordered metric space and every compact chain $\mathscr{C}$ in X^2 is Janhavi in view of Theorems 3.1 and 3.3. Furthermore, since the operator $\mathcal{Q}$ is a partially continuous, partially bounded, partially condensing and monotone nondecreasing on $(X^2, \alpha, \tilde{d})$ into itself and there exists an element $(x_0, y_0) \in X \times X$ such that $(x_0, y_0)\, \alpha\, \mathcal{Q}(x_0, y_0)$ or $\mathcal{Q}(x_0, y_0)\, \alpha\, (x_0, y_0)$, the desired conclusion follows by an application of a hybrid fixed point theorem for partial condensing mappings in a partially ordered metric space proved in Dhage [15, 16, 17, 18]. This completes the proof. □

Corollary 4.1. *Let $(X, \preceq, d)$ be a regular partially ordered metric space and let every compact chain C in X be Janhavi. Suppose that α and $\tilde{d}$ are the Kasu partial order and Kasu metric defined in $X^2 = X \times X$ respectively and suppose that $\mathcal{Q} : X^2 \to X^2$ is partially continuous, partially compact and monotone nondecreasing mapping. If there exists an element $(x_0, y_0) \in X \times X$ such that $(x_0, y_0)\, \alpha\, \mathcal{Q}(x_0, y_0)$ or $\mathcal{Q}(x_0, y_0)\, \alpha\, (x_0, y_0)$, then $\mathcal{Q}$ has a fixed point $(x^*, y^*) \in X \times X$ and the sequence $\{\mathcal{Q}^n(x_0, y_0)\}$ of successive iterations converges monotonically to (x^*, y^*).*

Remark 4.1. As mentioned in Dhage [15, 17] the condition

(A) every compact chain C in X is Janhavi,

of Theorem 4.1 may be replaced with a weaker condition that

(B) every compact chain $\mathscr{C}$ in $\mathcal{Q}(X \times X)$ is Janhavi.

We note that condition (A) ⇒ condition (B), however the converse may not be true. To see this, let us assume that the condition (A) holds. Then, by Theorem 3.3, every compact chain $\mathscr{C}$ in $X \times X$ is Janhavi. As $\mathcal{Q}$ is partially continuous, it is continuous on $\mathscr{C}$ and consequently $\mathcal{Q}(\mathscr{C})$ is also again a compact chain in $X \times X$ and so it is Janhavi. As $\mathscr{C}$ is an arbitrary chain in $X \times X$, every compact chain in $\mathcal{Q}(X \times X)$ is Janhavi.

In view of the above remark, Remark 4.1 we obtain the following applicable coupled hybrid fixed point result as a special case of Theorem 4.1.

Corollary 4.2. *Let $\big(X,\preceq,d\big)$ be a regular partially ordered metric space let α and $\tilde{d}$ be respectively the Kasu partial order and Kasu metric defined in X^2. Suppose that $\mathscr{Q}: X^2 \to X^2$ is a monotone nondecreasing mapping satisfying the condition of linear partial contraction,*

$$\tilde{d}(\mathscr{Q}Z,\mathscr{Q}W) \le k\,\tilde{d}(Z,W) \tag{4.11}$$

for all comparable elements $Z,W \in X^2$, where $0 \le k < 1$. If there exists an element $(x_0,y_0) \in X\times X$ such that $(x_0,y_0)\,\alpha\,\mathscr{Q}(x_0,y_0)$ or $\mathscr{Q}(x_0,y_0)\,\alpha\,(x_0,y_0)$, then $\mathscr{Q}$ has a unique comparable fixed point $(x^*,y^*)\in X\times X$ and the sequence $\{\mathscr{Q}^n(x_0,y_0)\}$ of successive iterations converges monotonically to (x^*,y^*). Moreover, the fixed point is unique if every pair of elements in X have a lower bound or an upper bound.

Proof. First we show that the condition (B) of Remark 4.1 holds. Let $\mathscr{C}$ be an arbitrary chain in $\mathscr{Q}(X\times X)$ and let $\{x_n\}$ be a monotone sequence in $\mathscr{C}$. Since $\mathscr{C}$ is compact, it has a convergent subsequence $\{x_{n_i}\}$ converging to a point, say x^*. Without loss of generality, we may assume that $x_n = \mathscr{Q}^n(x)$ for some $x\in\mathscr{C}$. After simple computation, by condition (4.11), it can be shown that $\{x_n\}$ is a Cauchy sequence of points in $\mathscr{C}$. As a result, the original sequence $\{x_n\}$ converges to x^* and that the compact chain $\mathscr{C}$ is Janhavi in $\mathscr{Q}(X\times X)$. Next, using the routine arguments, it can be shown that the operator $\mathscr{Q}$ is a k-set- contraction on $X\times X$ with respect to the Kasu partial Kuratowskii measure of noncompactness $\widetilde{\alpha_p}$ in X^2 with $k<1$. Now, by a direct application of Theorem 4.1 implies that operator $\mathscr{Q}$ has a fixed point $Z* = (x^*,y^*)\in E\times E$. If there is another fixed point $W^* = (u^*,v^*)$ of $\mathscr{Q}$ which is comparable to Z^*, then from the contraction condition (4.11) we get a contradiction. As a result, $\mathscr{Q}$ has a unique comparable fixed point.

To prove the uniqueness of fixed point, let $W^* = (u^*,v^*)$ be another fixed point of the operator $\mathscr{Q}$. Since given that every pair of elements of the partially ordered Banach space E have a lower or an upper bound, by Lemma 3.3, every pair of elements in E^2 also have a lower or an upper bound. Without loss of generality, we assume that there exists an upper bound U for the pair of elements $\{Z^*,W^*\}$ in E^2. Then, the elements Z^* and W^* are comparable to the element U. By nondecreasing nature of $\mathscr{Q}$, we obtain $Z^* = \mathscr{Q}^n Z^* \preceq \mathscr{Q}^n U$ and $W^* = \mathscr{Q}^n W^* \preceq \mathscr{Q}^n U$ for each $n\in\mathbb{N}$. Now, by contraction condition (4.11), we obtain

$$\begin{aligned}
\tilde{d}(Z^*,W^*) &= \tilde{d}(\mathscr{Q}^n Z^*,\mathscr{Q}^n W^*)\\
&\le \tilde{d}(\mathscr{Q}^n Z^*,\mathscr{Q}^n U) + \tilde{d}(\mathscr{Q}^n U,\mathscr{Q}^n W^*)\\
&\le k^n\big[\tilde{d}(Z^*,U) + \tilde{d}(U,W^*)\big]\\
&\to 0 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Hence $Z^* = W^*$ and consequently, the mapping $\mathscr{Q}$ has a unique fixed point. This completes the proof. □

Corollary 4.3. *Let (E, K) be an ordered Banach space and let α and $\|\cdot\|$ be the Kasu partial order and Kasu norm defined in $E^2 = E \times E$ respectively. Suppose that $\mathscr{Q} : E^2 \to E^2$ is partially continuous, partially condensing and monotone nondecreasing operator. If there exists an element $(x_0, y_0) \in E \times E$ such that $(x_0, y_0)\, \alpha\, \mathscr{Q}(x_0, y_0)$ or $\mathscr{Q}(x_0, y_0)\, \alpha\, (x_0, y_0)$, then $\mathscr{Q}$ has a fixed point $(x^*, y^*) \in E \times E$ and the sequence $\{\mathscr{Q}^n(x_0, y_0)\}$ of successive iterations converges monotonically to (x^*, y^*).*

Proof. By Lemma 2.1, the ordered Banach space (E, K) is regular. Again, every compact chain C in E is Janhavi in view of Lemma 2.2. Now the desired conclusion follows by an application of Theorem 4.1. □

Remark 4.2. We remark that the regularity of the partially ordered metric space X in above Theorem 4.1 may be relaxed and compensated with the continuity of the operators $\mathscr{Q}$ on X^2 into itself. See Dhage [15, 16, 17, 18] and the references therein. Furthermore, Corollary 4.2 includes the main coupled hybrid fixed point theorems of Berinde [4] and Dhage [22] for coupled mappings in a partially ordered metric space satisfying the symmetric partial contraction condition as the special cases.

As mentioned in Dhage [19], the above coupled hybrid fixed point results are useful to obtain a coupled hybrid fixed point theorem involving the sum of two coupled operators in a partially ordered Banach space. In the following section we prove our main coupled hybrid fixed point theorem of the paper on this line.

5. A coupled hybrid fixed point theorem

Given two mappings $\mathscr{F}, \mathscr{G} : E \times E \to E$, consider a couple of operator equations

$$x = \mathscr{F}(x, y) + \mathscr{G}(x, y) \tag{5.1}$$

and

$$y = \mathscr{F}(y, x) + \mathscr{G}(y, x) \tag{5.2}$$

for all $(x, y) \in E \times E$, where the operators $\mathscr{F}$ and $\mathscr{G}$ are not necessarily continuous.

The operators $\mathscr{F}$ and $\mathscr{G}$ involved in the coupled operator equations (5.1)-(5.2) are called the *coupled operators* on $E \times E$ into E. A pair of elements $(x^*, y^*) \in E \times E$ is called a *coupled fixed point* of the sum $\mathscr{F} + \mathscr{G}$ of two coupled operators $\mathscr{F}$ and $\mathscr{F}$ or *coupled solution* of the coupled operator equations (1.1) and (1.2) if

$$x^* = \mathscr{F}(x^*, y^*) + \mathscr{G}(x^*, y^*) \quad \text{and} \quad y^* = \mathscr{F}(y^*, x^*) + \mathscr{G}(y^*, x^*). \tag{5.3}$$

The existence of such coupled fixed points for coupled operators is generally obtained under certain monotonic condition of the coupled operator $\mathscr{T}$ on $E \times E$. See Heikkilá and Lakshmikantham [29], Sun [33], Bhaskar and Lakshmikantham [5] and Dhage and Dhage [25] and the references therein. A coupled operator $\mathscr{T}(x, y)$ is called **mixed monotone** if the map $x \mapsto \mathscr{T}(x, y)$ is nondecreasing for each $y \in E$ and the map $y \mapsto \mathscr{T}(x, y)$ is nonincreasing for each $x \in E$.

Before proving the main coupled hybrid fixed point theorem, we give some useful definitions in what follows.

Definition 5.1 (Dhage [9, 10])**.** An upper semi-continuous and monotonic nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathscr{D}$-function if $\psi(0) = 0$. The class of all $\mathscr{D}$-functions is denoted by $\mathfrak{D}$.

Definition 5.2 (Dhage [21])**.** A coupled operator $\mathscr{T} : E \times E \to E$ is called partial $\mathscr{D}$-Lipschitz if there exists a $\mathscr{D}$-function $\psi_{\mathscr{T}} \in \mathfrak{D}$ such that

$$\|\mathscr{T}(x, y) - \mathscr{T}(u, v)\| \le \frac{1}{2} \cdot \psi_{\mathscr{T}}\big(\|x - u\| + \|v - y\|\big) \tag{5.4}$$

for all comparable elements $(x, y), (u, v) \in E \times E$. If $\psi_{\mathscr{T}}(r) = k\,r$, $\mathscr{T}$ is called a partial Lipschitz on $E \times E$ with the Lipschitz constant k. Again, if $0 \le k < 1$, then $\mathscr{T}$ is called a partial contraction on $E \times E$ with contraction constant k. Furthermore, if $\psi_{\mathscr{T}}(r) < r$ for $r > 0$, then $\mathscr{T}$ is called a nonlinear partial $\mathscr{D}$- contraction on $E \times E$.

Definition 5.3 (Dhage [18])**.** A coupled operator $\mathscr{T} : E \times E \to E$ is called nonlinear symmetric partial $\mathscr{D}$-Lipschitz if there exists a $\mathscr{D}$-function $\psi \in \mathfrak{D}$ such that

$$\|\mathscr{T}(x, y) - \mathscr{T}(u, v)\| + \|\mathscr{T}(y, x) - \mathscr{T}(v, u)\| \le \psi_{\mathscr{T}}\big(\|x - u\| + \|y - v\|\big) \tag{5.5}$$

for all comparable elements $(x, y), (u, v) \in E \times E$. If $\psi(r) = k\,r$, $0 \le k < 1$, $\mathscr{T}$ is called a symmetric partial contraction on $E \times E$ with the contraction constant k. Furthermore, if $\psi_{\mathscr{T}}(r) < r$ for $r > 0$, then $\mathscr{T}$ is called a nonlinear symmetric partial $\mathscr{D}$- contraction on $E \times E$.

Remark 5.1. Note that partial contraction coupled operators are considered by Bhaskar and Lakshmikantham [5] whereas symmetric partial contraction coupled operators are considered by Berinde [4] in the study of coupled fixed point theorems in the partially ordered metric spaces with applications. It is clear that every nonlinear partial $\mathscr{D}$-contraction is nonlinear symmetric partial $\mathscr{D}$-contraction, but the converse may not be true.

The following lemma implicit in Dhage [15, 16, 17] is crucial in the proof of our main coupled hybrid fixed point theorem for the sum of two coupled operators in a partially ordered Banach space.

Lemma 5.1 (Dhage [15, 16, 17]). *Let $(X,\leq,d)$ be a partially ordered metric space and let $X^2 = X\times X$ be equipped with the Kasu partial order $\preceq$ and the Kasu metric $\tilde{d}$. Suppose that $\mathscr{Q}: X^2 \to X^2$ is a monotone mapping satisfying the inequality*

$$\tilde{d}\big(\mathscr{Q}Z,\mathscr{Q}W\big) \leq \psi\big(\tilde{d}(Z,W)\big) \tag{5.6}$$

for all comparable elements $Z,W \in X^2$, where $\psi \in \mathfrak{D}$. Then, for any bounded chain $\mathscr{C}$ in X^2,

$$\tilde{\alpha}_p\big(\mathscr{Q}(\mathscr{C})\big) \leq \psi\big(\tilde{\alpha}_p(\mathscr{C})\big) \tag{5.7}$$

where, $\tilde{\alpha}_p$ is a partial Kuratowskii measure of noncompactness in X^2.

Theorem 5.1. *Let $\big(E,\leq,\|\cdot\|_E\big)$ be a complete regular partially ordered normed linear space and let every compact chain C in E be Janhavi. Let $\mathscr{F},\mathscr{G}: E\times E \to E$ be two mixed monotone coupled operators satisfying the following conditions.*

(a) $\mathscr{F}$ is partially bounded and nonlinear symmetric partial $\mathscr{D}$-contraction, and

(b) $\mathscr{G}$ is a partially continuous and partially compact.

If there exists an element $(x_0,y_0)\in E\times E$ such that $x_0 \leq \mathscr{F}(x_0,y_0)+\mathscr{G}(x_0,y_0)$ and $y_0 \geq \mathscr{F}(y_0,x_0)+\mathscr{G}(y_0,x_0)$ or $x_0 \geq \mathscr{F}(x_0,y_0)+\mathscr{G}(x_0,y_0)$ and $y_0 \leq \mathscr{F}(y_0,x_0)+\mathscr{G}(y_0,x_0)$, then the coupled operator equations (5.1) and (5.2) have a coupled solution (x^*,y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$x_{n+1} = \mathscr{F}(x_n,y_n)+\mathscr{G}(x_n,y_n) \tag{5.8}$$

and

$$y_{n+1} = \mathscr{F}(y_n,x_n)+\mathscr{G}(y_n,x_n) \tag{5.9}$$

converge monotonically to x^* and y^* respectively. Moreover, the set of all comparable coupled solutions is compact.

Proof. Define a Kasu norm $\|\cdot\|$and a Kasu partial order $\preceq$ in E^2 by

$$\|(x,y)\| = \|x\|_E + \|y\|_E \tag{5.10}$$

and

$$X \preceq_m Y \iff x_1 \leq y_1 \wedge x_2 \geq y_2 \tag{5.11}$$

for $X=(x_1,y_1), Y=(y_1,y_2)\in E^2$. We define a Kasu partial Kuratowskii measure $\tilde{\alpha}$ of noncompatness in E^2 as follows. Let C and D be two chains in the partially ordered Banach space E and let $\mathscr{C} = C\times D$. Then $\mathscr{C}$ is a chain in E^2. Now define a partial Kuratowskii measure $\tilde{\alpha}$ by

$$\tilde{\alpha}(\mathscr{C}) = \alpha_p(C)+\alpha_p(D), \tag{5.12}$$

where α_p is a partial Kuratowskii measure of noncompactness in E.

Clearly, $(E^2, \preceq_m, \|\cdot\|)$ is a regular partial ordered Banach space and every compact chain $\mathscr{C}$ in E^2 is Janhavi in view of Theorems 3.1 and 3.3. Define an operator $\mathscr{Q} : E \times E \to E \times E$ by

$$\mathscr{Q}(x,y) = \Big(\mathscr{F}(x,y) + \mathscr{G}(x,y)\,,\, \mathscr{F}(y,x) + \mathscr{G}(y,x)\Big) \tag{5.13}$$

for all $(x,y) \in E^2$.

Now the operator $\mathscr{Q}$ may be written as

$$\begin{aligned}\mathscr{Q}(x,y) &= \Big(\mathscr{F}(x,y) + \mathscr{G}(x,y)\,,\, \mathscr{F}(y,x) + \mathscr{G}(y,x)\Big)\\ &= \Big(\mathscr{F}(x,y)\,,\, \mathscr{F}(y,x)\Big) + \Big(\mathscr{G}(x,y)\,,\, \mathscr{G}(y,x)\Big)\\ &= \mathscr{S}(x,y) + \mathscr{T}(x,y)\end{aligned} \tag{5.14}$$

where the operators $\mathscr{S}, \mathscr{T} : E^2 \to E^2$ are given by

$$\mathscr{S}(x,y) = \Big(\mathscr{F}(x,y)\,,\, \mathscr{F}(y,x)\Big) \tag{5.15}$$

and

$$\mathscr{T}(x,y) = \Big(\mathscr{G}(x,y)\,,\, \mathscr{G}(y,x)\Big). \tag{5.16}$$

Since $\mathscr{F}$ is a partial nonlinear $\mathscr{D}$-contraction, it is partially continuous on $E \times E$. As a result, the coupled operator $\mathscr{S}$ is well defined and partially continuous on $E \times E$ into $E \times E$. We show that $\mathscr{S}$ is a monotone nondecreasing coupled operator with respect to the order relation $\preceq_m$ in E^2. Let $Z = (x,y)$ and $W = (u,v)$ be two elements in E^2 such that $Z \preceq_m W$. Then, by definition of $\preceq_m$, we obtain $x \le u$ and $y \ge v$. Since $\mathscr{F}$ is mixed monotone, we have

$$\mathscr{F}(x,y) \le \mathscr{F}(u,y) \le \mathscr{F}(u,v)$$

and

$$\mathscr{F}(y,x) \ge \mathscr{F}(v,x) \ge \mathscr{F}(v,u).$$

Thus, we have

$$\mathscr{S}(x,y) = \big(\mathscr{F}(x,y)\,,\, \mathscr{F}(y,x)\big) \preceq_m \big(\mathscr{F}(u,v)\,,\, \mathscr{F}(v,u)\big) = \mathscr{S}(u,v)$$

and so, $\mathscr{S}$ is a monotone nondecrerasing operator on E^2 with respect to the order relation $\preceq_m$. Similarly, it can be shown that the operator $\mathscr{T}$ is partially bounded and monotone nondecreasing on $E \times E$ into itself. Then from (5.13) it follows that that $\mathscr{Q}$ is bounded and monotone nondecreasing bi-variate operator on E^2 into itself.

Now, we show that the operators $\mathscr{S}$ is a nonlinear partial $\mathscr{D}$-contraction on E^2. Let $Z = (x, y)$ and $W = (u, v)$ be any two elements of E^2 such that $Z \succeq_m W$. Then, by symmetric partial $\mathscr{D}$-contraction of $\mathscr{F}$, we obtain

$$\begin{aligned}
\|\mathscr{S}Z - \mathscr{S}W\| &= \|\mathscr{S}(x, y) - \mathscr{S}(u, v)\| \\
&= \Big\|\Big(\mathscr{F}(x, y), \mathscr{F}(y, x)\Big) - \Big(\mathscr{F}(u, v), \mathscr{F}(v, u)\Big)\Big\| \\
&= \Big\|\Big(\mathscr{F}(x, y) - \mathscr{F}(u, v), \mathscr{F}(y, x) - \mathscr{F}(v, u)\Big)\Big\| \\
&\leq \|\mathscr{F}(x, y) - \mathscr{F}(u, v)\|_E + \|\mathscr{F}(y, x) - \mathscr{F}(v, u)\|_E \\
&\leq \psi\left(\|x - u\|_E + \|v - y\|_E \right) \\
&= \psi\big(\|Z - W\|\big).
\end{aligned} \tag{5.17}$$

This shows that $\mathscr{S}$ is a nonlinear partial $\mathscr{D}$-contraction on E^2 into itself. This further in view of Lemma 5.1 yields that $\tilde{\alpha}_p\big(\mathscr{S}(\mathscr{C})\big) \leq \psi\big(\tilde{\alpha}_p(\mathscr{C})\big)$ for all bounded chains $\mathscr{C}$in E^2.

Next, we show that $\mathscr{Q}$ is a partially condensing operator with respect to the partial Kuratowski measure of noncompactneess in $E^2 = E \times E$. Let $\mathscr{C}$ be a chain in E^2. Then, by (5.14), we get $\mathscr{Q}(\mathscr{C}) \subset \mathscr{S}(\mathscr{C}) + \mathscr{T}(\mathscr{C})$. Now, from the properties (P_3) and (P_8) of partial Kuratowskii measure of noncompactness, it follows that

$$\widetilde{\alpha_p}(\mathscr{Q}(\mathscr{C})) \leq \widetilde{\alpha_p}(\mathscr{S}(\mathscr{C})) + \widetilde{\alpha_p}(\mathscr{T}(\mathscr{C})) \leq \phi\big(\widetilde{\alpha_p}(\mathscr{C})\big) < \widetilde{\alpha_p}\big(\mathscr{C}\big)$$

provided $\widetilde{\alpha_p}\big(\mathscr{C}\big) > 0$.

This shows that the operator $\mathscr{Q}$ is a partially condensing on E^2 into itself. Now, by an application of Theorem 4.1 gives that there is a point $Z^* = (x^*, y^*)$ in E^2 such that

$$(x^*, y^*) = Z^* = \mathscr{Q}(Z^*) = \mathscr{Q}(x^*, y^*)$$

which further yields that

$$x^* = \mathscr{F}(x^*, y^*) + \mathscr{G}(x^*, y^*)$$

and

$$y^* = \mathscr{F}(y^*, x^*) + \mathscr{G}(y^*, x^*).$$

This completes the proof. □

We note that Theorem 5.1 is a generalization of the following applicable coupled hybrid fixed point theorem of Dhage [20] with a different proof in view of Remark 5.1.

Corollary 5.1 (Dhage [22])**.** *Let $(E, \preceq, \|\cdot\|)$ be a complete regular partially ordered normed linear space and let every compact chain C in E is Janhavi. Let $\mathscr{F}, \mathscr{G} : E \times E \to E$ be two mixed monotone coupled operators satisfying the following conditions.*

(a) $\mathscr{F}$ is partially bounded and nonlinear partial $\mathscr{D}$-contraction, and

(b) $\mathscr{G}$ is a partially continuous and partially compact.

If there exists an element $(x_0, y_0) \in E \times E$ such that $x_0 \leq \mathscr{F}(x_0, y_0) + \mathscr{G}(x_0, y_0)$ and $y_0 \geq \mathscr{F}(y_0, x_0) + \mathscr{G}(y_0, x_0)$ or $x_0 \geq \mathscr{F}(x_0, y_0) + \mathscr{G}(x_0, y_0)$ and $y_0 \leq \mathscr{F}(y_0, x_0) + \mathscr{G}(y_0, x_0)$, then the coupled operator equations (5.1) and (5.2) have a coupled solution (x^*, y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ defined by (5.7) *and* (5.8) *converge monotonically to x^* and y^* respectively. Moreover, the set of all comparable coupled solutions is compact.*

Corollary 5.2 (Dhage [22])**.** *Let (E, K) be an ordered Banach space and let α and $\|\cdot\|$ be the Kasu order and Kasu norm defined in $E^2 = E \times E$ respectively. Let $\mathscr{F}, \mathscr{G} : E \times E \to E$ be two mixed monotone coupled operators satisfying the following conditions.*

(a) $\mathscr{F}$ is partially bounded and nonlinear symmetric partial $\mathscr{D}$-contraction, and

(b) $\mathscr{G}$ is a partially continuous and partially compact.

If there exists an element $(x_0, y_0) \in E \times E$ such that $x_0 \leq \mathscr{F}(x_0, y_0) + \mathscr{G}(x_0, y_0)$ and $y_0 \geq \mathscr{F}(y_0, x_0) + \mathscr{G}(y_0, x_0)$ or $x_0 \geq \mathscr{F}(x_0, y_0) + \mathscr{G}(x_0, y_0)$ and $y_0 \leq \mathscr{F}(y_0, x_0) + \mathscr{G}(y_0, x_0)$, then the coupled operator equations (5.1) and (5.2) have a coupled solution (x^*, y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ defined by (5.7) *and* (5.8) *converge monotonically to x^* and y^* respectively. Moreover, the set of all comparable coupled solutions is compact.*

Remark 5.2. The regularity of the partially ordered metric space E in above Theorem 5.1 may be relaxed and compensated with the continuity of the operators $\mathscr{F}$ and $\mathscr{G}$ on $E \times E$ into E. See Dhage [15, 16, 17, 18] and the references therein.

Remark 5.3. If $x = y$ in the coupled operator equations (5.1) and (5.2), then they reduce to the operator equation $\mathscr{A}x + \mathscr{B}x = x$, where $\mathscr{A}x = \mathscr{F}(x, x)$ and $\mathscr{B}x = \mathscr{G}(x, x)$, and consequently, Theorem 5.1 reduce to a hybrid fixed point theorem for the sum of two operators in E proved in Dhage [14, 15, 16].

We remark that the coupled hybrid fixed point theorems, Theorems 4.1 and 5.1 constitute a part of *Dhage iteration principle* for nonlinear equations whose central idea is "*the monotone convergence of the sequence of successive iterations or approximations to the solution of a noninear equation beginning with a lower or an upper solution as its first or initial approximation*" (see Dhage [14, 15, 16, 17]). The method of application of above principle to nonlinear equations is commonly known as *Dhage iteration method* which is widely used in the theory of nonlinear differential and integral equations for proving the existence and approximation theorems (see Dhage [18, 19, 20, 21, 22, 23, 24] and references therein).

The periodic boundary value problems are often discussed for different aspects of the solutions via applications of the tools form nonlinear functional analysis. In the following section we state a coupled periodic boundary value problem of second order linearly perturbed nonlinear differential equations to be discussed by an application of Theorem 5.1.

6. Periodic boundary value problems

Given a closed and bounded interval $J = [0, T]$ in $\mathbb{R}$, we consider the coupled hybrid periodic boundary value problems (in short coupled HPBVPs) of nonlinear second order ordinary differential equations,

$$\left.\begin{aligned} -x''(t)+\lambda^2 x(t) &= f(t,x(t),y(t))+g(t,x(t),y(t)),\ t\in J,\\ x(0)=x(T), &\quad x'(0)=x'(T), \end{aligned}\right\} \tag{6.1}$$

and

$$\left.\begin{aligned} -y''(t)+\lambda^2 y(t) &= f(t,y(t),x(t))+g(t,y(t),x(t)),\ t\in J,\\ y(0)=y(T), &\quad y'(0)=y'(T), \end{aligned}\right\} \tag{6.2}$$

for $\lambda \in \mathbb{R}$, $\lambda > 0$, where $f, g : J\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are continuous functions.

By a coupled solution of the coupled HPBVPs (6.1) and (6.2) we mean a pair of functions $(x^*, y^*) \in C^1(J,\mathbb{R}) \times C^1(J,\mathbb{R})$ that satisfies the equations (6.1) and (6.2), where $C^1(J,\mathbb{R})$ is the space of continuously differentiable real-valued functions defined on J.

The coupled hybrid PBVPs (6.1) and (6.2) are the linear perturbations of first kind of the following coupled HPBVPs of the form

$$\left.\begin{aligned} -x''(t)+\lambda^2 x(t) &= f(t,x(t),y(t)),\ t\in J,\\ x(0)=x(T), &\quad x'(0)=x'(T), \end{aligned}\right\} \tag{6.3}$$

and

$$\left.\begin{aligned} -y''(t)+\lambda^2 y(t) &= f(t,y(t),x(t)),\ t\in J,\\ y(0)=y(T), &\quad y'(0)=y'(T), \end{aligned}\right\} \tag{6.4}$$

which have been discussed in Dhage [13] for the existence and uniqueness theorem if the nonlinearity f satisfies a Lipschitz and compactness type condition and when f satisfies a partial compactness type condition it has been discussed in Dhage [16] for the existence and approximation of coupled solutions on J (see Dhage [13]). The purpose of the present study is to establish an existence and develop an algorithm for approximating the coupled solutions of the coupled HPBVPs (6.1) and (6.2) under some mixed hybrid conditions on the nonlinearities f and g.

The following useful lemma is obvious and may be found in Dhage [12] and the references therein.

Lemma 6.1. *For any real number $\lambda > 0$ and $\sigma \in L^1(J,\mathbb{R})$, x is a solution to the differential equation*

$$\left.\begin{aligned} -x''(t) + \lambda^2 x(t) &= \sigma(t) \text{ a.e. } t \in J, \\ x(0) = x(T),\ x'(0) &= x'(T), \end{aligned}\right\} \tag{6.5}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T G_\lambda(t,s)\sigma(s)\,ds \tag{6.6}$$

where,

$$G_\lambda(t,s) = \begin{cases} \dfrac{1}{2\lambda(e^{\lambda T}-1)}\left[e^{\lambda(t-s)} + e^{\lambda(T-t+s)}\right], & 0 \le s \le t \le T, \\ \\ \dfrac{1}{2\lambda(e^{\lambda T}-1)}\left[e^{\lambda(s-t)} + e^{\lambda(T-s+t)}\right], & 0 \le t < s \le T. \end{cases} \tag{6.7}$$

Notice that the Green's function G_λ is continuous and nonnegative on $J \times J$ and the numbers

$$\alpha = \min\{|G_\lambda(t,s)| : t,s \in [0,T]\} = \frac{e^{\lambda T}}{2\lambda(e^{\lambda T}-1)}$$

and

$$\beta = \max\{|G_\lambda(t,s)| : t,s \in [0,T]\} = \frac{e^{\lambda T}+1}{\lambda(e^{\lambda T}-1)}$$

exist for all positive real number λ. For the sake of convenience, we write $G_\lambda(t,s)$ as $G(t,s)$ in the subsequent part of the paper.

Other useful results for establishing the main existence result are as follows.

Lemma 6.2. *If there exists a function $u \in C^1(J,\mathbb{R})$ such that*

$$\left.\begin{aligned} -u''(t) + \lambda^2 u(t) &\le \sigma(t),\ \ t \in J, \\ u(0) \le u(T),\ \ u'(0) &\le u'(T), \end{aligned}\right\} \tag{6.8}$$

for all $t \in J$, where $\lambda \in \mathbb{R}$, $\lambda > 0$ and $\sigma \in L^1(J,\mathbb{R})$, then

$$u(t) \le \int_0^T G(t,s)\sigma(s)\,ds, \tag{6.9}$$

for all $t \in J$, where $G(t,s)$ is a Green's function given by the expression (6.7) on $J \times J$.

Proof. The proof of the lemma is obvious and follows from the maximum principle for BVPs of second order ordinary differential equations (see Protter and Weinberger [32] and Dhage and Heikkilä [27]). We omit the details. □

Similarly, we have the following result of differential inequality related to the second order periodic boundary value problems defined on J.

Lemma 6.3. *If there exists a differentiable function $v \in C(J,\mathbb{R})$ such that*

$$\left.\begin{aligned} -v''(t)+\lambda^2 v(t) \geq \sigma(t),\ t \in J,\\ v(0) \geq v(T),\ v'(0) \geq v'(T), \end{aligned}\right\} \tag{6.10}$$

for all $t \in J$, where $\lambda \in \mathbb{R}$, $\lambda > 0$ and $\sigma \in L^1(J,\mathbb{R})$, then

$$v(t) \geq \int_0^T G(t,s)\sigma(s)\,ds, \tag{6.11}$$

for all $t \in J$, where $G(t,s)$ is a Green's function given by the expression (6.7) on $J \times J$.

Now we are ready to apply our abstract coupled hybrid fixed point theorem to coupled HPBVPs (6.1) and (6.2) under suitable natural conditions. In the following section we prove our main existence and approximation theorem for coupled solutions of the coupled HPBVPs (6.1) and (6.2) defined on J.

7. Existence and approximation results

The equivalent integral forms of the HIDEs (6.1) and (6.1) will be considered in the function space $C(J,\mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J}|x(t)| \tag{7.1}$$

and

$$x \leq y \quad \text{if and only if} \quad x(t) \leq y(t) \quad \text{for all} \quad t \in J. \tag{7.2}$$

Clearly, $C(J,\mathbb{R})$ is a Banach space with respect to above supremum norm and is also partially ordered with respect to the above partial order relation $\leq$. It is known that the partially ordered Banach space $C(J,\mathbb{R})$ is regular and is a lattice, so every pair of elements in the space has an upper and a lower bound in the space. The next lemma concerning the $\mathscr{D}$-compatibility of sets in $C(J,\mathbb{R})$ follows by an application of the Arzelá-Ascoli theorem.

Lemma 7.1. *Let $\big(C(J,\mathbb{R}),\leq,\|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (7.1) and (7.2) respectively. Then, every partially compact subset of $C(J,\mathbb{R})$ possesses $\mathscr{D}$-compatibility property w.r.t. $\|\cdot\|$ and $\leq$ and so is Janhavi.*

Proof. The proof of the lemma is well-known and appears in the papers of Dhage [14, 15, 16, 17] and Dhage and Dhage [25, 26] and so we omit the details. □

We need the following definition in what follows.

Definition 7.1. A pair of functions $(u,v) \in C^1(J,\mathbb{R}) \times C^1(J,\mathbb{R})$ is said to be a lower coupled solution of the coupled equations (4.1) and (4.2) if

$$\left.\begin{aligned} -u''(t) + \lambda u(t) \le f(t,u(t),v(t)) + g(t,u(t),v(t)),\ t \in J, \\ u(0) \le u(T), \quad u'(0) \le u'(T), \end{aligned}\right\}$$

and

$$\left.\begin{aligned} -v''(t) + \lambda v(t) \ge f(t,v(t),u(t)) + f(t,v(t),u(t)),\ t \in J, \\ v(0) \ge v(T), \quad v'(0) \ge v'(T). \end{aligned}\right\}$$

Similarly, a pair of functions $(w,z) \in C^1(J,\mathbb{R}) \times C^1(J,\mathbb{R})$ is called a upper coupled solution of the coupled HPBVPs (6.1) and (6.2) if the above inequalities are satisfied with reverse sign.

The coupled HPBVPs (6.1) and (6.2) will be considered under the following set of assumptions:

(H_1) The function f is bounded on $J \times \mathbb{R} \times \mathbb{R}$ with bound M_f.

(H_2) There exists a $\mathscr{D}$-function $\varphi \in \mathfrak{D}$ such that

$$0 \le f(t,x_1,y_1) - f(t,x_2,y_2) \le \frac{1}{2}\cdot\varphi\big(x_1 - x_2 + y_2 - y_1\big)$$

for all $(x_1,y_1),(x_2,y_2) \in \mathbb{R}\times\mathbb{R}$ with $x_1 \ge x_2$ and $y_2 \ge y_1$. Moreover $\beta T\varphi(r) < r$, $r > 0$.

(H_3) $g(t,x,y)$ is nondecreasing in x and nonincreasing in y for each $t \in J$.

(H_4) The function g is bounded on $J \times \mathbb{R} \times \mathbb{R}$ with bound M_g.

(H_5) The coupled HPBVPs (6.1)-(6.2) have a lower coupled solution $(u,v) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$.

The hypotheses (H_1) through (H_5) are standard and have been widely used in the literature on nonlinear differential and integral equations. The special case of the hypothesis (H_2) with $\varphi(r) = \dfrac{Lr}{K+r}$, $L \le K$ is considered recently in Dhage [9, 12, 13, 14]. Now we formulate the main existence and approximation result for the coupled HPBVPs (6.1)-(6.2) under above mentioned natural conditions.

Theorem 7.1. *Assume that the hypotheses (H_1) through (H_5) hold. Then the coupled HPBVPs (6.1)-(6.2) have a coupled solution (x^*,y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ of successive approximations defined by*

$$\begin{aligned} x_0 &= u, \\ x_{n+1}(t) &= \int_0^T G(t,s) f(s,x_n(s),y_n(s))\,ds + \int_0^T G(t,s) g(s,x_n(s),y_n(s))\,ds,\ n \ge 0, \end{aligned} \tag{7.3}$$

and

$$y_0 = v,$$

$$y_{n+1}(t) = \int_0^T G(t,s)f(s,y_n(s),x_n(s))\,ds + \int_0^T G(t,s)g(s,y_n(s),x_n(s))\,ds,\ n \geq 0, \tag{7.4}$$

for $t \in J$, converge monotonically to x^* and y^* respectively.

Proof. Set $E = C(J,\mathbb{R})$. Then, by Lemma 2.2, every compact chain C in E is Janhavi. We introduce a Kasu norm $\|\cdot\|_{E^2}$ and a Kasu partial order $\preceq_m$ in $E^2 = E \times E$ by the relation

$$\|(x,y)\|_{E^2} = \|x\| + \|y\|$$

and

$$(x,y) \preceq_m (u,v) \iff x \leq u \wedge y \geq v$$

for $(x,y),(u,v) \in E \times E$. Clearly $\left(E^2, \preceq_m, \|\cdot\|_{E^2}\right)$ is a regular partially ordered Banach space with respect to above norm and partial order and every compact chain is E^2 is Janhavi in view of Theorem 3.3.

Next, by Lemma 6.1, the coupled HPBVPs (6.1) and (6.2) are equivalent to the nonlinear coupled integral equations of Fredholm type,

$$x(t) = \int_0^T G(t,s)f(s,x(s),y(s))\,ds + \int_0^T G(t,s)f(s,x(s),y(s))\,ds,\ t \in J, \tag{7.5}$$

and

$$y(t) = \int_0^T G(t,s)f(s,y(s),x(s))\,ds + \int_0^T G(t,s)f(s,y(s),x(s))\,ds,\ t \in J. \tag{7.6}$$

Now, consider the two coupled operators $\mathscr{F},\mathscr{G} : E \times E \to E$ defined by

$$\mathscr{F}(x,y)(t) = \int_0^T G(t,s)f(s,x(s),y(s))\,ds,\ t \in J, \tag{7.7}$$

and

$$\mathscr{G}(x,y)(t) = \int_0^T G(t,s)f(s,x(s),y(s))\,ds,\ t \in J. \tag{7.8}$$

Then the nonlinear coupled integral equations (7.5) and (7.6) are equivalent to the coupled operator equations,

$$x(t) = \mathscr{F}(x,y)(t) + \mathscr{G}(x,y)(t),\ t \in J, \tag{7.9}$$

and

$$y(t) = \mathscr{F}(y,x)(t) + \mathscr{G}(y,x)(t),\ t \in J. \tag{7.10}$$

We shall show that the coupled operators $\mathscr{F}$ and $\mathscr{G}$ satisfy all the conditions of Theorem 5.1 on $E \times E$ into E. This will be done in a series of following steps:

Step I: *$\mathscr{F}$ and $\mathscr{G}$ are mixed monotone.*

Let $(x,y),(u,v)\in E\times E$ be arbitrary and let $(x,y)\preceq_m(u,v)$. Then by definition of $\preceq_m$, we obtain $x\le u$ and $y\ge v$. Now, by hypotheses (H_2) and (H_4),

$$\begin{aligned}\mathscr{F}(x,y)(t)&=\int_0^T G(t,s)f(s,x(s),y(s))\,ds\\&\le\int_0^T G(t,s)f(s,u(s),v(s))\,ds\\&=\mathscr{F}(u,v)(t)\end{aligned}$$

and

$$\begin{aligned}\mathscr{G}(x,y)(t)&=\int_0^T G(t,s)f(s,x(s),y(s))\,ds\\&\le\int_0^T G(t,s)f(s,u(s),v(s))\,ds\\&=\mathscr{G}(u,v)(t)\end{aligned}$$

for all $t\in J$. Hence $\mathscr{F}$ and $\mathscr{G}$ are mixed monotone coupled operators on $E\times E$ into E.

Step II: *$\mathscr{F}$ is partially bounded and nonlinear symmetric partial $\mathscr{D}$-contraction.*

Let $(x,y)\in E\times E$ be arbitrary. Then,

$$|\mathscr{F}(x,y)(t)|\le\int_0^T G(t,s)|f(s,x(s),y(s))|\,ds\le\beta TM_f$$

for all $t\in J$. Taking the supremum over t in the above inequality yields $\|\mathscr{F}(x,y)\|\le\beta TM_f$ for all $x,y\in E$. So the coupled operator $\mathscr{F}$ is bounded and consequently partially bounded on $E\times E$.

Next, let $(x,y),(u,v)\in E\times E$ be any two elements such that $(x,y)\succeq_m(u,v)$. Then, by hypothesis (H_2),

$$\begin{aligned}|\mathscr{F}(x,y)(t)-\mathscr{F}(u,v)(t)|&\le\int_0^T G(t,s)|f(s,x(s),y(s))-f(s,u(s),v(s))|\,ds\\&\le\int_0^T G(t,s)\big[f(s,x(s),y(s))-f(s,u(s),v(s))\big]\,ds\\&\le\frac{1}{2}\int_0^T G(t,s)\varphi\big(x(s)-u(s)+v(s)-y(s)\big)\,ds\\&\le\frac{1}{2}\int_0^T G(t,s)\varphi\big(|x(s)-u(s)|+|v(s)-y(s)|\big)\,ds\\&\le\frac{1}{2}\beta T\varphi\big(\|x-u\|+\|v-y\|\big).\end{aligned}$$

Taking the supremum over t in the above inequality yields,

$$\|\mathscr{F}(x,y)-\mathscr{F}(u,v)\|\le\frac{1}{2}\beta T\varphi\big(\|x-u\|+\|v-y\|\big)\tag{7.11}$$

for all comparable elements $(x, y), (u, v) \in E \times E$. Similarly, we have

$$\|\mathscr{F}(x, y) - \mathscr{F}(u, v)\| \leq \frac{1}{2}\beta T\varphi\big(\|x-u\| + \|v-y\|\big). \tag{7.12}$$

Adding (5.11) and (5.12) together implies that

$$\|\mathscr{F}(x, y) - \mathscr{F}(u, v)\| + \|\mathscr{F}(y, x) - \mathscr{F}(v, u)\| \leq \beta T\varphi\big(\|x-u\| + \|v-y\|\big)$$

for all comparable elements $(x, y), (u, v) \in E \times E$. This shows that the coupled operator $\mathscr{F}$ is a nonlinear symmetric partial $\mathscr{D}$-contraction on $E \times E$.

Step III: *$\mathscr{G}$ is partially continuous on $E \times E$.*

Let C and D be any two chains in E and let $\{x_n\}$ and $\{y_n\}$ be two sequences in C and D respectively such that $x_x \to x$ and $y_n \to y$. Then, by continuity of the function g, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \mathscr{G}(x_n, y_n)(t) &= \lim_{n\to\infty} \int_0^T G(t,s) g(s, x_n(s), y_n(s))\, ds \\
&= \int_0^T G(t,s) \Big[\lim_{n\to\infty} g(s, x_n(s), y_n(s))\Big]\, ds \\
&= \int_0^T G(t,s) g(s, x(s), y(s))\, ds \\
&= \mathscr{G}(x, y)(t)
\end{aligned}$$

for all $t \in J$. This shows that the sequence $\{\mathscr{G}(x_n, y_n)\}$ converges to $\mathscr{G}(x, y)$ pointwise on J. We show that the convergence is uniform. To do so, it is enough o show that the sequence $\{F(x_n, y_n)\}$ is equicontinuous set of functions in E. Let $t_1, t_2 \in J$ be arbitrary. Then,

$$\begin{aligned}
&|\mathscr{G}(x_n, y_n)(t_1) - \mathscr{G}(x_n, y_n)(t_2)| \\
&\qquad \leq \int_0^T |G(t_1, s) - G(t_2, s)|\, |g(s, x_n(s), y_n(s))|\, ds \\
&\qquad \leq M_g \int_0^T |G(t_1, s) - G(t_2, s)|\, ds \\
&\qquad \to 0 \quad \text{as} \quad t_1 \to t_2,
\end{aligned}$$

uniformly for all $n \in \mathbb{N}$. This proves the equicontinuity of the sequence $\{\mathscr{G}(x_n, y_n)\}$ of functions in E. As a result, $\mathscr{G}(x_n, y_n) \to \mathscr{G}(x, y)$ uniformly. Hence $\mathscr{G}$ is continuous coupled operator on $C \times D$. Consequently, $\mathscr{G}$ is partially continuous on $E \times E$.

Step IV: *$\mathscr{G}$ is partially compact on $E \times E$.*

Let C and D be any two chains in E. We show that $\mathscr{G}(C \times D)$ is a compact subset of E. First we show that $\mathscr{G}(C \times D)$ is a uniformly bounded subset of E. Let $z \in \mathscr{G}(C \times D)$ be a fixed element. Then there exists a point $(x, y) \in C \times D$ such that $z = \mathscr{G}(x, y)$. Then,

$$|z(t)| = |\mathscr{G}(x, y)(t)| \leq \int_0^T G(t,s)|g(s, x(s), y(s))|\, ds \leq M_g K T$$

for all $t \in J$. Taking the supremum over t, $\|z\| \leq M_g KT$ for all $z \in \mathscr{G}(C \times D)$. Hence $\mathscr{G}(C \times D)$ is a uniformly bounded subset of E.

Next, we show that $\mathscr{G}(C \times D)$ is an equicontinuous subset of E. Let $t_1, t_2 \in J$ be arbitrary. Then,

$$\begin{aligned}
|z(t_1) - z(t_2)| &\leq |\mathscr{G}(x,y)(t_1) - \mathscr{G}(x,y)(t_2)| \\
&\leq \int_0^T |G(t_1,s) - G(t_2,s)||g(s,x(s),y(s))|\,ds \\
&\leq M_g \int_0^T |G(t_1,s) - G(t_2,s)|\,ds \\
&\to 0 \quad \text{as} \quad t_1 \to t_2,
\end{aligned}$$

uniformly for all $z \in \mathscr{G}(C \times D)$. This proves the equicontinuity of the set $\mathscr{G}(C \times D)$ in E. As a result, $\mathscr{G}(C \times D)$ is compact in view of Arzelá-Ascoli theorem. Hence $\mathscr{G}$ is a partially compact coupled operator on $E \times E$ into E.

Step V: *Coupled equations (5.9) and (5.10) have a lower coupled solution.*

Now, by hypothesis (H_5), there exists an element $(u,v) \in E \times E$ such that

$$\left.\begin{aligned}
-u''(t) + \lambda u(t) \leq f(t,u(t),v(t)) + g(t,u(t),v(t)), \\
u(0) \leq u(T),\ \ u'(0) \leq u'(T),
\end{aligned}\right\}$$

and

$$\left.\begin{aligned}
-v''(t) + \lambda v(t) \geq f(t,v(t),u(t)) + f(t,v(t),u(t)), \\
v(0) \geq v(T),\ \ v'(0) \geq v'(T),
\end{aligned}\right\}$$

for all $t \in J$. This further in view of Lemmas 6.2 and 6.3 implies that

$$u(t) \leq \int_0^T G(t,s) f(s,u(s),v(s))\,ds + \int_0^T G(t,s) f(s,u(s),v(s)),$$

and

$$v(t) \geq \int_0^T G(t,s) f(s,v(s),u(s))\,ds + \int_0^T G(t,s) f(s,v(s),u(s))$$

for all $t \in J$. Again, from the definition of the coupled operators $\mathscr{F}$ and $\mathscr{G}$ it follows that

$$u(t) \leq \mathscr{F}(u,v)(t) + \mathscr{G}(u,v)(t),\ t \in J,$$

and

$$v(t) \geq \mathscr{F}(v,u)(t) + \mathscr{G}(v,u)(t),\ t \in J.$$

Therefore, the coupled operator equations (7.9)-(7.10) have a coupled lower solution (u,v) in $E \times E$. Thus the coupled operators $\mathscr{F}$ and $\mathscr{G}$ satisfy all the conditions of Theorem 5.1 and hence the coupled operator equations and consequently the coupled HPBVPs (6.1)-(6.2) have a coupled solution (x^*, y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ defined by (7.3) and (**??**) converge monotonically to x^* and y^* respectively. $\square$

Remark 7.1. The conclusion of Theorem 5.1 also remains true if we replace the hypothesis (H_5) with the following one:

(H_6) The coupled HPBVPs (6.1) and (6.2) have a upper coupled solution $(u,v) \in C(J,\mathbb{R}) \times C(J,\mathbb{R})$.

The proof under this new hypothesis is obtained by giving similar arguments with appropriate modifications.

Example 7.1. Given a closed and bounded interval $J = [0,1]$ in $\mathbb{R}$, we consider the coupled hybrid periodic boundary value problems (in short coupled HPBVPs) of nonlinear first order ordinary differential equations,

$$\left.\begin{aligned} -x''(t) + x(t) = f_1(t,x(t),y(t)) + g_1(t,x(t),y(t)),\ t \in J, \\ x(0) = x(1),\ x'(0) = x'(1), \end{aligned}\right\} \tag{7.13}$$

and

$$\begin{aligned} -y''(t) + y(t) = f_1(t,y(t),x(t)) + g_1(t,y(t),x(t)),\ t \in J, \\ y(0) = y(1),\ y'(0) = y'(1), \end{aligned} \tag{7.14}$$

where $f_1, g_1 : J \times \mathbb{R} \times \mathbb{R}$ are continuous functions defined by

$$f_1(t,x,y) = \begin{cases} 0 & \text{if } -\infty < x, y \le 0, \\ \dfrac{1}{4} \cdot \left[\dfrac{x}{1+x} - \dfrac{y}{1+y} \right] & \text{if } 0 \le x, y < \infty, \end{cases}$$

and

$$g_1(t,x,y) = \tanh x - \tanh y$$

for all $t \in [0,1]$.

We shall show that the nonlinearities f_1 and g_1 satisfy the hypotheses (H_1) through (H_4). Clearly, the function f_1 is bounded on $[0,1] \times \mathbb{R} \times \mathbb{R}$ by 1. Next let $x_1, x_2, y_1, y_2 \in \mathbb{R}$ be such that $x_1 \ge x_2 > 0$ and $0 < y_1 \le y_2$. Then, we have

$$\begin{aligned} 0 &\le f_1(t,x_1,y_1) - f_1(t,x_2,y_2) \\ &= \frac{1}{4} \left[\frac{x_1}{1+x_1} - \frac{y_1}{1+y_1} - \frac{x_2}{1+x_2} + \frac{y_2}{1+y_2} \right] \\ &= \frac{1}{4} \left[\frac{x_1}{1+x_1} - \frac{x_2}{1+x_2} + \frac{y_2}{1+y_2} - \frac{y_1}{1+y_1} \right] \\ &= \frac{1}{4} \left[\frac{x_1 - x_2}{(1+x_1)(1+x_2)} + \frac{y_2 - y_1}{(1+y_2)(1+y_1)} \right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\left[\frac{x_1 - x_2}{1 + x_1 + x_2 + x_1x_2} + \frac{y_2 - y_1}{1 + y_2 + y_1 + y_2y_1}\right] \\
&\leq \frac{1}{4}\left[\frac{x_1 - x_2}{1 + x_1 + x_2} + \frac{y_2 - y_1}{1 + y_2 + y_1}\right] \\
&\leq \frac{1}{4}\left[\frac{x_1 - x_2}{1 + x_1 - x_2} + \frac{y_2 - y_1}{1 + y_2 - y_1}\right] \\
&\leq \frac{1}{4}\cdot\frac{x_1 - x_2 + y_2 - y_1}{1 + x_1 - x_2 + y_2 - y_1} \\
&= \frac{1}{4}\cdot\varphi(x_1 - x_2 + y_2 - y_1)
\end{aligned}
$$

where, $\varphi(r) = \dfrac{r}{1+r}$ for $r > 0$. Similarly, if $0 \geq x_1 \geq x_2$ and $y_1 \leq y_2 \leq 0$, then also the above inequality is satisfied. Therefore, the function f_1 satisfies the hypothesis (H_2).

Next, the function g_1 is bounded on $[0,1]\times\mathbb{R}\times\mathbb{R}$ with bound $M_{g_1} = 2$. Again, $g_1(t,x,y)$ os nondecreasing in x and nonincreasing in y for each $t \in [0,1]$. Here, $\lambda = 1$ and $T = 1$, so the Green's function $G_1(t,s)$ associated with the HBVPs (7.13) or (7.14) is given by

$$
G_1(t,s) = \begin{cases} \dfrac{1}{2(e-1)}\left[e^{(t-s)} + e^{(1-t+s)}\right], & 0 \leq s \leq t \leq 1, \\[2mm] \dfrac{1}{2(e-1)}\left[e^{(s-t)} + e^{(1-s+t)}\right], & 0 \leq t < s \leq 1. \end{cases} \tag{7.15}
$$

Therefore, $\beta = \sup\limits_{t,s\in J} G_1(t,s) = \dfrac{e+1}{2(e-1)} \leq 2$. Finally, the pair of functions (u,v) given by

$$
u(t) = -3\int_0^1 G_1(t,s)\,ds \quad \text{and} \quad v(t) = 3\int_0^1 G_1(t,s)\,ds
$$

is a lower coupled solution of the coupled HPBVPs (7.13) and (7.14) defined on $J = [0,1]$, Furthermore, $\beta\, T\varphi(r) \leq \dfrac{1}{2}\cdot\dfrac{e+1}{2(e-1)}\cdot\dfrac{r}{1+r} < r$ for $r > 0$. Thus the functions f_1 and g_1 satisfy all the hypotheses (H_1) through (H_4) of Theorem 7.1 and therefore, the coupled HPBVPs (7.13) and (7.14) have a coupled solution (x^*, y^*) and the sequences $\{x_n\}$ and $\{y_n\}$ defined by

$$
\begin{aligned}
x_0 &= -3\int_0^1 G_1(t,s)\,ds, \\
x_{n+1} &= \int_0^1 G_1(t,s) f_1(s, x_n(s), y_n(s))\,ds + \int_0^1 G_1(t,s) g_1(s, x_n(s), y_n(s))\,ds
\end{aligned}
$$

and

$$
\begin{aligned}
y_0 &= 3\int_0^1 G_1(t,s)\,ds, \\
y_{n+1} &= \int_0^1 G_1(t,s) f_1(s, y_n(s), x_n(s))\,ds + \int_0^1 G_1(t,s) g_1(s, y_n(s), x_n(s))\,ds
\end{aligned}
$$

converge monotonically to x^* and y^* respectively.

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