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EXTENDED A CONSTANT PART OF REDHEFFER'S TYPE INEQUALITIES

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Abstract. J-L. Li and Y-L. Li [4] gave the following Redheffer's type inequality;

$$\frac{1-\left(\frac{x}{\pi}\right)^2}{\sqrt{1+3\left(\frac{x}{\pi}\right)^4}} > \frac{\sin x}{x}$$

holds for $0 < x < \pi$, where the constant 3 is the best possible. In this paper, we establish two inequalities extended the constant part of the above inequality.

1. Introduction

Redheffer et al. [7], [8] established the following inequality; the inequality

$$\frac{\sin x}{x} \geq \frac{\pi^2 - x^2}{\pi^2 + x^2} \tag{1.1}$$

holds for $0 < x \leq \pi$. After that, mathematicians [1]–[4], [6]–[11] studied the Redheffer's type inequalities. Chen et al. [2] gave the Redheffer's type inequalities for $\cos x$ and $\frac{\sinh x}{x}$, Baricz et al. [1], [6], [11] established the Redheffer's type inequalities extended for Bessel functions and Zhu et al. [9] [10] showed the Redheffer's type inequalities for circular and hyperbolic functions. Especially, Li et al. [4] showed the following simple Redheffer's type inequality; the inequality

$$\frac{1-\left(\frac{x}{\pi}\right)^2}{\sqrt{1+3\left(\frac{x}{\pi}\right)^4}} > \frac{\sin x}{x} \tag{1.2}$$

holds for $0 < x < \pi$, where the constant 3 is the best possible. In this papar, we consider the above inequality (1.2) and our main results are followings.

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Theorem 1.1. *For $r > 3$ and $0 < x < \pi - \pi\sqrt{\frac{r-3}{r}}$, we have*

$$\frac{1-\left(\frac{x}{\pi}\right)^2}{\sqrt{1+r\left(\frac{x}{\pi}\right)^4}} > \frac{\sin x}{x}. \tag{1.3}$$

The above inequality (1.3) is extended the constant part of the inequality (1.2).

Theorem 1.2. *For $r > 3$ and $\pi\sqrt{\frac{2}{r-1}} < x < \pi$, we have*

$$\frac{\sin x}{x} > \frac{1-\left(\frac{x}{\pi}\right)^2}{\sqrt{1+r\left(\frac{x}{\pi}\right)^4}}. \tag{1.4}$$

The above inequality (1.4) is extended the constant part and the reversed type of the inequality (1.2).

Remark 1.3. It seem likely to that the inequality (1.3) holds for $r > 3$ and $0 < x < \pi - \pi\sqrt{\frac{r-3}{r+5}}$, which is a stronger than the condition of Theorem 1.1.

2. Proof of main theorems

Proof of Theorem 1.1. From $0 < x < \pi - \pi\sqrt{\frac{r-3}{r}}$, we have

$$3 < r < \frac{3\pi^2}{(2\pi - x)x}$$

and

$$\frac{1-\left(\frac{x}{\pi}\right)^2}{\sqrt{1+r\left(\frac{x}{\pi}\right)^4}} - \frac{\sin x}{x} > \frac{1-\left(\frac{x}{\pi}\right)^2}{\sqrt{1+\left(\frac{3\pi^2}{(2\pi-x)x}\right)\left(\frac{x}{\pi}\right)^4}} - \frac{\sin x}{x}$$
$$= \sqrt{\frac{(\pi-x)^2(x+\pi)^2(2\pi-x)}{\pi^2\left(3x^3-\pi^2 x+2\pi^3\right)}} - \frac{\sin x}{x}.$$

Hence, it suffices to show that

$$F(x) = \frac{(\pi-x)^2(x+\pi)^2(2\pi-x)}{\pi^2\left(3x^3-\pi^2 x+2\pi^3\right)} - \frac{\sin^2 x}{x^2} > 0.$$

First, we consider the case of $0 < x \le \frac{3\pi}{4}$. In this case, the following inequality is the important role of the proof. Li [5] showed that

$$\frac{x}{\sin x} = 1 + \sum_{n\in\mathbb{Z}\setminus 0} \frac{(-1)^{n+1}\left(\frac{x}{\pi}\right)^2}{n^2+n\left(\frac{x}{\pi}\right)} = 1 + 2\left(\frac{x}{\pi}\right)^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2-\left(\frac{x}{\pi}\right)^2} \tag{2.1}$$

for $0 < x < \pi$. It follows from the above inequality (2.1) that

$$\frac{x}{\sin x} \geq 1 + 2\left(\frac{x}{\pi}\right)^2 \left\{ \frac{1}{1-\left(\frac{x}{\pi}\right)^2} - \frac{1}{4-\left(\frac{x}{\pi}\right)^2} \right\}$$

for $0 < x < \pi$. Thus, we can get

$$\frac{\sin x}{x} \leq \frac{(\pi - x)(2\pi - x)(x+\pi)(x+2\pi)}{x^4 + \pi^2 x^2 + 4\pi^4}$$

for $0 < x < \pi$ and we have

$$\begin{aligned} F(x) \geq & \frac{(\pi - x)^2 (x+\pi)^2 (2\pi - x)}{\pi^2 \left(3x^3 - \pi^2 x + 2\pi^3\right)} - \left\{ \frac{(\pi - x)(2\pi - x)(x+\pi)(x+2\pi)}{x^4 + \pi^2 x^2 + 4\pi^4} \right\}^2 \\ = & \frac{(\pi - x)^3 (2\pi - x) x^2 (x+\pi) F_1(x)}{\pi^2 \left(3x^2 - 3\pi x + 2\pi^2\right)\left(x^4 + \pi^2 x^2 + 4\pi^4\right)^2}, \end{aligned}$$

where $F_1(x) = -x^5 - \pi x^4 - 6\pi^2 x^3 - 12\pi^3 x^2 - 8\pi^4 x + 16\pi^5$. Since $F_1(x)$ is strictly decreasing for $0 < x < \frac{3\pi}{4}$, we have

$$F_1(x) > F_1\left(\frac{3\pi}{4}\right) = \frac{169\pi^5}{1024} > 0$$

and

$$3x^2 - 3\pi x + 2\pi^2 \geq 3\left(\frac{\pi}{2}\right)^2 - 3\pi\left(\frac{\pi}{2}\right) + 2\pi^2 = \frac{5\pi^2}{4} > 0.$$

Thus, we obtain $F(x) > 0$ for $0 < x \leq \frac{3\pi}{4}$. Next, we consider the case of $\frac{3\pi}{4} < x < \pi$. By Taylor series, we have

$$\cos 2x > 1 - 2(x-\pi)^2 + \frac{2}{3}(x-\pi)^4 - \frac{4}{45}(x-\pi)^6$$

for $\frac{3\pi}{4} < x < \pi$, so we can get

$$\begin{aligned} F(x) &= \frac{-2x^6 + 6\pi x^5 - 2\pi^2 x^4 - 6\pi^3 x^3 + 4\pi^4 x^2 - 3\pi^2 x^2 + 3\pi^3 x - 2\pi^4}{2\pi^2 x^2 \left(3x^2 - 3\pi x + 2\pi^2\right)} + \frac{\cos 2x}{2x^2} \\ &> \frac{(\pi - x)^4 F_2(x)}{45\pi^2 x^2 \left(3x^2 - 3\pi x + 2\pi^2\right)}, \end{aligned}$$

where

$$\begin{aligned} F_2(x) = & -6\pi^2 x^4 + 18\pi^3 x^3 - 22\pi^4 x^2 + 45\pi^2 x^2 - 45x^2 \\ & +14\pi^5 x - 45\pi^3 x - 45\pi x - 4\pi^6 + 30\pi^4 - 90\pi^2. \end{aligned}$$

The derivatives of $F_2(t)$ are

$$F_2'(t) = -24\pi^2 x^3 + 54\pi^3 x^2 - 44\pi^4 x + 90\pi^2 x - 90x + 14\pi^5 - 45\pi^3 - 45\pi,$$

$$F_2''(x) = -72\pi^2x^2 + 108\pi^3x - 44\pi^4 + 90\pi^2 - 90$$

and

$$F_2'''(x) = 108\pi^3 - 144\pi^2x.$$

From $F_2'''(x) < 0$ for $\frac{3\pi}{4} < x < \pi$, $F_2''(x)$ is strictly decreasing for $\frac{3\pi}{4} < x < \pi$. By

$$F_2''(\pi) = -90 + 90\pi^2 - 8\pi^4 \cong 18.9917,$$

we have $F_2''(x) > 0$ for $\frac{3\pi}{4} < x < \pi$ and $F_2'(x)$ is strictly increasing for $\frac{3\pi}{4} < x < \pi$. From

$$F_2'\left(\frac{3\pi}{4}\right) = \frac{5}{4}\pi\left(-90 + 18\pi^2 + \pi^4\right) \cong 726.737,$$

we have $F_2'(x) > 0$ for $\frac{3\pi}{4} < x < \pi$ and $F_2(x)$ is strictly increasing for $\frac{3\pi}{4} < x < \pi$. By

$$F_2\left(\frac{3\pi}{4}\right) = \frac{1}{128}\left(-19080\pi^2 + 2760\pi^4 - 23\pi^6\right) \cong 456.446,$$

we obtain $F_2(x) > 0$ for $\frac{3\pi}{4} < x < \pi$. Thus, we obtain $F(x) > 0$ for $\frac{3\pi}{4} < x < \pi$. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. From $\pi\sqrt{\frac{2}{r-1}} < x < \pi$, we have

$$r > \frac{x^2 + 2\pi^2}{x^2}.$$

By the inequality (1.1), we have

$$\begin{aligned}
\frac{1-\left(\frac{x}{\pi}\right)^2}{\sqrt{1+r\left(\frac{x}{\pi}\right)^4}} - \frac{\sin x}{x} &< \frac{1-\left(\frac{x}{\pi}\right)^2}{\sqrt{1+\left(\frac{x^2+2\pi^2}{x^2}\right)\left(\frac{x}{\pi}\right)^4}} - \frac{\sin x}{x} \\
&= \frac{(\pi - x)(x+\pi)}{x^2+\pi^2} - \frac{\sin x}{x} \\
&< \frac{(\pi - x)(x+\pi)}{x^2+\pi^2} - \frac{\pi^2 - x^2}{\pi^2 + x^2} \\
&= 0.
\end{aligned}$$

The proof of Theorem 1.2 is complete. □

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