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GENERALIZED SEQUENCE SPACE $F(X, r)$

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Abstract. In this paper, we define and study vector valued sequence space $F(X, r)$. Few topological properties, inclusion relation, boundedness properties of subset are studied for this class.

1. Introduction

Rosier [3] has developed the theory of vector valued sequence space relative to scalar valued sequence space by introducing and studying a composite space $\Lambda\{E\}$. Barnes and Roy [1] also studied the boundedness in a topological linear space. Maddox [2] and Simons [4] used the idea of a sequence of a strictly positive numbers $p = (p_k)$ (not necessarily bounded in general) to generalize the classical spaces c, c_0, ℓ_∞ and strongly summable sequence spaces w_0, w, w_∞. In the present note, we introduce a more generalized space using a sequence $r = (r_k)$ of strictly positive real numbers which includes the corresponding work of Maddox [2], Simons [4] and Rosier [3].

2. Notions/Terminology

Throughout this paper, we consider X as a locally convex Hausdorff space equipped with a topology T generated by the family P of continuous seminorms p_u on X given by

$$p_u(z) = \sup_f\{|f(z)| : f \in u^0, z \in X\}$$

where u^0 is the polar of $u \in u(X)$ and $u(X)$ is the fundamental system of absolutely convex closed neighborhoods at origin of X. We denote the topological dual of X by X' which is always equipped with strong topology $\beta(X', X)$ generated by the family $p' = (p_{B^0})$ of seminorms p_{B^0} given by

$$p_{B^0}(f) = \sup\{|f(z)| : z \in B, f \in X'\}$$

where B is bounded subset of X and B^0 is the polar of B.

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3. The Space $F(X, r)$

Let F be a normal Banach space over the field of complex sequences with monotone norm $\| \quad \|_F$ and having Schauder basis (e_k) where $e_k = (0, 0, \ldots, 1, 0, \ldots)$ with 1 in the k-th place. Further, let $r = (r_k)$ be a sequence of positive real numbers such that $0 < r_k \leq 1$. We define

$$F(X, r) = \{x = (x_k) : x_k \in X \text{ for each } k \text{ and for every } u \in u(X), (p_u^{r_k}(x_k)) \in F\}. \quad (3.1)$$

For $x = (x_k) \in F(X, r)$, we define

$$g_u(x) = \|(p_u^{r_k}(x_k)\|_F. \quad (3.2)$$

One can easily verify that the space $(F(X, r), d_u)$ is a pseudometric space under the pseudometric d_u given by

$$d_u(x, y) = g_u(x - y) \text{ where } x = (x_k),\ y = (y_k) \in F(X, r) \text{ and } u \in u(X). \quad (3.3)$$

Consider the space $(F(X, r), Vg_u)$ where the topology induced by Vg_u is the supremum of the topologies induced by all the paranorms g_u, $u \in u(X)$. This means that

$$\begin{aligned}&\text{“a net } (x^\mu) \text{ converges to } x = (x_k) \text{ in } Vg_u \text{ if and only if}\\ &\qquad (x^\mu) \text{ converges to } x = (x_k) \text{ in each } g_u,\ u \in u(X)\text{”.}\end{aligned} \quad (3.4)$$

4. In this section, we study some topological properties and obtain some inclusion relations for the space $F(X, r)$.

Theorem 4.1. *$F(X, r)$ is paranormed space under the paranorm g_u given by* (3.2).

Proof is straightforward, so we omit it.

Theorem 4.2. *If X is complete, then $(F(X, r), Vg_u)$ is complete.*

The proof is straight forward, so we omit it.

Theorem 4.3. *Let $0 < r_k \leq s_k \leq 1$ for all k and $x = (x_k) \in F(X, r)$. Let $A = \{k : p_u(x_k) \geq 1\}$ and $B = \{k : p_u(x_k) < 1\}$. Then*
(i) *$F(X, r) \subseteq F(X, s)$ if $n'(A) < \infty$,*
(ii) *$F(X, s) \subseteq F(X, r)$ if $n'(B) < \infty$,*
where $n'(A)$ and $n'(B)$ denote the number of indices in A and B respectively and $F(X, s)$ is defined accordingly as in Section 3.

Proof. Let $n'(A) < \infty$ and let $x = (x_k) \in F(X, r)$. Define two sequences (y_k) and (z_k) as

$$y_k = \begin{cases} x_k & \text{if } p_u(x_k) \geq 1, \\ \theta & \text{if } p_u(x_k) < 1, \end{cases} \qquad z_k = \begin{cases} \theta & \text{if } p_u(x_k) \geq 1, \\ x_k & \text{if } p_u(x_k) < 1, \end{cases} \quad (4.1)$$

(where θ is the zero element of X).
Clearly from (4.1) it follows that

$$p_u^{s_k}(y_k) \geq p_u^{r_k}(y_k) \quad \text{and} \quad p_u^{s_k}(z_k) \leq p_u^{r_k}(z_k) \tag{4.2}$$

for each k. But we can find an integer n_k such that

$$p_u^{s_k}(y_k) \leq n_k p_u^{r_k}(y_k) \leq M p_u^{r_k}(y_k). \tag{4.3}$$

where $M = \max n_k (k \in n'(A))$.

Since F is normal space, so (4.2) and (4.3) imply that $x = (x_k) \in F(X, s)$. Hence

$$F(X, r) \subseteq F(X, s) \tag{4.4}$$

If $n'(B) < \infty$, then on the similar lines as used in above. Theorem 4.3 (i) we can prove that

$$F(X, s) \subseteq F(X, r). \tag{4.5}$$

This completes the proof.

Theorem 4.4. *Let $0 < r_k \leq 1$ for each k. If $\sum_{k=1}^{\infty} N^{\xi_k} < \infty$, for some integer $N(> 1)$ (where ξ_k is the conjugate index of r_k i.e. $(1/r_k) + (1/\xi_k) = 1$ and number of indices in $A = n'(A) < \infty$, (see Theorem 4.3), then $F(X, r) = F(X)$, where $F(X)$ is the vector space of all X-valued sequences $x = (x_k)$ such that sequence of scalars $(p_u(x_k)) \in F$, for each $u \in u(X)$.*

Proof. Given $n'(A) < \infty$. So by Theorem 4.3

$$F(X, r) \subseteq F(X). \tag{4.6}$$

Conversely, let $x = (x_k) \in F(X)$ and define two sequences $y = (y_k)$ and $z = (z_k)$ as in Theorem 4.3. Since

$p_u(y_k) \geq 1 (k \in A)$ and $p_u(z_k) < 1 (k \in B)$, we have

$$p_u^{r_k}(y_k) \leq p_u(y_k) = p_u(x_k) \quad (k \in A) \tag{4.7}$$

and on the same lines as used by Simons [4, Theorem 3, p.427] we have

$$p_u^{r_k}(z_k) \leq p_u(z_k)(1 + N \log N) = p_u(x_k)(1 + N \log N). \tag{4.8}$$

Since F is normal space and

$$p_u^{r_k}(x_k) = p_u^{r_k}(y_k) + p_u^{r_k}(z_k),$$

so we have $x = (x_k) \in F(X, r)$. Therefore

$$F(X) \subseteq F(X, r). \tag{4.9}$$

Hence from (4.6) and (4.9), we have $F(X,r) = F(X)$.

5. This section deals with the results related to the boundedness properties of subset of $F(X,r)$.

Let R be a normal subset of F and $u \in u(X)$. We define

$$[R,u] = \{x = (x_k) \in F(X,r) : (p_u^{r_k}(x_k)) \in R\}.$$

Theorem 5.1. *Let* $\inf r_k > 0$ *and* $0 < r_k \leq 1$. *Then following statements are equivalent:*

(i) *subset* $[R,u]$ *of* $F(X,r)$ *is metrically bounded;*

(ii) $[R,u]$ *is bounded.*

Using the same procedure as in Theorem 6 of Simons [4], proof follows.

Remark 5.2. In Theorem 5.1 the condition $\inf r_k > 0$ is not needed while proving (ii) $\Rightarrow$ (i).

Now we investigate the bounded set in $F(X,r)$ when $\lim_k r_k = 0$. We define $M_k[R,u] = \sup p_u^{r_k}(x_k)\|e_k\|_F$, where sup is taken over k-th component of $x = (x_k) \in [R,u]$.

Theorem 5.3. *Assuming that* $\lim_k r_k = 0$. *Then a set* $[R,u]$ *is bounded in* $F(X,r)$ *if and only if*

(i) $M_k[R,u] < \infty$ *for all* $k \geq 1$,

(ii) *Given any* $\varepsilon > 0$, *there exists an integer* m *such that* $\| \sum_{k=m}^{\infty} p_u^{r_k}(x_k)e_k\|_F < \varepsilon$,

for all $x = (x_k) \in [R,u]$.

Proof of the theorem is omitted as it can be proved using the same procedure as adopted by Barnes [1, Theorem 2.1].

Theorem 5.4. *Assuming that* $\lim r_k = 0$. *Then, if* $[R,u]$ *is bounded, then* $[R,u]$ *is totally bounded.*

Proof. Since $[R,u]$ is bounded and $r_k \to 0$ $(k \to \infty)$ so for given $\varepsilon(> 0)$ by Theorem 5.3 (ii)

$$\| \sum_{k > k_0}^{\infty} p_u^{r_k}(x_k)e_k\|_F < \varepsilon/2.$$

Let $X^{k_0} = \prod_{i=1}^{k_0} X_i$ = product of X_i, where $X_i = X$, $1 \leq i \leq k_0$ and $P_{k_0} : F(X,r) \to x^{k_0}$ such that $P_{k_0}(x) = (x_1, x_2, \ldots, x_{k_0})$, where $x = (x_k) \in F(X,r)$, it is easy to see that seminorm p_u of X induces a pseudometric

$$d_{u,k}(P_{k_0}(x), P_{k_0}(y)) = \sum_{k=1}^{k_0} P_u(x_k - y_k) \text{ on } X^{k_0}$$

where $x = (x_k)$ and $y = (y_k) \in F(X, r)$ which is equivalent to a pseudometric d'_{u,k_0} where

$$d'_{u,k_0}(P_{k_0}(x), P_{k_0}(y) = \sum_{k=1}^{k_0} p_u^{r_k}(x_k - y_k) \text{ on } X^{k_0}.$$

Since projection map p_{k_0} is continuous, so $p_{k_0}[R, u]$ is bounded. So $P_{k_0}[R, u]$ is totally bounded. Hence

$$p_{k_0}[R, u] \subset \bigcup_{i=1}^{m} S(z^i, \varepsilon/(2m_0))$$

where $m_0 > \max_{i \le k \le k_0} \|e_k\|_F$, and $S(z^i, \varepsilon/(2m_0)) = \{P_k x_0 : d'_{u,k}(z^i, P_k x) < \varepsilon/(2m_0)\}$.
$z^i = (z_1^i, z_2^i, \ldots, z_{k_0}^i)$, $1 \le i \le m$. Now let

$$D_{k_0} = \{b : b = (z_1^i, z_2^i, \ldots, z_{k_0}^i, \theta, \theta, \ldots),\ 1 \le i \le m\}$$

where θ is zero element of X. Clearly D_{k_0} is a finite set and $D_{k_0} \subset [R, u]$. If $x = (x_k) \in [R, u]$, then $P_{k_0}x \in P_{k_0}[R, u]$. But $p_{k_0}[R, u]$ is totally bounded as shown above, so there exists

$$b = (z_1^i, z_2^i, \ldots, z_{k_0}^i, 0, 0, \ldots) \in D_{k_0}$$

for some i such that

$$d'_{u,k_0}(p_{k_0}x, b) = \sum_{k=1}^{k_0} p_u^{r_k}(x_k - z_k^i) < \varepsilon/(2m_0). \qquad \text{Now consider}$$

$$\begin{aligned} d_u(x, b) &\le \sum_{k=1}^{k_0} p_u^{r_k}(x_k - z_k^i)\|e_k\|_F + \| \sum_{k>k_0}^{\infty} p_u^{r_k}(x_k)\|e_k\|_F \\ &< m_0(\varepsilon/2m_0) + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since $x \in [R, u]$ is arbitrary and D_{k_0} is finite, it follows that $[R, u]$ is totally bounded.

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