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RIQUIER PROBLEM IN A BIHARMONIC SPACE

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Abstract. After defining the notion of a biharmonic space Ω which generalizes $\mathbb{R}^n$, Riemann surfaces and Rimannian manifolds, we discuss the Riquier problem for an open set in Ω provided with the Wiener boundary.

1. Introduction

The Riquier problem in $\mathbb{R}^n$ is to find a biharmonic function b on an open set ω such that b and Δb tend respectively to previously given continuous functions f and g on the Euclidean boundary $\partial\omega$ in $\mathbb{R}^n$. The connection between the Riquier problem and the Dirichlet problem is easy to understand.

In this note, we discuss this problem in the general context of a biharmonic space. First we define the notion of a biharmonic space $(\omega, H, H^*, \lambda)$ where ω is a locally compact space provided with two harmonic sheaves H and H^* satisfying the three axioms of Brelot [2] and λ is a fixed Radon measure on ω. According to this definition, $\mathbb{R}^n$ for all $n \geq 1$, Riemann surfaces, Riemannian manifolds and domains ω in $\mathbb{R}^n$ with the solutions of certain elliptic differential operators of order 2 as harmonic functions on ω are all examples of biharmonic spaces.

The final section considers the solution to the Riquier problem on a relatively compact domain ω in a biharmonic space, where ω is endowed with the Wiener boundary.

2. Riquier Problem in Riemann Spaces

A locally integrable function b defined on an open set ω in $\mathbb{R}^n$, $n \geq 2$, is said to be biharmonic if and only if $\Delta^2 b = 0$ in the sense of distributions. It is known that if $\Delta^2 b = 0$, there exists a C^∞–function b_1 on ω such that $b = b_1$ a.e.. Consequently we always assume that a biharmonic function is a C^∞–function.

We shall say that $Q_y(x) > 0$ is the biharmonic Green potential on ω, with pole y, if $\Delta^2 Q_y = \delta_y$ where δ_y is the Dirac measure. Note that $\Delta Q_y(x) = -G_y(x)$ where $G_y(x) = G^\omega(x, y)$ is the symmetric Green function on ω with pole y. It is shown in [1] that the biharmonic Green potentials exist on an arbitrary domain in $\mathbb{R}^n$, if $n \geq 5$;

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however if ω is a relatively compact domain in $\mathbb{R}^n$, $n \geq 2$, then the biharmonic Green potentials exist on ω.

A relatively compact domain ω in $\mathbb{R}^n$ is said to be regular for the Dirichlet problem, if given a finite continuous function f on $\partial\omega$ there exists a harmonic function $H_f^\omega(x)$ on ω such that $\lim_{x\to y} H_f^\omega(x) = f(y)$ for every $y \in \partial\omega$. The Riquier problem for a relatively compact open set ω in $\mathbb{R}^n$is the following: Given two finite continuous functions f and g on $\partial\omega$, find a biharmonic function b on ω such that $\lim_{x\to y} b(x) = f(y)$ and $\lim_{x\to y} \Delta b(x) = g(y)$ for every y in $\partial\omega$.

Solution to the Riquier problem. Let ω be a relatively compact domain in $\mathbb{R}^n$. If ω is regular for the Dirichlet problem and if f and g are two finite continuous functions on $\partial\omega$, then there exists a biharmonic function b on ω such that b tends to f and Δb tends to g on $\partial\omega$.

Proof. Let X be a relatively compact domain in $\mathbb{R}^n$such that $\overline{\omega} \subset X$. Since ω is regular for the Dirichlet problem, there exists a unique continuous function H_g^ω on $\overline{\omega}$ such that $H_g^\omega = g$ on $\partial\omega$ and H_g^ω is harmonic on ω.

Let g_1 be a finite continuous function on X with compact support such that $g_1 = -H_g^\omega$ on $\overline{\omega}$. Since X is relatively compact, the harmonic Green potential $G^X(x, y)$ is defined on X. Let $u(x) = \int_X G^X(x, y)g_1(y)dy$. Then $u(x)$ is finite continuous on X and on ω, $\Delta u = \Delta(G^X \star g_1) = -\delta \star g_1 = -g_1$, and hence $\Delta^2 u = 0$ on ω. Define on ω, $b = u - H_u^\omega + H_f^\omega$. Then b is biharmonic on ω such that b tends to f on $\partial\omega$ and Δu tends to $-g_1 = H_g^\omega = g$ on $\partial\omega$.

The above problem has a counterpart in a Riemannian manifold. Let R be an oriented Riemannian manifold of dimension ≥ 2 with local parameters $x = (x^1, \ldots, x^n)$ and a C^∞–metric tensor g_{ij} such that $g_{ij}x^ix^j$ is positive definite. If g is the determinant of g_{ij}, let us denote the volume element by $dx = g^{\frac{1}{2}} dx^1 \cdots dx^n$; $\Delta = d\delta + \delta d$ denotes the Laplace–Beltrami operator which is also defined as $\Delta f = -div$ grad f. In local coordinates, Δ has an invariant expression $\Delta f = -g^{\frac{1}{2}} \frac{\partial}{\partial x^i}(g^{\frac{1}{2}} g^{ij} \frac{\partial f}{\partial x^j})$ and in the Euclidean case this reduces to $\Delta f = -\sum_{i=1}^n \frac{\partial^2 f}{\partial x^{i2}}$.

A function f is called harmonic (resp. biharmonic) if $\Delta f = 0$ (resp. $\Delta^2 f = 0$). Then we can solve the Riquier problem in R as we did in the Euclidean case $\mathbb{R}^n$. The result is as follows: Let ω be a relatively compact domain in a Riemannian manifold R as above. Suppose ω is regular for the Dirichlet problem. Then given a pair of finite continuous functions f and g on $\partial\omega$, there exists a unique biharmonic function b on ω such that $\lim_{x\to y} b(x) = f(y)$ and $\lim_{x\to y} \Delta b(x) = g(y)$ for every $y \in \partial\omega$.

However this procedure cannot be carried out in a Riemann surface. The difficulty arises from the fact that the Laplacian is not conformally invariant under a parametric change and hence the definition of a biharmonic function on a Riemann surface as a function b such that $\Delta^2 b = 0$ is not acceptable. To overcome this problem, we shall introduce the biharmonic functions on a Riemann surface using the following lemma (see Anandam [1]).

Lemma 2.1. *Let μ be a Radon measure on a domain ω in a Riemann surface R. Then there exists a superharmonic function s on ω such that the measure associated to s in a local Riesz representation is μ.*

Proof. Write $\omega = \cup_{n=1}^{\infty} K_n$ where K_n is compact, $K_n \subset \overset{0}{K}_{n+1}$ and each component of $\omega \setminus K_n$ is not relatively compact. For $n \geq 1$, let μ_n be the restriction of μ to $K_{n+1} \setminus K_n$ and let μ_0 be the restriction of μ to K_1.

Since μ_n has compact support, there exists a superharmonic function s_n on ω with associated measure μ_n in a local Riesz representation. By the approximation theorem of Pfluger [7] p.192, there exists a harmonic function v_n on ω, $n \geq 1$, such that $|s_{n+1} - v_{n+1}| < \frac{1}{2^n}$ on K_n. Set $v_0 = v_1 = 0$. Define $s = \sum_{n=0}^{\infty}(s_n - v_n)$ on ω. Then s is a superharmonic function on ω. For, let K be any compact set in ω; then $K \subset \overset{0}{K}_m$ for $m \geq 2$. Write $s = \sum_{n=0}^{m}(s_n - v_n) + \sum_{n=m+1}^{\infty}(s_n - v_n) = s_1 + s_2$.

Here s_2 is harmonic on $\overset{0}{K}_m$ and the measure associated with s_1 on K is μ. Thus s is superharmonic on a neighbourhood of K with associated measure μ on K. Since K is an arbitrary compact set in ω, s is superharmonic on ω with μ as its assoicated measure in a local Riesz representation.

Notation. If ω is an open set in a Riemann surface R and if μ is a Radon measure on ω, let s be a superharmonic function on ω with associated measure μ constructed as in Lemma 2.1. Note that s is constructed only up to an additive harmonic function on ω. Let us write $Ls = -\mu$ on ω to denote that s is a superharmonic function on ω with μ as its associated measure in a local Riesz represention. Let dx be the surface measure on R and f be a locally dx–integrable function on an open set ω. We write $Lu = -f$ on ω to denote that u is a δ–superharmonic function on ω with associated signed measure λ defined by $d\lambda = f^+ dx - f^- dx$.

With these notations, we shall say a function b defined on an open set ω in R is biharmonic if there exists a harmonic function h on ω such that $Lb = -h$ on ω. Clearly this way of defining a biharmonic function b, if carried out on a Riemannian monifold, coincides with the previous definition of $\Delta^2 b = 0$. Thus, the notion of biharmonic functions can be extended to a Riemann surface without the intervention of the Laplace operator Δ, and the solution to the Riquier problem can be sought in this case also.

This method suggests that biharmonic functions can also be considered in a general locally compact space ω provided with a sheaf of harmonic functions, where the notion of the derivatives does not exist.

3. Riquier Problem in a Harmonic Space

We shall consider now the Riquier problem in a very general set up that englobes the different cases mentioned in the previous section.

Let Ω be a connected locally compact space, with a countable base, but Ω is not compact. Let H and H^* be two sheaves on Ω satisfying the axioms 1, 2, 3 of M. Brelot

[2]. Assume the constants are harmonic in H and H^*. There may or may not exist any potential > 0 in H or H^*. (That is, in the harmonic classification of Rimannian spaces, H and H^* can be hyperbolic or parabolic). But we know that in any domain ω in Ω, there exists an H–potential if and only if $\Omega \setminus \omega$ is not locally polar in the H–sheaf (with a similar result for the H^*–sheaf). We assume three more things on the H–sheaf only:

1. H satisfies the local axiom of proportionality. That is, if ω is a relatively compact domain, $y \in \omega$, and if p and q are two H–potentials on ω with harmonic point support y, then p and q are proportional.

2. With the above assumption, in any domain ω such that $\Omega \setminus \omega$ is not locally polar, there always exists a Green H–potential $G(x, y)$. We assume that such a function is symmetric.

3. If ω is a domain and if h is an H–harmonic function on ω such that $h = 0$ on a neighbourhood of a point in ω, then $h \equiv 0$ on ω.

Definition 3.1. Let the above assumptions on Ω, H and H^* be satisfied. Let us fix a Radon measure $\lambda \geq 0$ on ω. Then we call $(\Omega, H, H^*, \lambda)$ a biharmonic space.

It appears that we have to assume too many things to define a biharmonic space $(\Omega, H, H^*, \lambda)$. But the fact that these conditions are satisfied in the following cases is reassuring.

1. $\Omega = \mathbb{R}^n$; $n \geq 1$, with the harmonic sheaves $H = H^*$ defined as usual by means of the Laplacian; λ is the Lebesgue measure.

2. Riemann surfaces and Riemannian manifolds with the local definition of harmonic functions, λ is the surface or the volume measure.

3. Let Ω be a domain in $\mathbb{R}^n$. Let

$$L = \sum_{i,j} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i}$$

be an elliptic differential operator of order 2; $a_{ij} = a_{ji}$ are of $C^{2,\lambda}$ class and b_i are of $C^{1,\lambda}$ class (here $C^{k,\lambda}$ denotes the class of functions f which are k–times continuously differentiable with their k-th order partial derivatives benig locally Lipschitzien); and the quadratic form $\sum_{i,j} a_{ij}\xi_i\xi_j$ is positive definite for all x in Ω. Then, the class of C^2–functions u in Ω satisfying the equation $Lu = 0$ form a harmonic sheaf H in the sense of Brelot (see Hervé [5]). Let H^* be the harmonic sheaf on Ω defined by a similar elliptic differential operator L^*. Then $(\Omega, H, H^*, \lambda)$ is a biharmonic space, with λ the Lebesgue measure.

Theorem 3.2. *Let (Ω, H) be a harmonic space with the sheaf H satisfying the above–mentioned bypotheses. Then, given a Radon measure μ on an open set ω in Ω, there exists an H–superharmonic function s on ω, denoted by $Ls = -\mu$, such that the measure associated with s in a local Riesz representation is μ.*

Proof. To prove this theorem, we follow the method given in the proof of Lemma 2.1, just replacing the approximation theorem of Pfluger by Theorem 10 of De la Pradelle [4].

Notation. Let ω be an open set in a biharmonic space $(\Omega, H, H^*, \lambda)$. Then given any continuous function f on ω, by Theorem 3.2, there exists a δ–superharmonic function u on ω with respect to the harmonic sheaf H such that the measure associated with u in a local Riesz representation is μ, given by $d\mu = f d\lambda$. We always assume that one such function u is continuous on ω and denote its dependence on f by the notation $Lu = -f$.

Definition 3.3. Let ω be an open set in a biharmonic space $(\Omega, H, H^*, \lambda)$. Then given an H^*–harmonic function on ω, there exists a δ–superharmonic function b on ω with respect to the harmonic sheaf H with associated signed measure μ given by $d\mu = h^* d\lambda$; that is $Lb = -h^*$ on ω. We say that b is biharmonic on ω.

Theorem 3.4. *Let $(\Omega, H, H^*, \lambda)$ be a biharmonic space. Let ω be a relatively compact domain with its boundary $\partial\omega$ regular for the Dirichlet problem in the two harmonic spaces (Ω, H) and (Ω, H^*). Then given two finite continuous functions f and g on $\partial\omega$, there exists a unique H–biharmonic function b on ω such that $\lim_{x\to y} b(x) = f(y)$ and $\lim_{x\to y} Lb(x) = g(y)$ for every $y \in \partial\omega$.*

Proof. Let h^* be the unique H^*–harmonic function on ω tending to $-g$ on $\partial\omega$. Let ω_0 be a relatively compact domain such that $\omega_o \supset \overline{\omega}$. Extend the function h^* as a continuous function g_1 on ω_0. Let $Lu = -g_1$ on ω_0. By hypothesis, u is continuous on ω_0. Let u_1 and f_1 be H–harmonic functions on ω tending to u and f respectively. Let $b(x) = u(x) - u_1(x) + f_1(x)$. Then b is biharmonic on ω, tending to f on $\partial\omega$; moreover on ω, $Lb = Lu = -g_1 = -h^*$ so that Lb tends to g on $\partial\omega$.

4. Riquier Problem with Wiener Boundary

Let R be a Riemannian manifold and ω a relatively compact domain in R. Let $\overline{\omega}$ be the Wiener compactification of ω and let $\partial\omega = \overline{\omega} \setminus \omega$. Suppose f and g are two finite continuous function on $\partial\omega$. Since $\partial\omega$ is resolutive, $h = H^{\partial\omega}_{-f}$ is a bounded harmonic function on ω. If $G(x, y)$ is the Green function on ω, $G \in L^1(\omega)$ and hence if $u(x) = \int_\omega G(x, y)h(y)dy$, $u(x)$ is bounded and $\Delta u = -h$; since h is in $C^\infty(\omega)$, we can assume u also is in $C^\infty(\omega)$.

Recall (Proposition 4.7[6]) that if μ is a measure in ω with $\|\mu\|$ finite, then $\int G(x, y) d\mu(y)$ is a potential on ω. Hence $u(x) = \int_\omega G(x, y)h^+(y)dy - \int_\omega G(x, y)h^-(y)dy$ is the difference of two potentials and hence harmonizable.

Thus u is a bounded Wiener function on ω and consequently u extends as a continuous function on the Wiener compactification $\overline{\omega}$.

Let $v = H^{\partial\omega}_{g-u}$. Then v is bounded harmonic on ω and $b = u + v$ is a bounded biharmonic function on ω, $\Delta b = -h$, b tending to g and Δb tending to f at the regular points of $\partial\omega$ (see Section 4 Chapter VIII [8]).

Proceeding in the same way, we prove the following theorem in the axiomatic case where the theory of Wiener compatification is due to Constantinescu and Cornea [3].

Theorem 4.1. *Let $(\Omega, H, H^*, \lambda)$ be a biharmonic space. Let ω be a relatively compact domain in Ω. Let Γ and Γ^* be the Wiener harmonic boundaries of ω in (ω, H) and (ω, H^*)*

respectively. Then if g and f^* are finite continuous on Γ and Γ^* respectively, there exists a unique biharmonic function b on ω such that b and Lb are bounded, b tends to g on Γ and Lb tends to f^* on Γ^*.

Proof. Extend f^* and g as finite continuous functions on the Wiener boundaries $\partial\omega^*$ and $\partial\omega$ respectively. Let $h^* = H^{\partial\omega^*}_{-f^*}$ on ω^*. Let ω_1 be an H–regular domain containing ω_c, the closure of ω in Ω. By the assumption on the continuity of the biharmonic functions, there exists a finite continuous function u on ω_1 such that $Lu = -1$; that is if $G_{\omega_1}(x,y)$ is the symmetric H–Green kernel on ω_1, $u(x) = \int_{\omega_1} G_{\omega_1}(x,y)d\lambda(y)$ is finite continuous on ω_1. Since u is finite continuous on $\omega_1 \supset \omega_c$, u is bounded on ω_c.

Hence, for $x \in \omega$,

$$\int_\omega G_\omega(x,y)d\lambda(y) \leq \int_\omega G_{\omega_1}(x,y)d\lambda(y) \leq \int_{\omega_1} G_{\omega_1}(x,y)d\lambda(y) = u(x).$$

Consequently, since h^* is bounded on ω and since $G_\omega(x,y)$ is symmetric by hypothesis, $\int_\omega G_\omega(x,y)h^{+*}(y)d\lambda(y)$ and $\int_\omega G_\omega(x,y)h^{-*}(y)d\lambda(y)$ are well defined bounded H^*–potentials on ω; hence $v(x) = \int_\omega G_\omega(x,y)h^*(y)d\lambda(y)$ is the difference of two bounded potentials and hence harmonizable; also v benig biharmonic, is continuous. Thus v is a bounded Wiener function on ω and hence v extends continuously on the Wiener compactification $\overline{\omega}$.

Since g is a finite continuous function on $\partial\omega$, there exists a bounded H–harmonic h_1 on ω tending to $g - v$ on Γ. Let $b = v + h_1$ on ω. Then b is a bounded biharmonic function on ω such that b tends to g on Γ and $Lb = -h^*$ tends to f^* on Γ^*.

For the uniqueness of b, notice that if u is bounded on ω and if Lu is bounded on ω^* such that u and Lu tend to 0 on Γ and Γ^* respectively, then $u \equiv 0$. For, Lu is bounded H^*–harmonic on ω and tends to 0 on Γ^* and hence $Lu \equiv 0$; this means that u is bounded H–harmonic on ω and tends to 0 on Γ and hence $u \equiv 0$.

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