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SHEFFER POLYNOMIALS AND APPROXIMATION OPERATORS

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Abstract. In this paper we are studying the sequence of linear positive operators $(P_n^{(Q,S)})$ defined in (2). Using the Bohman-Korovkin uniform convergence criterion we are proving that the sequence $(P_n^{(Q,S)})$ converges uniformly to the identity operator.
In addition we give some estimates. Finally we consider two examples $(P_n^{(A,S)})$ and $(P_n^{(\nabla,S)})$ defined in (25), (27).

1. Introduction

Let Π be the algebra of all polynomials in one variable, with real coefficients and let $Q:\Pi\to\Pi$ be a delta operator with basic polynomial set (p_n).
Operators $L_n: C[0,1]\to C[0,1]$, $n=0,1,2,\dots$ of the form

$$(L_nf)(x)=\frac{1}{p_n(1)}\sum_{k=0}^{n}\binom{n}{k}p_k(x)p_{n-k}(1-x)f\left(\frac{k}{n}\right) \tag{1}$$

have been studied in [1]-[2], [4]-[8], [10]-[11].
Next we consider a Sheffer set relative to Q, namely $s_0(x)=c\neq 0$ and $Qs_n(x)=ns_{n-1}(x)$.
Let S be an invertible shift invariant operator such that $Ss_n(x)=p_n(x)$.

Definition. (Q,S) belongs to the class W if the following conditions are satisfied

i) $p_n'\geq 0$, $s_n(0)\geq 0$, $s_0(x)=1$, $s_n(1)\neq 0$ for $n=1,2,3,\dots$.
ii) $\lim_{n\to\infty}\dfrac{\sum_{i=2}^{n}\binom{n-2}{i-2}s_{n-i}(0)(Q'^{-2}p_{i-2})(1)}{s_n(1)}=1.$

where Q' is the Pincherle derivative of the operator Q.
The identity

$$s_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}p_k(x)s_{n-k}(y)$$

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which characterizes the Sheffer polynomials, suggest us to consider the sequence of linear polynomial operators $P_n^{(Q,S)} : C[0,1] \to C[0,1]$ defined by

$$(P_n^{(Q,S)} f)(x) = \frac{1}{s_n(1)} \sum_{k=0}^{n} \binom{n}{k} p_k(x) s_{n-k}(1-x) f\left(\frac{k}{n}\right) \tag{2}$$

where $(Q,S) \in W,\ f \in C[0,1]$.
For example if $Q = DE^{-\beta},\ p_n(x) = x(x+n\beta)^{n-1},\ s_n(x) = (x+n\beta)^n$ we obtain the Cheney-Sharma operator (1964, see [2])

$$(S_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{x(x+k\beta)^{k-1}(1-x+\overline{n-k}\beta)^{n-k}}{(1+n\beta)^n} f\left(\frac{k}{n}\right)$$

where $E^{-\beta}$ is the shift operator and D the derivative.

Lemma 1. *If $(Q,S) \in W$ then $P_n^{(Q,S)}$, $n = 1,2,3,\ldots$ are positive operators.*

Proof. From the identities (see [9])

$$p_n(x) = x \sum_{k=0}^{n-1} \binom{n-1}{k} p_{n-k-1}(x) p'_{k+1}(0) \tag{3}$$

$$s_n(x) = \sum_{k=0}^{n} \binom{n}{k} s_k(0) p_{n-k}(x) \tag{4}$$

on obtain $p_n(x) \geq 0,\ s_n(x) \geq 0,\ n = 1,2,\ldots$ for every $x \in [0,1]$.

Lemma 2. *If $p'_n(0) \geq 0$ for $n = 1,2,\ldots$, then*

$$0 < (Q'^{-2} p_{n-2})(1) \leq p_n(1), \quad n = 2,3,\ldots$$

Proof. According to theorem 9 from [4], and lemma 1 we have

$$\frac{p_{n-1}(x)}{x} p'_1(0) \leq (Q'^{-2} p_{n-2})(x) \leq \frac{p_n(x)}{x^2}, \quad x > 0,\ n = 2,3,\ldots.$$

For $x = 1$ this gives

$$0 < (Q'^{-2} p_{n-2})(1) \leq p_n(1) \quad \text{for} \quad n = 2,3,\ldots$$

2. Results

If m is a natural number, let us denote

$$S_m(x,y,n) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) s_{n-k}(y) \left(\frac{k}{n}\right)^m$$

and let P be the linear operator defined by $P = xQ'^{-1}$.
Using the method from [4] or [6] and the identity

$$s_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) s_{n-k}(y)$$

we obtain

$$S_m(x,y,n) = \frac{1}{n^m} \sum_{k=0}^{m} k! \binom{n}{k} S(m,k) P^k E^y s_{n-k}(x) \tag{5}$$

where $S(m,k)$ are the Stirling numbers of second kind and E^y is the shift operator $(E^y f)(x) = f(x+y)$.
Taking into account that

$$PE^y p_{n-1}(x) = \frac{x}{x+y} p_n(x+y) \tag{6}$$

$$s_{n-1}(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} s_{n-k-1}(0) p_k(x) \tag{7}$$

(see [6]) we obtain

$$PE^y s_{n-1}(x) = \frac{x}{x+y} \sum_{i=1}^{n} \binom{n-1}{i-1} s_{n-i}(0) p_i(x+y) \tag{8}$$

Now using the fact that

$$P^2 E^y p_{n-2}(x) = \frac{x}{x+y} p_n(x+y) - xyQ'^{-2} p_{n-2}(x+y)$$

(see [6]) and

$$s_{n-2}(x) = \sum_{k=0}^{n-2} \binom{n-2}{k} s_{n-k-2}(0) p_k(x)$$

we obtain

$$P^2 E^y s_{n-2}(x) = \frac{x}{x+y} \sum_{i=2}^{n} \binom{n-2}{i-2} s_{n-i}(0) p_i(x+y)$$

$$-xy \sum_{i=2}^{n} \binom{n-2}{i-2} s_{n-i}(0) Q'^{-2} p_{i-2}(x+y) \tag{9}$$

Next we have

$$S_0(x,y,n) = s_n(x+y)$$

$$S_1(x,y,n) = \frac{x}{x+y} \sum_{i=1}^{n} \binom{n-1}{i-1} s_{n-i}(0) p_i(x+y)$$

$$S_2(x,y,n)=\frac{1}{n}\frac{x}{x+y}\sum_{i=1}^{n}\binom{n-1}{i-1}s_{n-i}(0)p_i(x+y)+\frac{n-1}{n}\frac{x}{x+y}\sum_{i=2}^{n}\binom{n-2}{i-2}s_{n-i}(0)p_i(x+y)$$
$$-\frac{n-1}{n}xy\sum_{i=2}^{n}\binom{n-2}{i-2}s_{n-i}(0)(Q'^{-2}p_{i-2})(x+y)$$

Hence

$$(P_n^{(Q,S)}e_0)(x)=e_0 \tag{10}$$
$$(P_n^{(Q,S)}e_1)(x)=\frac{\sum_{i=1}^{n}\binom{n-1}{i-1}s_{n-i}(0)p_i(1)}{s_n(1)}x \tag{11}$$
$$(P_n^{(Q,S)}e_2)(x)=\frac{1}{n}\frac{\sum_{i=1}^{n}\binom{n-1}{i-1}s_{n-i}(0)p_i(1)}{s_n(1)}x+\frac{n-1}{n}\frac{\sum_{i=2}^{n}\binom{n-2}{i-2}s_{n-i}(0)p_i(1)}{s_n(1)}x$$
$$-\frac{n-1}{n}\frac{\sum_{i=2}^{n}\binom{n-2}{i-2}s_{n-i}(0)(Q'^{-2}p_{i-2})(1)}{s_n(1)}x(1-x) \tag{12}$$

where $e_k(x)=x^k$, $k=0,1,2,\ldots$.
Let us denote

$$a_n=\frac{\sum_{i=1}^{n}\binom{n-1}{i-1}s_{n-i}(0)p_i(1)}{s_n(1)}$$
$$b_n=\frac{\sum_{i=2}^{n}\binom{n-2}{i-2}s_{n-i}(0)p_i(1)}{s_n(1)}$$
$$c_n=\frac{s_{n-1}(0)p_1(1)+\sum_{i=2}^{n}\binom{n-2}{i-2}s_{n-i}(0)p_i(1)}{s_n(1)}$$
$$d_n=\frac{\sum_{i=2}^{n}\binom{n-2}{i-2}s_{n-i}(0)(Q'^{-2}p_{i-2})(1)}{s_n(1)}$$

Therefore

$$(P_n^{(Q,S)}e_1)(x)=a_nx \tag{13}$$
$$(P_n^{(Q,S)}e_2)(x)=b_nx^2+\frac{1}{n}c_nx+(x-x^2)\left(b_n-\frac{n-1}{n}d_n\right) \tag{14}$$

Taking into account that

$$\binom{n-2}{i-2}\le\binom{n-1}{i-1}\le\binom{n}{i},\quad (Q,S)\in W$$

and using lemma 2, we obtain

$$d_n\le\frac{\sum_{i=2}^{n}\binom{n-2}{i-2}s_{n-i}(0)p_i(1)}{s_n(1)}=b_n\le\frac{\sum_{i=2}^{n}\binom{n-1}{i-1}s_{n-i}(0)p_i(1)}{s_n(1)}\le a_n\le\frac{\sum_{i=1}^{n}\binom{n}{i}s_{n-i}(0)p_i(1)}{s_n(1)}\le 1 \tag{15}$$

whence

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 1.$$

Now using $\frac{n-i}{i-1}\binom{n-2}{i-2} \le \binom{n}{i}$ for $i = 2, 3, \dots, n$, we have

$$0 \le c_n \le \frac{ns_{n-1}(0)p_1(1) + \sum_{i=2}^{n}\binom{n}{i}s_{n-i}(0)p_i(1)}{s_n(1)} = \frac{\sum_{i=1}^{n}\binom{n}{i}s_{n-i}(0)p_i(1)}{s_n(1)} \le 1$$

Hence

$$\lim_{n\to\infty}\left(\frac{1}{n}c_n\right) = 0.$$

Theorem 1. *If* $(Q,S) \in W$ *and* $f \in C[0,1]$, *then*

$$\lim_{n\to\infty} \|f - P_n^{(Q,S)} f\| = 0$$

where $\|f\| = \max_{t\in[0,1]} |f(t)|$.

Proof. According to the Bohman-Korovkin theorem it is sufficient to show that

$$\lim_{n\to\infty} \|e_k - P_n^{(Q,S)} e_k\| = 0, \quad k = 0, 1, 2.$$

In our case, from (10), (13), (14)

$$\|e_0 - P_n^{(Q,S)} e_0\| = 0$$

$$\|e_1 - P_n^{(Q,S)} e_1\| = \|x - a_n x\| = 1 - a_n$$

and $\lim_{n\to\infty} \|e_1 - P_n^{(Q,S)} e_1\| = 0$.
Finally

$$\begin{aligned}\|e_2 - P_n^{(Q,S)} e_2\| &= \left\|(1-b_n)x^2 - \frac{1}{n}c_n x - (x - x^2)\left(b_n - \frac{n-1}{n}d_n\right)\right\| \\ &\le (1-b_n) + \frac{1}{n}c_n + \frac{1}{4}\left(b_n - \frac{n-1}{n}d_n\right)\end{aligned}$$

where $1 - \frac{1}{n} \le 1 \le \frac{b_n}{d_n}$.
Therefore

$$\lim_{n\to\infty} \|e_2 - P_n^{(Q,S)} e_2\| = 0$$

Theorem 2. *Let* $(P_n^{(Q,S)})$, $(Q,S) \in W$ *be the sequence of linear positive operators defined in* (2). *If* $f \in C^2[0,1]$, $m_f = \min_{x\in[0,1]} f''(x)$, $M_f = \max_{x\in[0,1]} f''(x)$, *then for* $x \in [0,1]$

$$\frac{1}{2}m_f\theta_n(x) \le (P_n^{(Q,S)} f)(x) - f(a_n x) \le \frac{1}{2}M_f\theta_n(x) \tag{16}$$

where $\theta_n = (P_n^{(Q,S)} e_2)(x) - a_n^2 x^2$.

Proof. If $h \in C[0,1]$ is convex on I, $c_1, c_1, \dots, c_n$ are non-negative numbers with $c_0 + c_1 + \dots + c_n = 1$, then for every system of points $x_0, x_1, \dots, x_n$ from $[0,1]$

$$h\left(\sum_{k=0}^{n} c_k x_k\right) \le \sum_{k=0}^{n} c_k h(x_k).$$

Let us consider

$$c_k = \frac{1}{s_n(1)} \binom{n}{k} p_k(x) s_{n-k}(1-x)$$

$$x_k = \frac{k}{n}$$

x being arbitrary in $[0,1]$. Then

$$\sum_{k=0}^{n} c_k x_k = (P_n^{(Q,S)} e_1)(x) = a_n x$$

and hence

$$h(a_n x) \le (P_n^{(Q,S)} h)(x) \tag{17}$$

Now we observe that:

$$h_1(x) = \frac{1}{2} M_f x^2 - f(x); \quad h_2(x) = f(x) - \frac{1}{2} m_f x^2$$

are convex on $[0,1]$.
Hence

$$h_1(a_n x) \le (P_n^{(Q,S)} h_1)(x)$$
$$h_2(a_n x) \le (P_n^{(Q,S)} h_2)(x)$$

and

$$(P_n^{(Q,S)} h_1)(x) = \frac{1}{2} M_f (P_n^{(Q,S)} e_2)(x) - (P_n^{(Q,S)} f)(x)$$
$$(P_n^{(Q,S)} h_2)(x) = (P_n^{(Q,S)} f)(x) - \frac{1}{2} m_f (P_n^{(Q,S)} e_2)(x)$$

Now using (17)

$$\frac{1}{2} M_f a_n^2 x^2 - f(a_n x) \le \frac{1}{2} M_f (P_n^{(Q,S)} e_2)(x) - (P_n^{(Q,S)} f)(x)$$
$$f(a_n x) - \frac{1}{2} m_f a_n^2 x^2 \le (P_n^{(Q,S)} f)(x) - \frac{1}{2} m_f (P_n^{(Q,S)} e_2)(x)$$

Finally

$$\frac{1}{2} m_f [(P_n^{(Q,S)} e_2)(x) - a_n^2 x^2] \le (P_n^{(Q,S)} f)(x) - f(a_n x) \le \frac{1}{2} M_f [(P_n^{(Q,S)} e_2)(x) - a_n^2 x^2]$$

Theorem 3. *Let* $(Q, S) \in W$, $f \in C[0,1]$ *and denote by* $w(f;\delta)$ *the modulus of continuity of the function* f. *If* $x \in [0,1]$, *then*

$$|f(x) - (P_n^{(Q,S)} f)(x)| \leq 2w(f; \sqrt{A_n}) \tag{18}$$

$$\|f - P_n^{(Q,S)} f\| \leq 2w(f; \sqrt{B_n}) \tag{19}$$

where

$$A_n = (b_n - 2a_n + 1)x^2 + \frac{1}{n}c_n x + (x - x^2)\left(b_n - \frac{n-1}{n}d_n\right)$$

$$B_n = (b_n - 2a_n + 1) + \frac{1}{n}c_n + \frac{1}{4}\left(b_n - \frac{n-1}{n}d_n\right)$$

Proof. If $L : C[0,1] \to C[0,1]$ is a linear positive operator, then (see for instance theorem 4.2 and 4.5 from [3])

$$|f(x) - (L_n f)(x)| \leq 2w(f; \sqrt{(L\Omega_2)(x)} \tag{20}$$

$$\|f - Lf\| \leq \inf_{m=1,2,\ldots} \{1 + \delta^{-m} \|L\Omega_m\|\} w(f, \delta) \tag{21}$$

where $\delta > 0$, $\Omega_j(t, x) = \Omega_j(t) = |t - x|^j$.
We have

$$(P_n^{(Q,S)} \Omega_2)(x) = (1 - 2a_n + b_n)x^2 + \frac{1}{n}c_n x + (x - x^2)\left(b_n - \frac{n-1}{n}d_n\right)$$

We observe that

$$1 - 2a_n + b_n = s_n(0) + (n-2)s_{n-1}(0)p_1(1) + \sum_{i=2}^{n}\left[\binom{n-2}{i-2} + \binom{n}{i} - 2\binom{n-1}{i-1}\right] s_{n-i}(0)p_i(1) \geq 0$$

for $n = 2, 3, \ldots$.
Therefore

$$(P_n^{(Q,S)} \Omega_2)(x) \geq 0 \quad \text{for} \quad x \in [0,1]$$

and

$$|f(x) - (P_n^{(Q,S)} f)(x)| \leq 2w(f; \sqrt{A_n}).$$

For $m = 2$ in (19) we obtain

$$\|f - P_n^{(Q,S)} f\| \leq 2w(f; \sqrt{B_n}).$$

3. Examples

Let us consider the following examples

I. The Abel operator $A = DE^{-\alpha}$ is a delta operator with basic polynomials

$$p_n^\alpha = x(x+n\alpha)^{n-1}$$

and we consider that

$$\alpha = \alpha(n) > 0, \quad \lim_{n\to\infty} n^3\alpha(n) = 0 \tag{22}$$

Let (s_n^α) be the Sheffer set

$$s_n^\alpha = (x+\overline{n-1}\alpha)^n, \quad s_0(x) = 1$$

namely $As_n = ns_{n-1}$, $n = 1, 2, 3, \ldots$.
We have

i) $[p_n^\alpha]'(0) = (n\alpha)^{n-1} > 0$, $s_n(0) = (n-1)^n\alpha^n \geq 0$, $s_n^\alpha(1) = (1+\overline{n-1}\alpha)^n \neq 0$.
ii) According to theorem 9 (from [4]) we have

$$\frac{p_{n-1}^\alpha(1)}{p_n^\alpha(1)}[p_1^\alpha]'(0) \leq \frac{(Q'^{-2}p_{n-2}^\alpha)(1)}{p_n^\alpha(1)} \leq 1, \quad n = 2, 3, \ldots$$

From (22)

$$\lim_{n\to\infty} \alpha(n) = \lim_{n\to\infty} n\alpha(n) = \ln n^2\alpha(n) = 0.$$

Hence

$$\lim_{n\to\infty} \frac{p_{n-1}^\alpha(1)}{p_n^\alpha(1)} = \lim_{n\to\infty} \frac{(1+\overline{n-1}\alpha)^{n-1}}{(1+n\alpha)^{n-1}} = 1$$

and using $(p_1^\alpha)'(0) = 1$ we have

$$\lim_{n\to\infty} \frac{(Q'^{-2}p_{n-2}^\alpha)(1)}{p_n^\alpha(1)} = 1 \tag{23}$$

Now

$$\lim_{n\to\infty} \frac{p_n^\alpha(1)}{s_n^\alpha(1)} = \lim_{n\to\infty} \frac{(1+n\alpha)^{n-1}}{(1+\overline{n-1}\alpha)^n} = 1 \tag{24}$$

From (23), (24) we obtain

$$\lim_{n\to\infty} \frac{(Q'^{-2}p_{n-2}^\alpha)(1)}{s_n^\alpha(1)} = 1.$$

Now using (15) we observe that

$$\frac{(Q'^{-2}p_{n-2}^\alpha)(1)}{s_n^\alpha} \leq d_n \leq 1$$

namely $\lim_{n\to\infty} d_n = 1$.
Therefore $(A, S) \in W$ and the sequence of linear polinomial operator

$$P_n^{(A,S)} : C[0,1] \to C[0,1]$$

defined by

$$(P_n^{(A,S)}f)(x) = \frac{1}{(1+\overline{n-1}\alpha)^n}\sum_{k=0}^{n}\binom{n}{k}x(x+k\alpha)^{k-1}(1-x+\overline{n-k-1}\alpha)^{n-k}f\left(\frac{k}{n}\right) \quad (25)$$

with $\alpha = \alpha(n) > 0$, $\ln n^3\alpha(n) = 0$, verify

$$\lim_{n\to\infty} \|f - P_n^{(A,S)}f\| = 0,$$

for every $C[0,1]$, where

$$\|f\| = \max_{t\in[0,1]} |f(t)|.$$

II. Backward difference operator

$$\nabla = \frac{1}{\alpha}(I - E^{-\alpha})$$

where $\alpha = \alpha(n) > 0$, $\lim_{n\to\infty}\alpha(n) = 0$, with the basic sequence

$$p_n^{\alpha}(x) = x(x+\alpha)\cdots(x+\overline{n-1}\alpha), \quad p_0(x) = 1. \quad (26)$$

Let (s_n^{α}) be a Sheffer set relative to ∇ defined by

$$s_n^{\alpha}(x) = x(x+\alpha)\cdots(x+\overline{n-2}\alpha)(x+\overline{n-1}\alpha+n), \quad s_0(x) = 1$$

We have

i) $[p_n^{\alpha}]'(0) = (n-1)!\alpha^{n-1} > 0$, $s_n(0) \geq 0$, $n = 1, 2, \ldots, s_n^{\alpha}(1) \neq 0$;
ii) We observe that

$$\frac{(\nabla'^{-2}p_{n-2}^{\alpha})(1) + (n-2)(\nabla'^{-2}p_{n-3}^{\alpha})(1)}{s_n^{\alpha}(1)} \leq d_n \leq 1, \quad n = 3, 4, \ldots$$

But

$$(\nabla'^{-2}p_{n-2}^{\alpha})(1) = (1+2\alpha)(1+3\alpha)\cdots(1+\overline{n-1}\alpha),$$
$$(\nabla'^{-2}p_{n-3}^{\alpha})(1) = (1+2\alpha)(1+3\alpha)\cdots(1+\overline{n-2}\alpha),$$

whence

$$\lim_{n\to\infty}\frac{(\nabla'^{-2}p_{n-2}^{2})(1) + (n-2)(\nabla'^{-2}p_{n-3}^{\alpha})(1)}{s_n^{\alpha}(1)} = \lim_{n\to\infty}\frac{1+\overline{n-1}\alpha+n-2}{(1+\alpha)(1+\overline{n-1}\alpha+n)} = 1$$

and

$$\lim_{n\to\infty} d_n = 1.$$

Therefore $(\nabla, S) \in W$ and the sequence of linear polinomial operators

$$P_n^{(\nabla,S)} : C[0,1] \to C[0,1]$$

defined by

$$(P_n^{(Q,S)} f)(x) = \frac{1}{1 + \overline{n-1}\alpha + n} \sum_{k=0}^{n} w_{n,k}(x,\alpha) f\left(\frac{k}{n}\right) \tag{27}$$

where

$$w_{n,k}(x,\alpha) = \binom{n}{k} \frac{\prod_{i=0}^{k-1}(x + i\alpha) \prod_{i=0}^{n-k-2}(1 - x + i\alpha)}{(1+\alpha)(1+2\alpha)\cdots(1+\overline{n-2}\alpha)} (1 - x + \overline{n-k-1}\alpha + n - k)$$

$\alpha = \alpha(n) > 0$, $\lim_{n\to\infty} \alpha(n) = 0$, verify

$$\lim_{n\to\infty} \|f - P_n^{(\nabla,S)} f\| = 0, \quad \text{for every} \quad f \in C[0,1].$$

Finally we observe that $P_n^{(D,I)}$ is the Bernstein operator where D is the derivative operator and I the identity

$$(P_n^{(D,I)} f)(x) = (B_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^k f\left(\frac{k}{n}\right)$$

Operators such as $P_n^{(Q,I)}$ had been studied by Brass H. [1], Cheney E. W., Sharma A. [2], Manole C. [6], Moldovan Gr. [7], Mühlbach G. [8], Stancu D. D. [10], [11]

$$(P_n^{Q,I} f)(x) = \frac{1}{p_n(1)} \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(1-x) f\left(\frac{k}{n}\right)$$

For this operators condition ii) of definition becomes

$$\lim_{n\to\infty} \frac{(Q'^{-2} p_{n-2})(1)}{p_n(1)} = 1.$$

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