TAMKANG JOURNAL OF MATHEMATICS
Volume 49, Number 2, 155-182, June 2018
doi:10.5556/j.tkjm.49.2018.2586

This paper is available online at http://journals.math.tku.edu.tw/index.php/TKJM/pages/view/onlinefirst

GLOBAL ANALYSIS OF STOCHASTIC SIR MODEL WITH VARIABLE DIFFUSION RATES

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Abstract. In this article, a stochastic SIR epidemic model with treatment rate in a population of varying size is proposed and investigated. For the stochastic version, we briefly discuss the existence of global unique solutions and using the Lyapunov function, the disease free equilibrium solution is globally asymptotic stabe if $\mathscr{R}_0 \leq 1$ and the endemic equilibrium solution is obtained when $\mathscr{R}_0 > 1$. The main attention is paid to the pth-moment exponentially stable on the system, proved under suitable assumptions on the white noise perturbations and the optimal control for the deterministic model. Finally numerical simulations are done to show our theoretical results and to demonstrate the complicated dynamics of the model.

1. Introduction

Diseases can affect people not only physically but also emotionally as contracting and living with a disease can affect the person's perspective of life. Most infectious diseases could be driven toward eradication by undergoing proper treatment. With years mathematical models have become an important tool in describing the dynamics of the spread of an infectious disease and the effect of vaccination and treatments on its dynamics. As such, these models can be particularly useful in comparing the effect of various prevention, therapy and control programs. Since a variety of these programs are available, it is a natural objective to design optimal programs in terms of some pre-assumed criterion. This brings the application of the tools of optimal control to these problems. Optimal control has a long history of being applied to problems in biomedicine, particularly, to models for cancer chemotherapy.

For some diseases, medical treatments can be given to patients to cure the infection but there may not be vaccine to immunize susceptible individuals (e.g. Malaria, Cold). For few other diseases there is no cure but individuals can be vaccinated against getting infected (e.g. Polio). Nevertheless, diseases like Measles, Cholera, and Tuberculosis all have approved

Received September 13, 2017, accepted December 27, 2017.
2010 *Mathematics Subject Classification.* 60H10. 60H30. 92D30.
Key words and phrases. SIR model, Stochastic Asymptotic stability, pth-moment exponentially stability, Lyapunov function, Optimal control.
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medical treatment options and vaccine. From then on, many authors have developed and investigated the SIR epidemic model.

SIR models have been around for many years. The first one was introduced and published in 1927, in “Contribution to the Mathematical Theory of Epidemics", written by William Kermack and Anderson McKendrick [7]. In this paper we improve the work by taking into account one additional control measure treatment to minimize infection on both susceptible and infected individuals. We want to minimize the total amount of control measures used for the control strategy in the form of vaccination and treatment. We also assume that the susceptible individuals are vaccinated before the infection and treatment should be provided to the infected individual. The concept behind vaccination is to reduce the proportion of susceptible individuals until the disease cannot survive.

The SIR model is one of the most important models in epidemiological patterns and disease control. Various SIR models have been investigated by many researchers due to their theoretical and practical significance and the study of the model will continue to be one of the most interesting hot topics in both epidemiology and mathematical ecology.
The classical SIR model (1.1) without treatment in the infection compartment can be expressed as follows

$$\begin{aligned} S'(t) &= \Pi - \mu S - \beta SI, \\ I'(t) &= \beta SI - (\mu + \alpha) I, \\ R'(t) &= \alpha I - \mu R, \end{aligned} \tag{1.1}$$

The classical SIR model (1.2) with treatment in the infection compartment can be expressed as follows

$$\begin{aligned} S'(t) &= \Pi - \mu S - \beta SI, \\ I'(t) &= \beta SI - (\mu + \alpha + r) I, \\ R'(t) &= (\alpha + r) I - \mu R, \end{aligned} \tag{1.2}$$

where $S(t)$ denotes the number of the individuals susceptible to the disease, $I(t)$ denotes the number of the individuals who are infectious and $R(t)$ denotes the number of the individuals who are recovered at time t respectively.The parameter Π represents the recruitment rate of the population, μ represents the natural mortality rate(due to other factors without the studied disease), α represents the recovery rate of infected individuals, β represents the disease transmission rate and r represents the treatment rate of the infected individuals. All the parameters are assumed to be positive constants.

All infectious diseases are subject to randomness in terms of the nature of transmission. So, we perturbed the deterministic system (1.2) by a white noise and obtained a stochastic

counterpart by replacing the rates β by $\beta + F_1(S,I,R)\frac{dW_1}{dt}$ and α by $\alpha + F_2(S,I,R)\frac{dW_2}{dt}$, where F_i, $i = 1,2$ are locally Lipschitz-continuous functions on $\mathbb{D}$ and W_i, $i = 1,2$ are $i.i.d$. Wiener processes defined on a filtered complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, where

$$\mathbb{D} = \left\{(S,I,R) \in \mathbb{R}^3 : S \geq 0, I \geq 0, R \geq 0, S + I + R \leq \frac{\Pi}{\mu}\right\}.$$

The stochastic version corresponding to the system (1.2) takes the following form,

$$\begin{aligned} dS &= \left(\Pi - \mu S - \beta SI\right)dt - SIF_1(S,I,R)dW_1, \\ dI &= \left(\beta SI - (\mu + \alpha + r)I\right)dt + SIF_1(S,I,R)dW_1 - IF_2(S,I,R)dW_2, \\ dR &= \left((\alpha + r)I - \mu R\right)dt + IF_2(S,I,R)dW_2. \end{aligned} \tag{1.3}$$

Since the diffusion coefficients F_i, $i = 1,2$ are arbitrary local Lipschitz-continuous functions, we have a family of stochastic SIR model.

The rest of the paper is organized as follows. In Section 3, we discuss the existence of a unique global solution for the stochastic SIR model (1.3). In Section 4, we discuss the stochastic asymptotic stability of infection free equilibrium and the endemic equilibrium with the help of Lyapunov functions. In Section 5, we discuss the pth-moment exponentially stability on the stochastic SIR model. In Section 6, we formulate an optimal control problem subject to the SIR model, we discuss the existence of an optimal control and the characterize the optimal control and derive the optimality system using Pontryagin's maximum principle. In Section 7, We visualize our result.

2. Preliminaries

Consider the d-dimensional stochastic differential equation of the form

$$dX(t) = f(X(t),t)dt + g(X(t),t)dW(t), \quad X(t_0) = X_0, \tag{2.1}$$

with $t_0 \leq t \leq T < \infty$, where $f : \mathbb{R}^d \times [t_0, T] \to \mathbb{R}^d$ and $g : \mathbb{R}^d \times [t_0, T] \to \mathbb{R}^{d\times m}$ are Borel measurable, $W = \{W(t)\}_{t\geq t_0}$ is an $\mathbb{R}^m$ - valued Wiener process, and X_0 is an $\mathbb{R}^d$ - valued random variable defined on a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$.

The infinitesimal generator $\mathscr{L}$ associated with the SDE (2.1) is given by

$$\mathscr{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(x,t)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{m}\left(g(x,t)g^T(x,t)\right)_{ij}\frac{\partial^2}{\partial x_i \partial x_j}. \tag{2.2}$$

Theorem 2.1 ($\mathbb{D}$-invariance. Khas'minskii [4] as appears in [3]). *Let $\mathbb{D}$ and $\mathbb{D}_n$ be open sets in $\mathbb{R}^d$ with*

$$\mathbb{D}_n \subseteq \mathbb{D}_{n+1}, \qquad \overline{\mathbb{D}}_n \subseteq \mathbb{D}, \qquad \text{and} \quad \mathbb{D} = \bigcup_n \mathbb{D}_n$$

and suppose f and g satisfy the existence and uniqueness conditions for solutions of (2.1), on each set $\{(t,x) : t > t_0, x \in \mathbb{D}_n\}$. Suppose there is a non-negative continuous function $V : \mathbb{D} \times [t_0, T] \to \mathbb{R}_+$ with continuous partial derivatives and satisfying $\mathscr{L}V \le cV$ for some positive constant c and $t > t_0$, $x \in \mathbb{D}$. If also,

$$\inf_{t>t_0,\ x\in\mathbb{D}\setminus\mathbb{D}_n} V(x,t) \to \infty \quad as \quad n \to \infty,$$

then, for any X_0 independent of $\sigma(W)$ such that $\mathbb{P}(X_0 \in \mathbb{D}) = 1$, there is a unique Markovian, continuous time solution $X(t)$ of (2.1) with $X(0) = X_0$, and $X(t) \in \mathbb{D}$ for all $t > 0$ (a.s.).

Assume that f and g satisfy, in addition to the existence and uniqueness assumptions, $f(x^*, t) = 0$ and $g(x^*, t) = 0$, for equilibrium solution x^*, for $t \ge t_0$. Furthermore, let's assume that x_0 be a non-random constant with probability 1.

Definition 2.2. The equilibrium solution x^* of the SDE (2.1) is stochastically stable (stable in probability) if for every $\epsilon > 0$ and $s \ge t_0$

$$\lim_{x_0 \to x^*} \mathbb{P}\left(\sup_{s \le t < \infty} \|X_{s,x_0}(t)\| \ge \epsilon \right) = 0 \tag{2.3}$$

where $X_{s,x_0}(t)$ denotes the solution of (2.1), satisfying $X(s) = x_0$, at time $t \ge s$.

Definition 2.3. The equilibrium solution x^* of the SDE (2.1), is said to be stochastically asymptotically stable if it is stochastically stable and

$$\lim_{x_0 \to x^*} \mathbb{P}\left(\lim_{t\to\infty} X_{s,x_0}(t) = x^* \right) = 1. \tag{2.4}$$

Definition 2.4. The equilibrium solution x^* of the SDE (2.1) is said to be globally stochastically asymptotically stable if it is stochastically stable and for every x_0 and every s

$$\mathbb{P}\left(\lim_{t\to\infty} X_{s,x_0}(t) = x^* \right) = 1. \tag{2.5}$$

Definition 2.5. pth moment exponentially stable if there is a pair of positive constants C_1 and C_2 such that for all $X_0 \in \mathbb{R}^d$

$$\mathbb{E}\left(|X(t, X_0)|^p \right) \le C_1 |X_0|^p e^{-C_2 t} \quad \text{on} \quad t \ge 0. \tag{2.6}$$

The following theorem is a useful criterion for stochastic stability of equilibrium solutions in terms of Lyapunov function.

Theorem 2.6 (Arnold [1])**.** *Assume that f and g satisfy the existence and uniqueness assumptions and they have continuous coefficients with respect to t.*

(i) *Suppose that there exist a positive definite function* $V \in C^{2,1}(U_h \times [t_0,\infty))$, *where* $U_h = \{x \in \mathbb{R}^d : \|x - x^*\| < h\}$, *for* $h > 0$, *such that*

$$\text{for all } t \geq t_0, \quad x \in U_h : \quad \mathscr{L}V(x,t) \leq 0. \tag{2.7}$$

Then, the equilibrium solution x^* *of* (2.1) *is stochastically stable.*

(ii) *If, in addition,* V *is decrescent (there exists a positive definite function* V_1 *such that* $V(x,t) \leq V_1(x)$ *for all* $x \in U_h$*) and* $\mathscr{L}V(x,t)$ *is negative definite, then the equilibrium solution* x^* *is stochastically asymptotically stable.*

(iii) *If the assumptions of part* (ii) *hold for a radially unbounded function* $V \in C^{2,1}\left(\mathbb{R}^d \times [t_0,\infty)\right)$ *defined everywhere, then the equilibrium solution* x^* *is globally stochastically asymptotically stable.*

It is important for us to show whether this stochastic SIR model has a unique global solution, before we begin to investigate the dynamical behaviour of the stochastic SIR model (1.3). It is well known that,in order for an stochastic differential equations to have a unique global solution for any given initial value, the coefficients of the system are generally required to satisfy the linear growth condition and the locally Lipschitz conditions. However, the coefficients of the system (1.3)do not satisfy the linear growth condition, though they are locally Lipschitz continuous, so the solution of the the system (1.3) may explode at a finite time.

3. Existence of a unique global solution

In this section, using Lyapunov analysis method, we discuss existence of a unique global solution of the model (1.3).

Theorem 3.1. *Let* $(S(t_0), I(t_0), R(t_0)) = (S_0, I_0, R_0) \in \mathbb{D}$, *and* (S_0, I_0, R_0) *is independent of* W. *Then the stochastic SIR model* (1.3) *admits a unique continuous time, Markovian global solution* $(S(t), I(t), R(t))$ *on* $t \geq t_0$ *and this solution is invariant (a.s) with respect to* $\mathbb{D}$.

Proof. We use Theorem 2.1 and follow ideas of [8]. Since the coefficients of the system (1.3) are locally Lipschitz-continuous on $\mathbb{D}$, for any initial value $(S_0, I_0, R_0) \in \mathbb{D}$, there is a unique local solution on $t \in [t_0, \tau(\mathbb{D}))$, where $\tau(\mathbb{D})$ is the random time of first exit of stochastic process $(S(t), I(t), R(t))$ from the domain $\mathbb{D}$, started in $(S(s), I(s), R(s)) = (S_0, I_0, R_0) \in \mathbb{D}$ at the initial time $s \in [t_0, \infty)$. To make this solution global, we need to prove that

$$\mathbb{P}(\tau(\mathbb{D}) = \infty) = 1 \quad \text{a.s.}$$

Let

$$\mathbb{D}_n := \Big\{(S,I,R) : e^{-n} < S < \frac{\Pi}{\mu} - e^{-n}, e^{-n} < I < \frac{\Pi}{\mu} - e^{-n},$$
$$e^{-n} < R < \frac{\Pi}{\mu} - e^{-n}, S + I + R \leq \frac{\Pi}{\mu}\Big\},$$

for $n \in \mathbb{N}$. The system (1.3), has a unique solution up to stopping time $\tau(\mathbb{D}_n)$. Let

$$\begin{aligned} V(S,I,R) = S - \ln S + I - \ln I + \left(\frac{\Pi}{\mu} - S\right) - \ln\left(\frac{\Pi}{\mu} - S\right) \\ + \left(\frac{\Pi}{\mu} - R\right) - \ln\left(\frac{\Pi}{\mu} - R\right), \end{aligned} \tag{3.1}$$

defined on $\widetilde{\mathbb{D}} = \left\{(S,I,R) \in \mathbb{R}^3, t \geq t \geq t_0 : S > 0, I > 0, R > 0, S + I + R \leq \frac{\Pi}{\mu}\right\}$ and assume that $\mathbb{E}(V(S,I,R)) < \infty$. Note that $V(S,I,R) \geq 4$ for $(S,I,R) \in \widetilde{\mathbb{D}}$.
Let $W(S,I,R,t) = e^{-c(t-s)}V(S,I,R)$, defined on $\widetilde{\mathbb{D}} \times [s,\infty)$, where

$$\begin{aligned} c = \frac{1}{4}\left(3\mu + \alpha + r + \left(\beta + \beta\frac{\pi}{\mu} + \mu\right)\frac{\Pi}{\mu}\right) \\ + \sup_{(S,I,R)\in\widetilde{\mathbb{D}}} \frac{1}{4}\left(\frac{3}{2}\frac{\Pi^2}{\mu^2}F_1^2(S,I,R) + F_2^2(S,I,R)\right) \end{aligned} \tag{3.2}$$

Apply the infinitesimal operator $\mathscr{L}$ on equation (3.1), we obtain

$$\begin{aligned} \mathscr{L}V(S,I,R) = \left(\Pi - \mu S - \beta SI\right)\frac{\partial V}{\partial S} + \left(\beta SI - (\mu + \alpha + r)I\right)\frac{\partial V}{\partial I} \\ + \left((\alpha + r)I - \mu R\right)\frac{\partial V}{\partial R} + \frac{1}{2}S^2I^2F_1^2(S,I,R)\left(\frac{\partial^2 V}{\partial S^2} - 2\frac{\partial^2 V}{\partial S \partial I} + \frac{\partial^2 V}{\partial I^2}\right) \\ + \frac{1}{2}I^2F_2^2(S,I,R)\left(\frac{\partial^2 V}{\partial I^2} - 2\frac{\partial^2 V}{\partial I \partial R} + \frac{\partial^2 V}{\partial R^2}\right) \end{aligned} \tag{3.3}$$

$$\begin{aligned} \mathscr{L}V(S,I,R) = \left(\Pi - \mu S - \beta SI\right)\left(\frac{1}{\frac{\Pi}{\mu} - S} - \frac{1}{S}\right) \\ + \left(\beta SI - (\mu + \alpha + r)I\right)\left(1 - \frac{1}{I}\right) + \left((\alpha + r)I - \mu R\right)\left(\frac{1}{\frac{\Pi}{\mu} - R} - 1\right) \\ + \frac{1}{2}S^2I^2F_1^2(S,I,R)\left(\frac{1}{\left(\frac{\Pi}{\mu} - S\right)^2} + \frac{1}{S^2} + \frac{1}{I^2}\right) \\ + \frac{1}{2}I^2F_2^2(S,I,R)\left(\frac{1}{I^2} + \frac{1}{\left(\frac{\Pi}{\mu} - R\right)^2}\right). \end{aligned} \tag{3.4}$$

After a little algebra, we have

$$\begin{aligned}\mathscr{L}V(S,I,R) =& \mu - \frac{\beta SI}{\left(\frac{\Pi}{\mu} - S\right)} - \frac{\Pi}{S} + \mu + \beta I + \beta SI - (\mu + \alpha + r)I \\ & - \beta S + (\mu + \alpha + r) + \frac{(\alpha + r)I}{\left(\frac{\Pi}{\mu} - R\right)} - (\alpha + r)I - \frac{\mu R}{\left(\frac{\Pi}{\mu} - R\right)} \\ & + \mu R + \frac{1}{2}S^2 I^2 F_1^2(S,I,R)\left(\frac{1}{\left(\frac{\Pi}{\mu} - S\right)^2} + \frac{1}{S^2} + \frac{1}{I^2}\right) \\ & + \frac{1}{2}I^2 F_2^2(S,I,R)\left(\frac{1}{I^2} + \frac{1}{\left(\frac{\Pi}{\mu} - R\right)^2}\right).\end{aligned}$$

Since $S + I + R \leq \dfrac{\Pi}{\mu}$, we have,

$$\begin{aligned}\mathscr{L}V(S,I,R) \leq & 3\mu + \alpha + r + \beta I + \beta SI + \mu R \\ & + \frac{1}{2}S^2 I^2 F_1^2(S,I,R)\left(\frac{1}{\left(\frac{\Pi}{\mu} - S\right)^2} + \frac{1}{S^2} + \frac{1}{I^2}\right) \\ & + \frac{1}{2}I^2 F_2^2(S,I,R)\left(\frac{1}{I^2} + \frac{1}{\left(\frac{\Pi}{\mu} - R\right)^2}\right).\end{aligned}$$

$$\begin{aligned}\mathscr{L}V(S,I,R) \leq & 3\mu + \alpha + r + \left(\beta + \beta\frac{\pi}{\mu} + \mu\right)\frac{\Pi}{\mu} \\ & + \sup_{(S,I,R)\in\widetilde{\mathbb{D}}}\left(\frac{3}{2}\frac{\Pi^2}{\mu^2}F_1^2(S,I,R) + F_2^2(S,I,R)\right) \\ = & 4c.\end{aligned} \tag{3.5}$$

So, $\mathscr{L}V(S,I,R) \leq cV(S,I,R)$, since $V(S,I,R) \geq 4$, for $(S,I,R) \in \widetilde{\mathbb{D}}$.

Hence $\mathscr{L}W(S,I,R,t) = e^{-c(t-s)}(-cV(S,I,R) + \mathscr{L}V(S,I,R)) \leq 0$.

Note that , $\inf_{(S,I,R)\in\mathbb{D}\setminus\mathbb{D}_n} V(S,I,R) > n+1$, for $n \in \mathbb{N}$. Now define $\tau_n := \min\{t, \tau(\mathbb{D}_n)\}$ and apply

Dynkin's formula to get

$$\begin{aligned}\mathbb{E}[W(S(\tau_n),I(\tau_n),R(\tau_n),\tau_n)] =& \mathbb{E}[W(S(s),I(s),R(s),s)]\\ &+\mathbb{E}\left[\int_s^{\tau_n}\mathscr{L}W(S(u),I(u),R(u),u)du\right]\\ \leq& \mathbb{E}[W(S(s),I(s),R(s),s)]\\ =& \mathbb{E}[V(S(s),I(s),R(s))] = \mathbb{E}[V(S_0,I_0,R_0)].\end{aligned} \tag{3.6}$$

Next, to show that $\mathbb{P}(\tau(\mathbb{D}_n) < t) = 0$, we take the expected value of $e^{c(t-\tau_n)}V(S(\tau_n),I(\tau_n),R(\tau_n))$.

$$\begin{aligned}\mathbb{E}\left[e^{c(t-\tau_n)}V(S(\tau_n),I(\tau_n),R(\tau_n))\right] =& \mathbb{E}\left[e^{c(t-s)}e^{-c(\tau_n-s)}V(S(\tau_n),I(\tau_n),R(\tau_n))\right]\\ =& \mathbb{E}\left[e^{c(t-s)}W(S(\tau_n),I(\tau_n),R(\tau_n),\tau_n)\right]\\ \leq& e^{c(t-s)}\mathbb{E}[V(S_0,I_0,R_0)],\end{aligned} \tag{3.7}$$

and obtain

$$\begin{aligned}0 \leq& \mathbb{P}(\tau(\mathbb{D}) < t) \leq \mathbb{P}(\tau(\mathbb{D}_n) < t), \qquad \text{since} \quad \mathbb{D}_n \subseteq \widetilde{\mathbb{D}}\\ =& \mathbb{P}(\tau_n < t)\\ =& \mathbb{E}(\mathbf{1}_{\tau_n<t}), \qquad \text{where } \mathbf{1} \text{ is the indicator function}\\ \leq& \mathbb{E}\left(e^{c(t-\tau_n)}\frac{V(S(\tau(\mathbb{D}_n)),I(\tau(\mathbb{D}_n)),R(\tau(\mathbb{D}_n)))}{\inf\limits_{(S,I,R)\in\widetilde{\mathbb{D}}\setminus\mathbb{D}_n}V(S,I,R)}\mathbf{1}_{\tau_n<t}\right)\\ \leq& e^{c(t-s)}\left(\frac{\mathbb{E}(V(S_0,I_0,R_0))}{\inf\limits_{(S,I,R)\in\widetilde{\mathbb{D}}\setminus\mathbb{D}_n}V(S,I,R)}\right)\\ \leq& e^{c(t-s)}\left(\frac{\mathbb{E}(V(S_0,I_0,R_0))}{n+1}\right)\\ \leq& e^{c(t-s)}\left(\frac{\mathbb{E}(V(S_0,I_0,R_0))}{n+1}\right) \to 0, \quad \text{as} \quad n\to\infty\end{aligned} \tag{3.8}$$

for all $(S_0,I_0,R_0) \in \mathbb{D}_n$ (for large n), and for all fixed $t \in [s,\infty)$.

Thus $\mathbb{P}(\tau(\mathbb{D}) < t) = \mathbb{P}(\tau(\mathbb{D}_n) < t) = 0$, for $(S_0,I_0,R_0) \in \mathbb{D}$ and $t \geq t_0$, that is, $\mathbb{P}(\tau(\mathbb{D}) = \infty) = 1$.

This proves the invariance property and the global existence of the solution $(S(t),I(t),R(t))$ on $\mathbb{D}$. Uniqueness and continuity of the solution is obtained by Theorem 3.1.

Note that $I = 0, S = 0$ and $R = 0$ are not in the domain $\widetilde{\mathbb{D}}$. We study these cases separately.

1. If $I(t) = 0$ the system (1.3) become as ODE

$$\begin{aligned}dS(t) =& (\Pi - \mu S)dt\\ dR(t) =& -\mu R dt\end{aligned} \tag{3.9}$$

with initial condition $(S_0, R_0) \in \mathbb{D}_1 = \left\{(S,R) : S > 0, R > 0, S + R \leq \frac{\Pi}{\mu}\right\}$. Since the right hand side of ODE is continuous on $\mathbb{D}_1$. The solution $(S(t), R(t))$ globally exist on $\mathbb{D}_1$.

2. If $S(t) = 0$ the system (1.3) becomes

$$\begin{aligned} dI(t) &= (-(\mu + \alpha + r)I)dt - IF(I,R)dW(t) \\ dR(t) &= ((\alpha + r)I - \mu R)dt + IF(I,R)dW(t) \end{aligned} \tag{3.10}$$

with initial condition $(I_0, R_0) \in \mathbb{D}_2 = \left\{(I,R) : I > 0, R \geq 0, I + R \leq \frac{\Pi}{\mu}\right\}$ then the above SDE has a unique global solution on $\mathbb{D}_2$. It can be proven with using the function $V(I,R) = I - \ln I + \left(\frac{\Pi}{\mu} - R\right) - \ln\left(\frac{\Pi}{\mu} - R\right)$ defined on $\mathbb{D}_2$ in similar calculations.

3. If $R(t) = 0$ then the model becomes

$$\begin{aligned} dS(t) &= (\Pi - \mu S - \beta SI)dt - SIF(I,S)dW(t) \\ dI(t) &= (\beta SI - (\mu + r)I)dt + SIF(I,S)dW(t) \end{aligned} \tag{3.11}$$

If the initial condition $(S_0, I_0) \in \mathbb{D}_3 = \left\{(S,I) : S \geq 0, I > 0, S + I \leq \frac{\Pi}{\mu}\right\}$ then the above SDE has a unique global solution on $\mathbb{D}_3$. One can be prove it using the function $V(S,I) = S - \ln S + I - \ln I + \left(\frac{\Pi}{\mu} - S\right) - \ln\left(\frac{\Pi}{\mu} - S\right)$ defined on $\mathbb{D}_3$ defined on $\mathbb{D}_3$.

Hence the proof is complete. The unique solution (S, I, R) exists globally and invariant with respect to the domain $\mathbb{D} = \left\{(S,I,R) : S \geq 0, I \geq 0, R \geq 0, S + I + R \leq \frac{\Pi}{\mu}\right\}$ for all $(S_0, I_0, R_0) \in \mathbb{D}$ and $t \geq t_0$. □

4. Stochastic asymptotic stability of infection-free and endemic equilibrium states

A.M.Lyapunov developed a methods for determining stability without solving the equation. The stability means insensitivity of the state of the system to small changes in the initial state or the parameters of the system. For a stable system, the trajectories which are close to each other at a specific instant should therefore remain close to each other at all subsequent instants.

In this section, we study the stochastic asymptotic stability of equilibrium solutions of (1.3).

The SIR model (1.3) can have at most two equilibrium solutions, namely an infection-free equilibrium solution $E_1 = (S_1, I_1, R_1) = \left(\frac{\Pi}{\mu}, 0, 0\right)$, which corresponds to a steady state of

the population without disease and an endemic equilibrium solution

$$E_2 = (S_2, I_2, R_2) = \left(\frac{\mu+\alpha+r}{\beta}, \frac{\Pi}{\mu+\alpha+r} - \frac{\mu}{\beta}, \frac{(\alpha+r)}{\mu}\left(\frac{\Pi}{\mu+\alpha+r} - \frac{\mu}{\beta}\right)\right). \\ \left(\frac{\Pi}{\mathscr{R}_0}, \frac{\mu}{\beta}(\mathscr{R}_0 - 1), \frac{\alpha+r}{\beta}(\mathscr{R}_0 - 1)\right) \tag{4.1}$$

where $\mathscr{R}_0 = \dfrac{\beta\Pi}{\mu(\mu+\alpha+r)}$ is the basic reproduction number.

The basic reproduction $\mathscr{R}_0$ is computed using the next generation matrix approach. $\mathscr{R}_0$ is the average number of secondary infections produced when one single infected individual is introduced into a host population where everyone is susceptible. When $\mathscr{R}_0 \le 1$, the disease die out without any medical interventions but when $\mathscr{R}_0 > 1$ the disease becomes endemic.

Theorem 4.1. *The disease-free equilibrium solution* $E_1 = \left(\frac{\Pi}{\mu}, 0, 0\right)$ *of* (1.3) *is globally stochastically asymptotically stable on* $\mathbb{D}$, *if* $\mathscr{R}_0 \le 1$.

Proof. We use Theorem 4.1 and define a Lyapunov function

$$V_1(S, I, R) = \frac{1}{2}\left(S - \frac{\Pi}{\mu} + I + R\right)^2 + \frac{\Pi}{\mu}I + \frac{\Pi}{\mu}R, \tag{4.2}$$

The infinitesimal generator $\mathscr{L}$ acting on the Lyapunov function can be written as:

$$\begin{aligned}
\mathscr{L}V_1(S, I, R) &= \left(\Pi - \mu S - \beta SI\right)\left(S - \frac{\pi}{\mu} + I + R\right) \\
&\quad + \left(\beta SI - (\mu+\alpha+r)I\right)\left(S - \frac{\Pi}{\mu} + I + R + \frac{\Pi}{\mu}\right) \\
&\quad + \left((\alpha+r)I - \mu R\right)\left(S - \frac{\Pi}{\mu} + I + R + \frac{\Pi}{\mu}\right) \\
&= \left(\Pi - \mu S - \mu I - \mu R\right)\left(S - \frac{\Pi}{\mu} + I + R\right) + \frac{\Pi}{\mu}\beta SI \\
&\quad - \frac{\Pi}{\mu}(\mu+\alpha+r)I + \frac{\Pi}{\mu}(\alpha+r)I - \Pi R \\
&= -\mu\left(S - \frac{\Pi}{\mu} + I + R\right)^2 + \frac{\Pi}{\mu}\beta SI - \frac{\Pi}{\mu}\mu I - \Pi R
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
\mathscr{L}V_1(S, I, R) &\le -\mu\left(S - \frac{\Pi}{\mu} + I + R\right)^2 + \left(\frac{\Pi}{\mu}\right)^2\beta I - \frac{\Pi}{\mu}\mu I - \Pi R \\
&\le -\mu\left(S - \frac{\Pi}{\mu} + I + R\right)^2 - \frac{\Pi}{\mu}I\left(\mu - \frac{\Pi}{\mu}\beta\right) - \Pi R
\end{aligned} \tag{4.4}$$

Therefore if $\mu - \dfrac{\Pi}{\mu}\beta \ge 0$ then $\mathscr{L}V_1(S, I, R)$ becomes negative definite on $\mathbb{D}$. □

Remark 4.1. The above theorem concludes that if $\mu - \frac{\Pi}{\mu}\beta \geq 0$ then disease will die out. This statement does not contradict with the fact $\mathscr{R}_0 \leq 1$. Because the stability condition $\mu - \frac{\Pi}{\mu}\beta \geq 0$ can be written in terms of the basic reproduction number as follows

$$\frac{\Pi}{\mu}\beta \leq \mu < \mu + \alpha + r \Rightarrow \frac{\beta\Pi}{\mu(\mu+\alpha+r)} = \mathscr{R}_0 \leq 1$$

Example 4.2. The dynamical behaviour of the SIR model describing by the stochastic SIR model (1.3), stabilizes at the disease - free level, whenever $\mathscr{R}_0 \leq 1$. The disease-free equilibrium solution $(S_1, I_1, R_1) = (13.3333, 0, 0)$ is globally asymptotically stochastically stable, Figure for the parameters: $\Pi = 10, \alpha = 0.15, \beta = 0.09, \mu = 0.75$ and $r = 0.3$ ($R_0 = 1 \leq 1$). Here we use the initial value $(S_0, I_0, R_0) = (0.6, 0.2, 0.2)$

Remark 4.3. Theorem 4.1 completely determines the globally stochastically asymptotically stable of our model when $\mathscr{R}_0 \leq 1$. The epidemic logical consequence is that the number of the infected, no matter how large initially, will vanish in time so that the disease dies out. In contrast, the disease will persist when $\mathscr{R}_0 > 1$. To investigate the resulted long term dynamics, we turn to the endemic equilibrium solution in what follows.

Now, we will be studying the global stability of the endemic equilibrium solution if $\mathscr{R}_0 > 1$ by Lyapunov function.

Theorem 4.2. *The endemic equilibrium solution, $E_2 = (S_2, I_2, R_2)$ of the system (1.3) is stochastically asymptotically stable on $\widetilde{\mathbb{D}}$ if $\mathscr{R}_0 > 1$ and satisfies $\eta(S, I, R) \leq 0$, where*

$$\begin{aligned}\eta(S,I,R) = &-\frac{\mu^2}{\Pi}(S - S_2 + I - I_2 + R - R_2)^2 - \left(b\beta I + b\beta S_2 + b\mu - a\beta\right)(S - S_2)^2 \\ &-(b\beta S_2 - a\beta)(I - I_2)^2 + \frac{1}{2}\left(aI_2 + bI^2\right)S^2F_1^2(S,I,R) + \frac{1}{2}aI_2F_2^2(S,I,R).\end{aligned} \tag{4.5}$$

Proof. Note that the condition $\mathscr{R}_0 > 1$ and $F_i(S_2, I_2, R_2) = 0$, are needed for the existence of the endemic equilibrium solution. Define a Lyapunaov function.

$$\begin{aligned}V_2(S,I,R) = &S - S_2 + I - I_2 + R - R_2 - (S_2 + I_2 + R_2)\ln\left(\frac{S+I+R}{S_2+I_2+R_2}\right) \\ &+ a\left(I - I_2 - I_2\ln\left(\frac{I}{I_2}\right)\right) + \frac{b}{2}(S - S_2)^2\end{aligned} \tag{4.6}$$

for $a = \left(\frac{1}{2}\right)bS_2$, b is positive constant. The infinitesimal generator $\mathscr{L}$ acting on the Lyapunov

function V_2 can be written as:

$$\begin{aligned}
\mathscr{L}V_2(S,I,R) &= \left(\Pi-\mu S-\beta SI\right)\left(1-\frac{S_2+I_2+R_2}{S+I+R}+b(S-S_2)\right)\\
&\quad+\left(\beta SI-(\mu+\alpha+r)I\right)\left(1-\frac{S_2+I_2+R_2}{S+I+R}+a(1-\frac{I_2}{I}\right)\\
&\quad+\left((\alpha+r)I-\mu R\right)\left(1-\frac{S_2+I_2+R_2}{S+I+R}\right)\\
&\quad+\frac{1}{2}S^2I^2F_1^2(S,I,R)\left(\frac{S_2+I_2+R_2}{(S+I+R)^2}(1-2+1)+b+a\frac{I_2}{I^2}\right)\\
&\quad+\frac{1}{2}I^2F_2^2(S,I,R)\left(\frac{S_2+I_2+R_2}{(S+I+R)^2}(1-2+1)+a\frac{I_2}{I^2}\right)\\
\mathscr{L}V_2(S,I,R) &= \left(1-\frac{S_2+I_2+R_2}{S+I+R}\right)\left(\Pi-\mu S-(\mu+\alpha+r)I+(\alpha+r)I-\mu R\right)\\
&\quad+b\left(\Pi-\mu S-\beta SI\right)(S-S_2)+a\left(\beta S-\left(\mu+\alpha+r\right)\right)(I-I_2)\\
&\quad+\frac{1}{2}\left(aI_2+bI^2\right)S^2F_1^2(S,I,R)+\frac{1}{2}aI_2F_2^2(S,I,R)
\end{aligned}\tag{4.7}$$

The following identities to help to simplify $\mathscr{L}V_2(S,I,R)$

(i) $\mu(S_2+I_2+R_2)=\Pi$

(ii) $\Pi-\mu S-\beta SI=-\beta(S-S_2)I-\beta S_2(I-I_2)-\mu(S-S_2)$

(iii) $\beta S-(\mu+\alpha+r)=\beta S-\beta S_2=\beta(S-S_2)$

After substituting the above identities into last equation, we get

$$\begin{aligned}
\mathscr{L}V_2(S,I,R) &= \frac{S-S_2+I-I_2+R-R_2}{S+I+R}\left(-\mu(S-S_2+I-I_2+R-R_2)\right)\\
&\quad+b(S-S_2)\left(-\beta(S-S_2)I-\beta S_2(I-I_2)-\mu(S-S_2)\right)\\
&\quad+a\beta(S-S_2)(I-I_2)+\frac{1}{2}\left(aI_2+bI^2\right)S^2F_1^2(S,I,R)\\
&\quad+\frac{1}{2}aI_2F_2^2(S,I,R).\\
&= -\mu\frac{(S-S_2+I-I_2+R-R_2)^2}{S+I+R}-b\beta I(S-S_2)^2\\
&\quad-b\beta S_2(S-S_2)(I-I_2)-b\mu(S-S_2)^2+a\beta(S-S_2)(I-I_2)\\
&\quad+\frac{1}{2}\left(aI_2+bI^2\right)S^2F_1^2(S,I,R)+\frac{1}{2}aI_2F_2^2(S,I,R).\\
\mathscr{L}V_2(S,I,R) &\le -\mu\frac{(S-S_2+I-I_2+R-R_2)^2}{S+I+R}-b\beta I(S-S_2)^2\\
&\quad-2\left(b\beta S_2-a\beta\right)(S-S_2)(I-I_2)-b\mu(S-S_2)^2\\
&\quad+\frac{1}{2}\left(aI_2+bI^2\right)S^2F_1^2(S,I,R)+\frac{1}{2}aI_2F_2^2(S,I,R)
\end{aligned}\tag{4.8}$$

since $2ab \le a^2 + b^2$, we have
$2(S-S_2)(I-I_2) \le (S-S_2)^2 + (I-I_2)^2$

$$\begin{aligned}
\mathscr{L}V_2(S,I,R) \le & -\mu\frac{(S-S_2+I-I_2+R-R_2)^2}{S+I+R} - \left(b\beta I + b\beta S_2 + b\mu - a\beta\right)(S-S_2)^2 \\
& - \left(b\beta S_2 - a\beta\right)(I-I_2)^2 + \frac{1}{2}\left(aI_2 + bI^2\right)S^2F_1^2(S,I,R) + \frac{1}{2}aI_2F_2^2(S,I,R) \\
\mathscr{L}V_2(S,I,R) \le & -\frac{\mu^2}{\Pi}(S-S_2+I-I_2+R-R_2)^2 - \left(b\beta I + b\beta S_2 + b\mu - a\beta\right)(S-S_2)^2 \\
& - (b\beta S_2 - a\beta)(I-I_2)^2 + \frac{1}{2}\left(aI_2 + bI^2\right)S^2F_1^2(S,I,R) + \frac{1}{2}aI_2F_2^2(S,I,R).
\end{aligned} \tag{4.9}$$

By assumptions $\mathscr{L}V_2(S,I,R)$ becomes negative definite on $\mathbb{D}$ for some suitable $F_i(S,I,R)$, $i = 1,2$ and $\mathscr{L}V_2(S,I,R) = 0$ only at (S_2,I_2,R_2) and by the choice of suitable functions $F_i(S,I,R)$. □

Remark 4.4. We conclude that the stability of the equilibrium solutions that the disease-free and endemic equilibrium solution cannot be exist simultaneously. If $\mathscr{R}_0 \le 1$, then the disease - free equilibrium solution is globally asymptotically stable. If $\mathscr{R}_0 > 1$ the disease - free equilibrium solution is unstable, while the endemic equilibrium is globally asymptotically stable.

Example 4.5. Assume that the parameters of system (1.3) are given by $\Pi = 2, \alpha = 0.2, \beta = 0.6, \mu = 0.3$ and $r = 0.1(\mathscr{R}_0 = 6.6667 > 1)$. Here we use the initial value $(S_0, I_0, R_0) = (0.6, 0.2, 0.2)$. In the view of Theorem 4.2, the system (1.3) stabilize the endemic equilibrium level. Figure 6

We present the following theorem which gives some conditions for the pth- moment exponential stability of the disease- free equilibrium of stochastic system (1.3) in terms of Lyapunov function.

5. Moment exponential stability

Theorem 5.1. *Let p, c_1, c_2 and c_3 be positive numbers. Suppose that there exist a function $V \in C^{2,1}(\mathbb{R}_+, \mathbb{R}^n)$, such that*

$$c_1|x|^p \le V(t,x) \le c_2|x|^p, \quad \text{and} \quad V(t,x) \le -c_3|x|^p, \quad t \ge 0 \tag{5.1}$$

the equilibrium of system (1.3) is pth- moment exponentially stable. When p =2, it is usually said to be mean square exponentially stable and the equilibrium $x = 0$ is globally asymptotically stable.

From Young's inequality,i.e., $xy \le \frac{x^p}{p} + \frac{y^q}{q}$ for $x, y > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$,we have the following inequalities:

Lemma 5.2. *Set $p \geq 2$ and $\epsilon, x, y > 0$ then*

$$\begin{aligned} x^{p-1}y &\leq \frac{(p-1)\epsilon}{p}x^p + \frac{1}{p\epsilon^{p-1}}y^p, \\ x^{p-2}y^2 &\leq \frac{(p-2)\epsilon}{p}x^p + \frac{1}{p\epsilon^{\frac{p-2}{2}}}y^p, \end{aligned} \tag{5.2}$$

This lemma can be proved easily. We use the Lemma, to prove the Theorem 5.3

Theorem 5.3. *Let $p \geq 2$. If the condition $\mathscr{R}_0 \leq 1$ and $\frac{1}{2}(p-1)\sigma^2 < 2(\mu+\alpha+r) - \frac{\beta\Pi}{\mu}$ where $\sigma^2 = (\frac{\mu}{\Pi})^4\left[(\frac{\Pi}{\mu}-S_2)^2 + (\frac{\Pi}{\mu}-I_2)^2\right]$ hold, the disease-free equilibrium E_1 of the system (1.3) is pth - moment exponentially stable in $\mathbb{D}$.*

Proof. Set $p \geq 2$ and $(S_0, I_0, R_0) \in \mathbb{D}$, in the view of 3.1 the solution of system (1.3) remains in $\mathbb{D}$. Consider the Lyapunov function $V_3 = \left(\frac{\Pi}{\mu} - S\right)^p + \frac{1}{p}I^P + R^p$. It is easy to check that inequalities (5.1) are true. Furthermore,

$$\begin{aligned} \mathscr{L}V_3(S,I,R) = &-p\mu\left(\frac{\Pi}{\mu}-S\right)^p + p\beta SI\left(\frac{\Pi}{\mu}-S\right)^{p-1} + \beta SI^p \\ &-(\mu+\alpha+r)I^p + p(\alpha+r)IR^{p-1} - p\mu R^p \\ &+\frac{1}{2}p(p-1)S^2I^2\left(\frac{\Pi}{\mu}-S\right)^{p-2}F_1^2(S,I,R) + \frac{1}{2}(p-1)S^2I^pF_1^2(S,I,R) \\ &+\frac{1}{2}(p-1)I^pF_2^2(S,I,R) + \frac{1}{2}p(p-1)I^2R^{p-2}F_2^2(S,I,R) \end{aligned} \tag{5.3}$$

In $\mathbb{D}, \{S,I,R\} \leq \frac{\Pi}{\mu}$, hence

$$\begin{aligned} \mathscr{L}V_3(S,I,R) \leq &-p\mu\left(\frac{\Pi}{\mu}-S\right)^p + p\beta\frac{\Pi}{\mu}I\left(\frac{\Pi}{\mu}-S\right)^{p-1} + \frac{\Pi}{\mu}\beta I^p - (\mu+\alpha+r)I^p \\ &+p(\alpha+r)IR^{p-1} - p\mu R^p + \frac{1}{2}p(p-1)\left(\frac{\Pi}{\mu}\right)^2 I^2\left(\frac{\Pi}{\mu}-S\right)^{p-2}F_1^2(S,I,R) \\ &+\frac{1}{2}(p-1)\left(\frac{\Pi}{\mu}\right)^2 I^pF_1^2(S,I,R) \\ &+\frac{1}{2}(p-1)I^pF_2^2(S,I,R) + \frac{1}{2}p(p-1)I^2R^{p-2}F_2^2(S,I,R) \end{aligned} \tag{5.4}$$

Which can be simplified to

$$\begin{aligned} \mathscr{L}V_3(S,I,R) \leq &-p\mu\left(\frac{\Pi}{\mu}-S\right)^p + p\beta\frac{\Pi}{\mu}I\left(\frac{\Pi}{\mu}-S\right)^{p-1} + \frac{\Pi}{\mu}\beta I^p - (\mu+\alpha+r)I^p \\ &+p(\alpha+r)IR^{p-1} - p\mu R^p + \frac{1}{2}p(p-1)\left(\frac{\Pi}{\mu}\right)^2 I^2\left(\frac{\Pi}{\mu}-S\right)^{p-2}F_1^2(S,I,R) \end{aligned}$$

$$+\frac{1}{2}(p-1)I^p\left[\left(\frac{\Pi}{\mu}\right)^2 F_1^2(S,I,R)+F_2^2(S,I,R)\right]$$
$$+\frac{1}{2}p(p-1)I^2R^{p-2}F_2^2(S,I,R)$$

$$\begin{aligned}\mathscr{L}V_3(S,I,R)\leq &-p\mu\left(\frac{\Pi}{\mu}-S\right)^p+p\beta\frac{\Pi}{\mu}I\left(\frac{\Pi}{\mu}-S\right)^{p-1}+\frac{\Pi}{\mu}\beta I^p-(\mu+\alpha+r)I^p\\&+p(\alpha+r)IR^{p-1}-p\mu R^p+\frac{1}{2}p(p-1)\left(\frac{\Pi}{\mu}\right)^2I^2\left(\frac{\Pi}{\mu}-S\right)^{p-2}F_1^2(S,I,R)\\&+\frac{1}{2}(p-1)I^p\left(\frac{\mu}{\Pi}\right)^4\left[(\frac{\Pi}{\mu}-S_2)^2+(\frac{\Pi}{\mu}-I_2)^2\right]\\&+\frac{1}{2}p(p-1)I^2R^{p-2}F_2^2(S,I,R)\end{aligned}\tag{5.5}$$

Note that, $F_1=\left(\frac{\mu}{\Pi}\right)^3(S-S_2)$, $F_2=\left(\frac{\mu}{\Pi}\right)^2(I-I_2)$ and

take $\sigma^2=(\frac{\mu}{\Pi})^4\left[(\frac{\Pi}{\mu}-S_2)^2+(\frac{\Pi}{\mu}-I_2)^2\right]$

Now apply the Lemma 5.2

$$\begin{aligned}I\left(\frac{\Pi}{\mu}-s\right)^{p-1}&\leq\frac{p-1}{p}\epsilon\left(\frac{\Pi}{\mu}-S\right)^p+\frac{1}{p}\epsilon^{1-p}I^p\\I^2\left(\frac{\Pi}{\mu}-s\right)^{p-2}&\leq\frac{p-2}{p}\epsilon\left(\frac{\Pi}{\mu}-S\right)^p+\frac{2}{p\epsilon^{\frac{p-2}{2}}}I^p\\IR^{p-1}&\leq\frac{p-1}{p}\epsilon R^p+\frac{1}{p}\epsilon^{1-p}I^P\\I^2R^{p-2}&\leq\frac{p-2}{p}\epsilon R^p+\frac{2}{p\epsilon^{\frac{p-2}{2}}}I^p\end{aligned}\tag{5.6}$$

Inject these four inequalities in (5.5), we get

$$\begin{aligned}\mathscr{L}V_3(S,I,R)\leq &-\left[p\mu-(p-1)\beta\frac{\Pi}{\mu}\epsilon+\frac{1}{2}(p-1)(p-2)(\frac{\Pi}{\mu})^2\epsilon F_1^2\right]\left(\frac{\Pi}{\mu}-S\right)^p\\&-\left[(\mu+\alpha+r)-\frac{\Pi}{\mu}\beta-\frac{1}{2}(p-1)\sigma^2\right]I^p-[p\mu-\frac{1}{2}(p-1)(p-2)\epsilon F_2^2]R^P\\&-\left[-\beta\frac{\Pi}{\mu}\epsilon^{1-p}-(\alpha+r)\epsilon^{1-p}-\frac{1}{2}(p-1)\frac{\Pi^2}{\mu^2}\epsilon^{\frac{2-p}{p}}F_1^2-(p-1)\epsilon^{\frac{2-p}{p}}F_2^2\right]I^p\end{aligned}\tag{5.7}$$

We choose ϵ sufficiently small such that the coefficients of $\left(\frac{\Pi}{\mu}-S\right)^p$ and R^p be negative, and since $(\mu+\alpha+r)-\frac{\Pi}{\mu}\beta-\frac{1}{2}(p-1)\sigma^2>0$, we must have $2(\mu+\alpha+r)-\frac{\Pi}{\mu}\beta-\frac{1}{2}(p-1)\sigma^2>0$ the coefficient of I^p must be negative. According to Theorem 5.1 the proof is completed. □

Under the Theorems 5.1 and Theorem 5.3, we have in the case p=2 the following corollary:

Corollary 5.4. *If the condition* $\mathscr{R}_0 \le 1$ *and* $\frac{1}{2}(p-1)\sigma^2 < 2(\mu+\alpha+r) - \frac{\beta\Pi}{\mu}$ *hold, the disease-free equilibrium* E_1 *of the system* (1.3) *is globally asymptotically stable in* $\mathbb{D}$.

Example 5.1. Verify the Theorem 5.1 which states, if $\mathscr{R}_0 \le 1$ then the disease - free equilibrium solution $(S_3, I_3, R_3) = (1.25, 0, 0)$ of the system (1.3) is globally stochastically asymptotically stable Figure 8, under the assumption for the parameters: $\Pi = 0.5, \alpha = 0.05, \beta = 0.4, \mu = 0.4$ and $r = 0.05(\mathscr{R}_0 = 1 \le 1)$. Here we use the initial value $(S_0, I_0, R_0) = (0.6, 0.2, 0.2)$. Also,

$$\sigma = \left(\frac{\mu}{\Pi}\right)^2 \sqrt{(\frac{\Pi}{\mu} - S_2)^2 + (\frac{\Pi}{\mu} - I_2)^2} = 0.8 < \sqrt{4\left(\mu+\alpha+r\right) - 2\frac{\beta\Pi}{\mu}} = 1.$$

We use the optimal vaccination strategies to control the total number of susceptible and recovered individuals and also to minimize the probability that the infected individuals spread the disease in the population. Vaccination is regarded as one of the most primary strategies used by public health authorities to control human infectious diseases, and these strategies can also provide with some several clear benefits.

6. Optimal control

Consider the control variable $u(t) \in \mathbb{U}$ to be the percentage of susceptible individuals being vaccinated per unit time. Hence

$$\mathbb{U} = \{u : u(t) \text{ is measurable } 0 \le u(t) \le u_{\max} < \infty, t \in [0, t_{end}]\} \tag{6.1}$$

indicates an admissible control.

We define our objective functional as

$$J(u) = \int_0^{t_{end}} [S(t) + I(t) + \frac{1}{2}cu^2(t)]\,dt \tag{6.2}$$

Subject to

$$\begin{aligned} S'(t) &= \Pi - (\mu+u)S - \beta SI, \\ I'(t) &= \beta SI - (\mu+\alpha)I - rI, \\ R'(t) &= \alpha I + rI + uS - \mu R, \end{aligned} \tag{6.3}$$

with initial conditions $S(0) = S_0, I(0) = I_0, R(0) = R_0$. Here c is the weight factor (positive constants) which is associated with the control $u(t)$. The objective of our work is to minimize the

infected and susceptible individuals and to maximize the total number of recovered individuals by using possible minimal control variables $u(t)$.

6.1. Existence of an optimal control

For existence, we consider the control system (6.3) with initial conditions, then we can write (6.3) in the following form:

$$V_t = AV + F(V) \tag{6.4}$$

where $V = \begin{bmatrix} S(t) \\ I(t) \\ R(t) \end{bmatrix}$ $A = \begin{bmatrix} -(\mu+u) & 0 & 0 \\ 0 & -(\mu+\alpha) & 0 \\ u & \alpha & -\mu \end{bmatrix}$ $F(V) = \begin{bmatrix} \Pi - \beta SI \\ \beta SI - rI \\ rI \end{bmatrix}$ and V_t denotes the derivative of V with respect to the time t. The system (6.3) is nonlinear system with a bounded coefficient.

We set

$$G(V) = AV + F(V) \tag{6.5}$$

$$\begin{aligned} F(V_1) - F(V_2) &= \begin{bmatrix} \Pi - \beta S_1 I_1 \\ \beta S_1 I_1 - r I_1 \\ r I_1 \end{bmatrix} - \begin{bmatrix} \Pi - \beta S_2 I_2 \\ \beta S_2 I_2 - r I_2 \\ r I_2 \end{bmatrix} \\ &= \begin{bmatrix} -\beta(S_1 I_1 - S_2 I_2) \\ \beta(S_1 I_1 - S_2 I_2) - r(I_1 - I_2) \\ r(I_1 - I_2) \end{bmatrix} \end{aligned} \tag{6.6}$$

$$\begin{aligned} |F(V_1) - F(V_2)| &= |-\beta(S_1 I_1 - S_2 I_2)| + |\beta(S_1 I_1 - S_2 I_2) - r(I_1 - I_2)| + |r(I_1 - I_2)| \\ &\le \beta|S_1 I_1 - S_2 I_2| + \beta|S_1 I_1 - S_2 I_2| + r|I_1 - I_2| + r|I_1 - I_2| \\ &\le 2\beta|S_1 I_1 - S_2 I_2 - S_2 I_1 + S_2 I_1| + 2r|I_1 - I_2| \\ &\le 2\beta|I_1||S_1 - S_2| + (2\beta|S_2| + 2r)|I_1 - I_2| \\ &\le 2\beta\frac{\Pi}{\mu}|S_1 - S_2| + \left(2\beta\frac{\Pi}{\mu} + 2r\right)|I_1 - I_2| \\ &\le M_1|S_1 - S_2| + M_2|I_1 - I_2| \end{aligned} \tag{6.7}$$

where $M_1 = 2\beta\dfrac{\Pi}{\mu}$ and $M_2 = 2\beta\dfrac{\Pi}{\mu} + 2r$.

Also, we get

$$|G(V_1) - G(V_2)| \le L|V_1 - v_2|$$

where $L = M_1 + M_2 + ||A|| < \infty$

Thus, it follows that the function G is uniformly Lipschitz continuous. From the definition of the control $u(t)$ and the restriction on $S(t), I(t)$ and $R(t) \ge 0$. We see that a solution of

the system (6.4) exists(Birkhoff and Rota 1989). Let us go back to the optimal control problem (6.2)-(6.3). In order to find as optimal solution. First we find the Lagranges and Hamiltonian for the optimal control problem (6.2)-(6.3). In fact, the Lagrangian of the optimal problem is given by

$$L(S,I,u) = S(t) + I(t) + \frac{1}{2}cu^2(t). \tag{6.8}$$

To accomplish this, we define the Hamiltonian H for the control problem

$$H(S,I,R,u,\lambda_1,\lambda_2,\lambda_3,t) = L(S,I,u) + \lambda_1(t)\frac{dS(t)}{dt} + \lambda_2(t)\frac{dI(t)}{dt} + \lambda_3\frac{dR(t)}{dt} \tag{6.9}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the adjoint functions to be determined suitably.

6.2. Characterization of the optimal control

We shall now characterize the optimal control $u^*(t)$ such that $J(u^*(t)) = \min_{u\in\mathbb{U}} J(u(t))$ subject to the control system (6.3) with initial conditions. The existence of an optimal control is guaranteed by the compactness of the control and the state spaces, and the convexity in the problem based on Theorem 4.1 of chapter III in Flemming and Rishel [14]. The nontrivial requirement from Flemming and Rishel's theorem are listed below

(1) The set of all solutions to system eq (6.3) with corresponding control functions $\mathbb{U}$(as given in (6.1)) is nonempty.

(2) The state system can be written as a linear function of the control variables with coefficients depend on time and the state variables.

(3) The integrand L in eq (6.2) is convex on $\mathbb{U}$ and additionally satisfies $L(t,S,I,u) \geq k_1|u|^\delta - k_2$ where $k_1 > 0$, $k_2 > 0$ and $\delta > 1$

We applied Pontriagin's maximum principle [13] to obtain the following results.

Theorem 6.1. *If u^* is an optimal control corresponding to states S^*, I^* and R^* which minimize the objective functional equation (6.2) then there exist an adjoint variables λ_1, λ_2 and λ_3 satisfy*

$$\begin{aligned}
\lambda_1' &= -\frac{\partial H}{\partial S} = -1 + \lambda_1(\mu + u + \beta I) - \lambda_2\beta I - \lambda_3 u \\
\lambda_2' &= -\frac{\partial H}{\partial I} = -1 + \lambda_1(\beta S) - \lambda_2(\beta S - (\mu + \alpha + r)) - \lambda_3(\alpha + r) \\
\lambda_3' &= -\frac{\partial H}{\partial R} = \lambda_3\mu
\end{aligned} \tag{6.10}$$

with transversality condition

$$\lambda_i(t_{end}) = 0, \quad i = 1,2,3 \tag{6.11}$$

Furthermore, the optimal control $u^*(t)$ is given by

$$u^*(t) = \min\left\{\max(0, \frac{(\lambda_1 - \lambda_3)S^*}{c}), u_{\max}\right\}.$$

Proof. We from the Hamiltonian H as below

$$\begin{aligned} H =& \frac{1}{2}cu^2 + \lambda_1(\Pi - (\mu+u)S - \beta SI) \\ &+ \lambda_2(\beta SI - (\mu+\alpha+r)I) + \lambda_3((\alpha+r)I + uS - \mu R) \end{aligned} \tag{6.12}$$

Using Pontriagin's maximum principle, we derive the system (6.10) from

$$\lambda_1' = -\frac{\partial H}{\partial S}, \quad \lambda_2' = -\frac{\partial H}{\partial I}, \quad \lambda_3' = -\frac{\partial H}{\partial R}. \tag{6.13}$$

The transversality conditions give $\lambda_i(t_{end}) = 0, i = 1,2,3$ as in (6.11). The Hamiltonian is maximized with respect to the controls at the optimal control, thus we differentiate H with respect to u in the interior of $\mathbb{U}$ to obtain the optimality condition that follows.

$$\frac{\partial H}{\partial u} = cu - \lambda_1 S + \lambda_3 S = 0. \tag{6.14}$$

Substituting $u(t) = u^*(t)$, setting $S = S^*$, $I = I^*$, $R = R^*$ we get

$$u^*(t) = \frac{(\lambda_1 - \lambda_3)S^*}{c} \tag{6.15}$$

using the property of the control space, we obtain

$$u^*(t) = \begin{cases} 0 \text{ if } \dfrac{(\lambda_1-\lambda_3)S^*}{c} \le 0 \\ \dfrac{(\lambda_1-\lambda_3)S^*}{c} \text{ if } 0 < \dfrac{(\lambda_1-\lambda_3)S^*}{c} < u_{\max} \\ u_{\max} \text{ if } \dfrac{(\lambda_1-\lambda_3)S^*}{c} \ge u_{\max} \end{cases}$$

So the optimal control is characterized as

$$u^*(t) = \min\left\{\max\left(0, \frac{(\lambda_1-\lambda_3)S^*}{c}\right), u_{\max}\cdot\right\}$$

Therfore our resulting optimality system is

$$\begin{aligned} S'(t) &= \Pi - \mu S + u^* S - \beta SI \\ I'(t) &= \beta SI - (\mu+\alpha)I - rI \\ R'(t) &= \alpha I + rI + u^* S - \mu R \\ \lambda_1' &= -1 + \lambda_1(\mu + u + \beta I) - \lambda_2 \beta I - \lambda_3 u \\ \lambda_2' &= -1 + \lambda_1(\beta S) - \lambda_2(\beta S - (\mu+\alpha+r)) - \lambda_3(\alpha+r) \\ \lambda_3' &= \lambda_3 \mu \\ \lambda_i(t_{end}) &= 0, \quad i = 1,2,3 \end{aligned} \tag{6.16}$$

□

We use Matlab to run the forward - backward sweep on the Deterministic SIR model (1.2) with each control seperately. In the Section 7, Figure (9) shows the population for the Susceptible, Infected and Recovered compartments, respectively.

7. Numerical simulation

In this section we visualize our results.Consider the deterministic SIR model (1.2) as

$$\begin{aligned} S'(t) &= \Pi - \mu S - \beta SI, \\ I'(t) &= \beta SI - (\mu + \alpha + r)I, \\ R'(t) &= (\alpha + r)I - \mu R, \end{aligned}$$

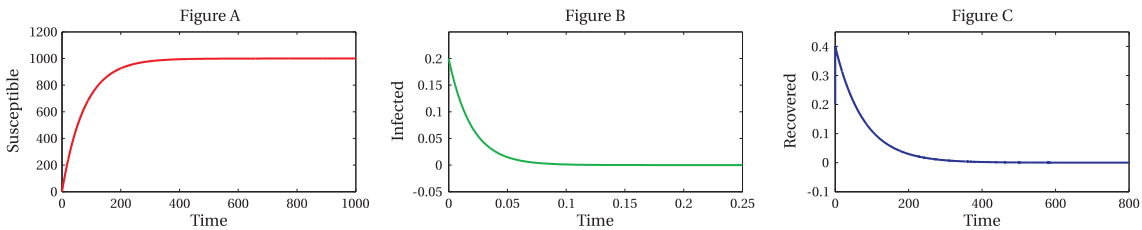

Figure 1: Deterministic trajectories of SIR epidemic model (1.2) for the parameters: $\Pi = 13, \alpha = 52, \beta = 0.03, \mu = 0.013$ and $r = 0.04 (R_0 = 0.5763 \leq 1)$. Here we use the initial value $(S_0, I_0, R_0) = (0.6, 0.2, 0.2)$.

In Figure 1(A), 1(B) and 1(C), dynamics of Susceptible, Infected and Recovered versus time are plotted. If $\mathscr{R}_0 \leq 1$ then the disease free equilibirium solution $(S, I, R) = (1000, 0, 0)$ of the system (1.2) is globally asymptotically stable on $\mathbb{D}$.

Consider the stochastic SIR model.

$$\begin{aligned} dS &= \left(\Pi - \mu S - \beta SI+\right) dt - \left(\frac{\mu}{\Pi}\right)^3 SI(S - S_2) dW_1, \\ dI &= \left(\beta SI - (\mu + \alpha + r)I\right) dt + \left(\frac{\mu}{\Pi}\right)^3 SI(S - S_2) dW_1 - \left(\frac{\mu}{\Pi}\right)^2 I(I - I_2) dW_2, \\ dR &= \left((\alpha + r)I - \mu R\right) dt + \left(\frac{\mu}{\Pi}\right)^2 I(I - I_2) dW_2, \end{aligned} \tag{7.1}$$

where Π, α, β, r, and μ are positive constants and an endemic equilibrium

$$E_2 = (S_2, I_2, R_2) = \left(\frac{\Pi}{\mathscr{R}_0}, \frac{\mu}{\beta}(\mathscr{R}_0 - 1), \frac{\alpha + r}{\beta}(\mathscr{R}_0 - 1)\right) \tag{7.2}$$

Global existence of a unique solution of the system (7.1) in

$$\mathbb{D} = \left\{(S, I, R) \in \mathbb{R}^3 : S \geq 0, I \geq 0, R \geq 0, S + I + R \leq \frac{\Pi}{\mu}\right\}$$

is proven by Theorem 3.1.

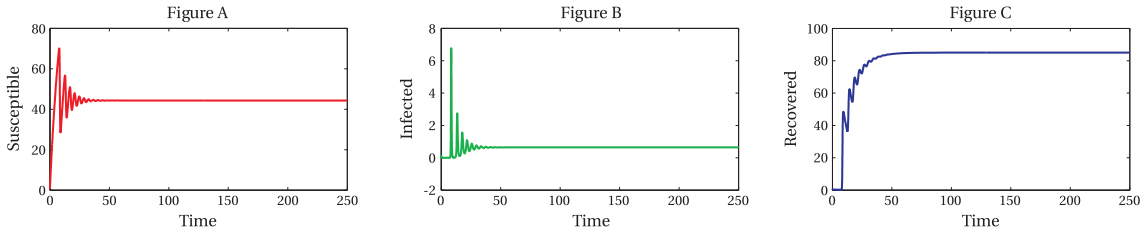

Figure 2: Deterministic trajectories of SIR epidemic model (1.2) for the parameters: $\Pi = 13, \alpha = 13, \beta = 0.3, \mu = 0.1$ and $r = 0.2 (R_0 = 2.9323 \geq 1)$. Here we use the initial value $(S_0, I_0, R_0) = (0.6, 0.2, 0.2)$.

In Figure 2(A), 2(B) and 2(C), dynamics of Susceptible, Infected and Recovered versus time are plotted. If $\mathscr{R}_0 > 1$ then the endemic equilibirium solution $(S, I, R) = (44.34, 0.6442, 85.02)$ of the system (1.2) is globally asymptotically stable on $\mathbb{D}$.

In Figure 3(A), 3(B), 3(C) and 5(A), 5(B), 5(C), dynamics of expected values of Susceptible, Infected and Recovered versus time are plotted. They show that Susceptible, Infected and Recovered populations, in average, settle around the equilibria. Figure 3(D), 3(E), 3(F) and 5(D), 5(E), 5(F) display the evaluation of the variances of Susceptible, Infected and Recovered versus time. As it is seen, variances rapidly go to zero. Hence the equilibrium solutions are approached.

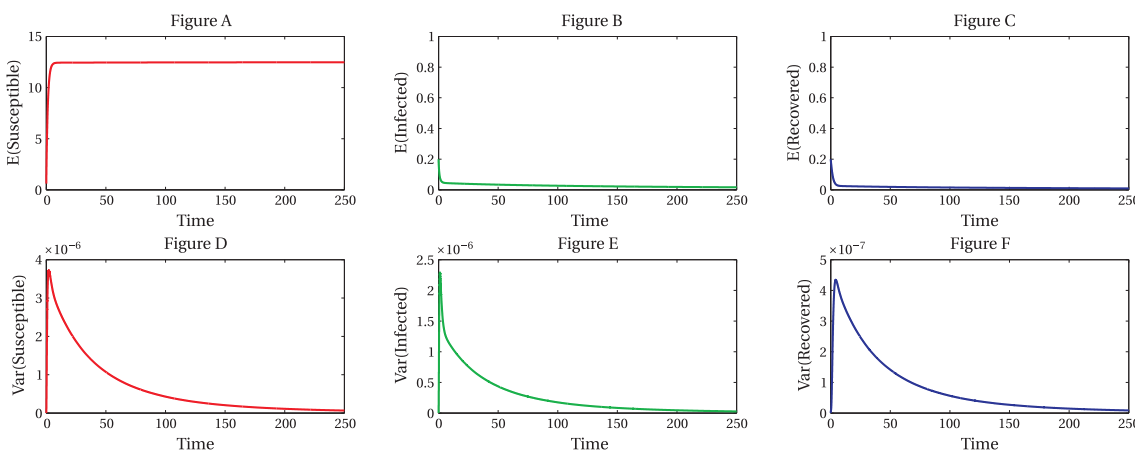

Figure 3: The disease free equilibrium $(S_1, I_1, R_1) = (12.5, 0, 0)$ is globally asymptotically stochastically stable for the parameters: $\Pi = 10, \alpha = 0.25, \beta = 0.1, \mu = 0.8$ and $r = 0.2 (R_0 = 1 \leq 1)$. Here we use the initial value $(S_0, I_0, R_0) = (0.6, 0.2, 0.2)$ and step size $\triangle = 10^{-2}$ for simulations. Expectations and variances are taken for 10,000 trajectories.

Figure 3 verifies Theorem 4.1 which states, if $\frac{\beta}{\mu(\mu+\alpha+r)} \leq 1$ then the disease free equilibrium solution $(S_1, I_1, R_1) = (12.5, 0, 0)$ of the system (7.1) is globally stochastically asymptotically stable on $\mathbb{D}$.

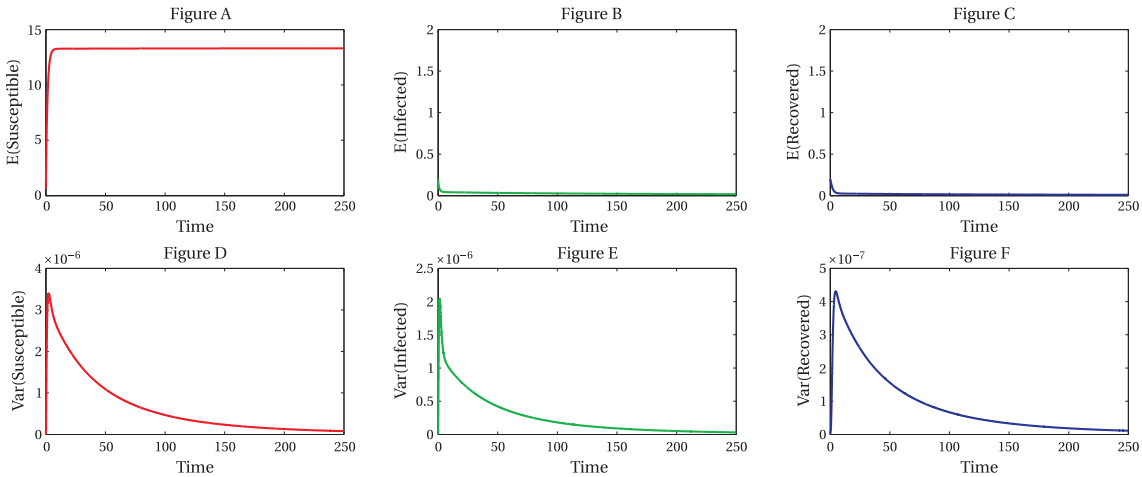

Figure 4: The disease free equilibrium $(S_1, I_1, R_1) = (13.3333, 0, 0)$ is globally asymptotically stochastically stable for the parameters: $\Pi = 10, \alpha = 0.15, \beta = 0.09, \mu = 0.75$ and $r = 0.3(R_0 = 1 \leq 1)$. Here we use the initial value $(S_0, I_0, R_0) = (0.6, 0.2, 0.2)$ and step size $\triangle = 10^{-2}$ for simulations. Expectations and variances are taken for 10,000 trajectories.

Figure 5 agrees to Theorem 4.2 which proves stochastic asymptotic stability of the endemic equilibrium solution $(S_2, I_2, R_2) = \left(\frac{\Pi}{\mathscr{R}_0}, \frac{\mu}{\beta}(\mathscr{R}_0 - 1), \frac{\alpha+r}{\beta}(\mathscr{R}_0 - 1)\right)$ to the system (7.1) on $\mathbb{D}$ under the assumption $\mathscr{R}_0 = 1.25 > 1$ and η is negative definite, which requires non-negative of the constant $\phi := \left(b\beta\frac{\Pi}{\mu} + b\beta S_2 + b\mu - a\beta\right) - \frac{1}{2}\left(\frac{\mu}{\Pi}\right)^4 \left(aI_2 + b\left(\frac{\Pi}{\mu}\right)^2\right)$ and $\xi := (b\beta S_2 - a\beta) - \frac{1}{2}\left(\frac{\mu}{\Pi}\right)^4 aI_2$, where

$$\eta(S, I, R) = -\frac{\mu^2}{\Pi}(S - S_2 + I - I_2 + R - R_2)^2 - \phi(S - S_2)^2 - \xi(I - I_2)^2,$$

for $a = \left(\frac{1}{2}\right) bS_2$, taking $b = 1 > 0$.

7.1. Optimal control results

We use the backward-forward sweep method to solve the optimal system. The process begins with a initial guess on the control variable. Then, the state equations are simultaneously solved with an initial guess forward sweep method in time and the adjoint equations are solved backward sweep method in time because of the transversality conditions. The control is updated by inserting the new values of states and adjoints into its characterization, and the process is repeated until convergence occurs.

In Figure 10(A), the plot represent the population of susceptible individuals (S) with optimal control vaccination. In Figure 10(B), the plot represent the population of infected indi-

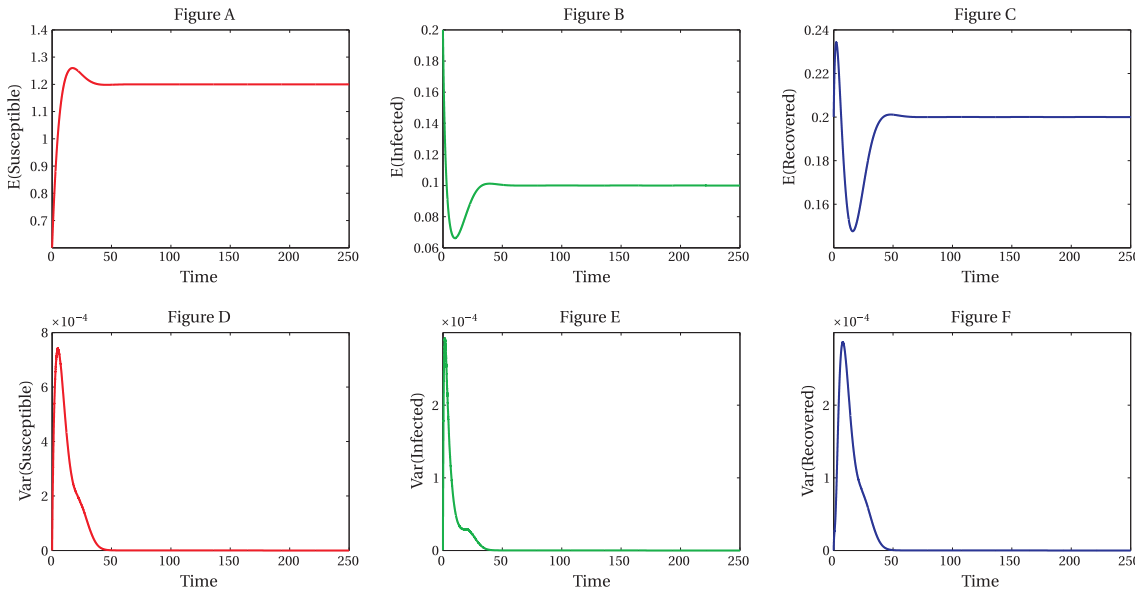

Figure 5: The endemic equilibrium solution $E_2 = (S_2, I_2, R_2) = (1.2, 0.1, 0.2)$ is asymptotically stochastically stable for the parameters: $\Pi = 0.3, \alpha = 0.2, \beta = 0.5, \mu = 0.2$ and $r = 0.2$ Since $\mathscr{R}_0 = 1.25 > 1$ and $\eta \leq 0$. We use initial value $(S_0, I_0, R_0) = (0.6, 0.2, 0.2)$ and step size $\triangle = 10^{-2}$ for simulations. Expectations and variances are taken for 10,000 trajectories.

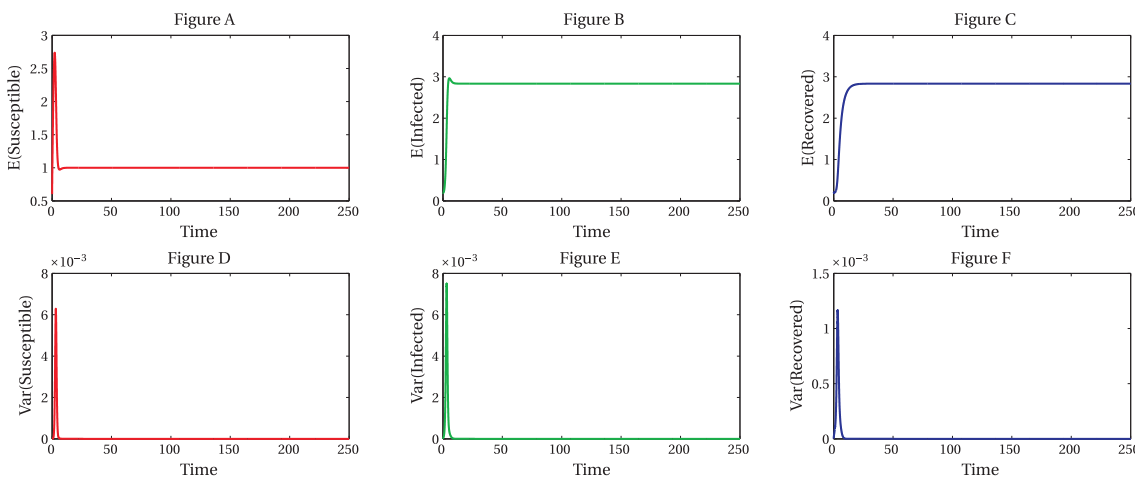

Figure 6: The endemic equilibrium solution $E_2 = (S_2, I_2, R_2) = (1, 2.8333, 2.8333)$ is asymptotically stochastically stable for the parameters: $\Pi = 2, \alpha = 0.2, \beta = 0.6, \mu = 0.3$ and $r = 0.1$ Since $\mathscr{R}_0 = 1.25 > 1$ and $\eta \leq 0$. We use initial value $(S_0, I_0, R_0) = (0.6, 0.2, 0.2)$ and step size $\triangle = 10^{-2}$ for simulations. Expectations and variances are taken for 10,000 trajectories.

viduals (I) with optimal control vaccination and in Figure 10(C), the plot represent the population of recovered individuals (R) with optimal control vaccination.

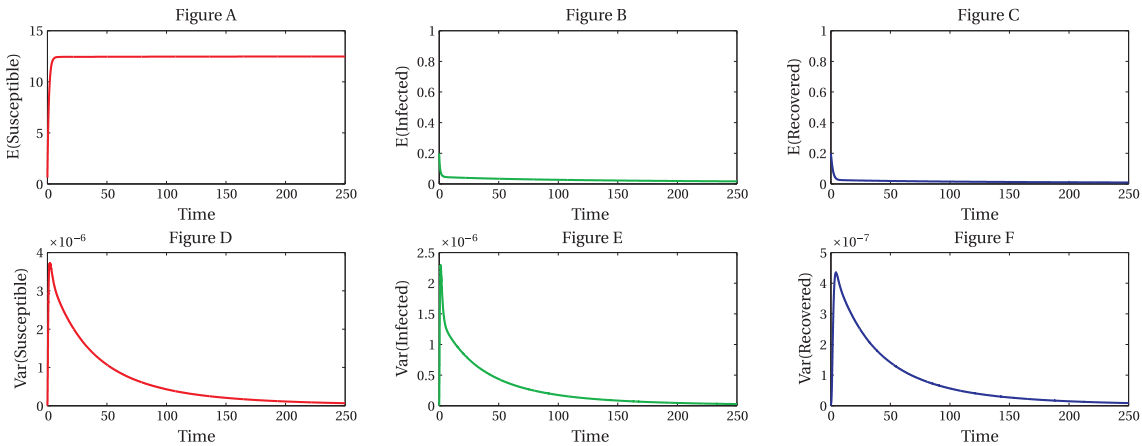

Figure 7: The disease free equilibrium $(S_3, I_3, R_3) = (6.25, 0, 0)$ is pth - moment exponentially stable for the parameters: $\Pi = 5, \mu = 0.8, \alpha = 0.2, \beta = 0.2$, and $r = 0.25$ $(R_0 = 1 \leq 1)$. We use initial value $(S_0, I_0, R_0) = (0.6, 0.2, 0.2)$ and step size $\triangle = 10^{-2}$ for simulations. Expectations and variances are taken for 10,000 trajectories.

Figure 7 agrees to Theorem 5.3 which proves pth- moment exponentially stable of the disease free equilibrium solution (S_3, I_3, R_3) to the system (7.1) on $\mathbb{D}$ under the assumption $\mathscr{R}_0 = 1 \leq 1, p = 2$ and

$$\sigma = \left(\frac{\mu}{\Pi}\right)^2 \sqrt{(\frac{\Pi}{\mu} - S_2)^2 + (\frac{\Pi}{\mu} - I_2)^2} = 0.1600 < \sqrt{4\left(\mu + \alpha + r\right) - 2\frac{\beta\Pi}{\mu}} = 1.5811.$$

Figure 9 does not hold the Theorem 5.3 the solution (S_4, I_4, R_4) to the system (7.1) on $\mathbb{D}$ under the assumption $\mathscr{R}_0 = 2 > 1, p = 2$ and

$$\sigma = \left(\frac{\mu}{\Pi}\right)^2 \sqrt{(\frac{\Pi}{\mu} - S_2)^2 + (\frac{\Pi}{\mu} - I_2)^2} = 0.2581 > \sqrt{4\left(\mu + \alpha + r\right) - 2\frac{\beta\Pi}{\mu}} = 0.$$

In Figure 10(A), 10(B), 10(C), we plot the susceptible, infected and recovered individuals using system (6.3). In our graph, note that the solid line represent the population of susceptible, infected and recovered individuals with control. The dotted line represent the population of susceptible, infected and recovered individuals without control.

Now we consider Figure 11 and Figure 12. The graphs represent the adjoint variables λ_1, λ_2 and λ_3 in the optimal system.

Figure 12 shows the control variable u(t) plotted as a function of S and λ_1 for the weight factor $c = 1$ in the population. We observe that the control variable u at a time t play a significant role in minimizing the probability that the infected individuals spread the disease in the population.

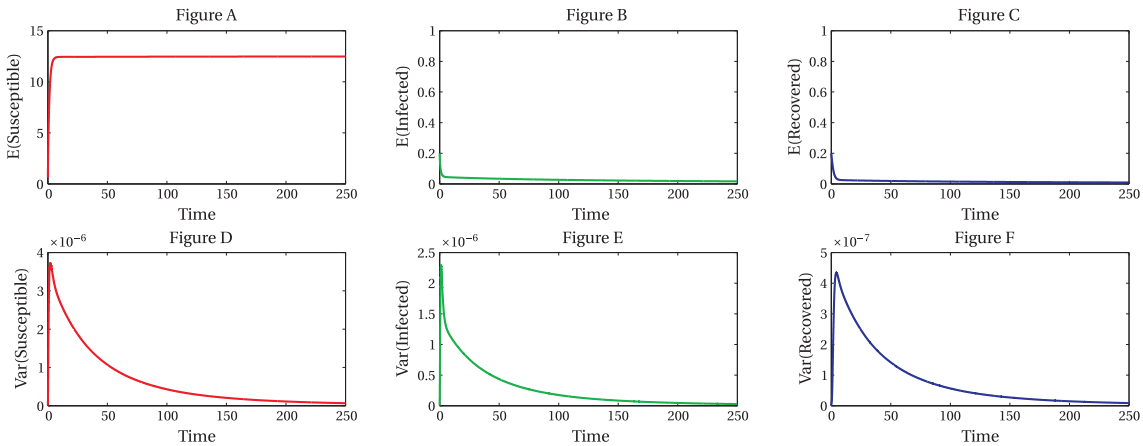

Figure 8: The disease free equilibrium $(S_3, I_3, R_3) = (1.25, 0, 0)$ is pth - moment exponentially stable for the parameters: $\Pi = .5, \mu = 0.4, \alpha = 0.05, \beta = 0.4$, and $r = 0.05$ ($R_0 = 1 \leq 1$). We use initial value $(S_0, I_0, R_0) = (0.6, 0.2, 0.2)$ and step size $\triangle = 10^{-2}$ for simulations. Expectations and variances are taken for 10,000 trajectories.

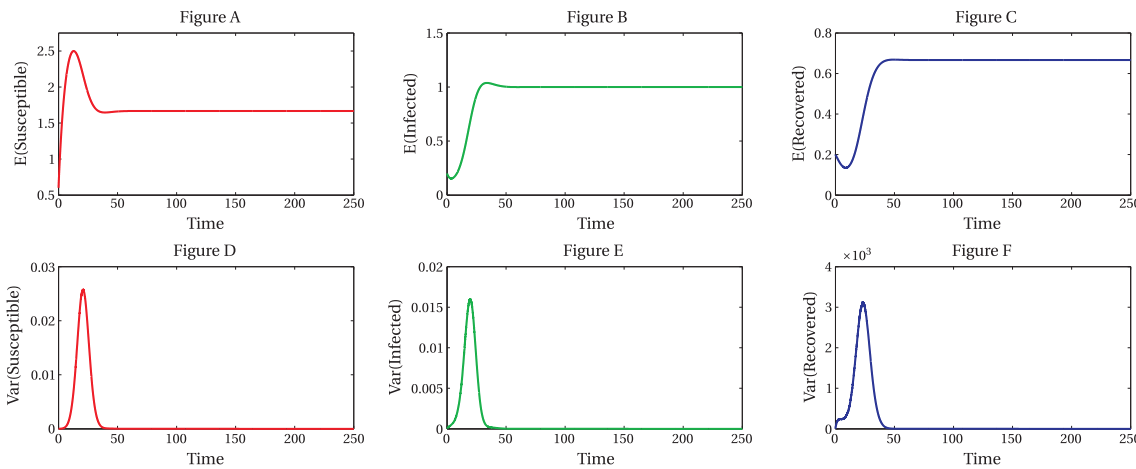

Figure 9: Stochastic trajectories of SIR epidemic model (7.1) for the parameters: $\Pi = 0.5, \mu = 0.15, \alpha = 0.05, \beta = 0.15$, and $r = 0.05$ ($R_0 = 2 > 1$) and step size $\triangle = 10^{-2}$ for simulations. Expectations and variances are taken for 10,000 trajectories.The solution $(S_4, I_4, R_4) = (1.6667, 1, 0.6667)$ is globally asymptotically stochastically stable.

8. Summary

In this paper, we have formulated a stochastic SIR model for the spread of disease throughout a homogeneous community of a fixed size. Assume that the times that an individual spends in distinct states are exponentially distributed. We consider general diffusion term, hence, we have a family of stochastic SIR model. In Section 3, we established a positive unique

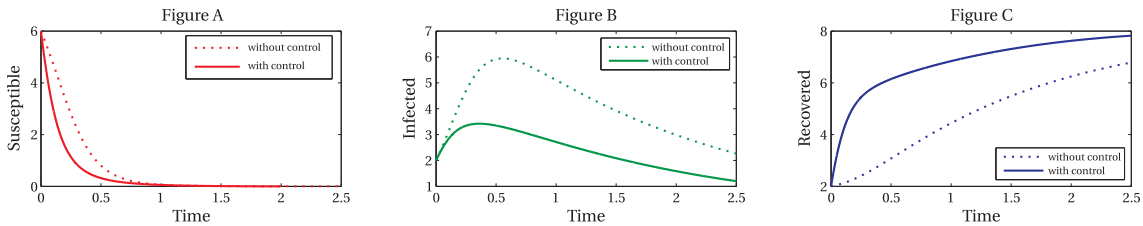

Figure 10: Deterministic trajectories of SIR epidemic model (1.3) and optimal control model (6.3) for the parameters : $S_0 = 6, I_0 = 2, R_0 = 2, \Pi = 0.001, \mu = 0.04, \alpha = 0.01, \beta = 0.9$ and $r = 0.5$ and step size $\triangle = 10^{-2}$ for simulations. Susceptiable, Infected and Recovered are taken for 10,000 trajectories.

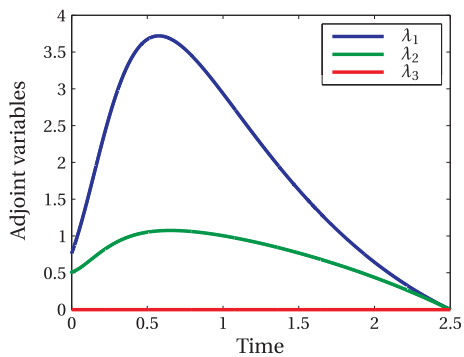

Figure 11: The plot represent the adjoint variables λ_1, λ_2 and λ_3 for the system of equations (6.10).

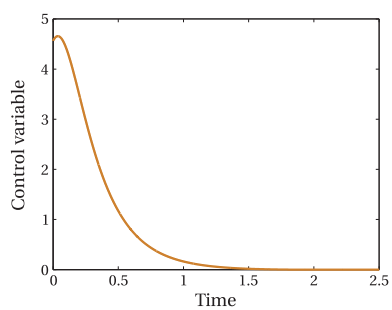

Figure 12: The plot represent the poppulation of suceptible individuals(S) with optimal control vaccination.

solution for the model (1.3), which is essential for any population dynamics models. In epidemic theory, stochastic dynamical system often starts with the basic reproductive number $\mathscr{R}_0$. A sufficient condition for stochastic asymptotic stability is found in terms of parameters and functional dependence on the variable. In Section 4, we discussed stochastic asymptotic stability of infection - free equilibrium with the help of invariance principle and Lyapunov's

method. In example 4.2, If $\mathscr{R}_0 \leq 1$, the disease - free equilibrium has no infectives, as seen Figure . Hence, the disease - free equilibrium is globally asymptotically stable. We have indicated in section 4.2, the stochastic SIR model (1.3), stochastic asymptotic stability of endemic equilibrium solution in example 4.5 if $\mathscr{R}_0 > 1$, Figure 6. We obtain, the disease - free equilibrium is unstable. A remarkable fact of the criteria $\eta(S, I, R) \leq 0$ on $\mathbb{D}$ is that a sufficient condition for stability can be found even for general local Lipschitz continuous F_i's.

We have investigated pth - moment exponential stability of stochastic SIR model (1.3). By using the Lyapunov function, Lemma 5.2, and stochastic analysis techniques, some sufficient conditions are derived to ensure pth - moment exponential stability. In example 5.1, verify the Theorem 5.1, which states, if $\mathscr{R}_0 \leq 1$ then the disease - free equilibrium solution of the system (1.3) is globally stochastically asymptotically stable as seen in Figure 8. In Sections 3, 4 and 5 using numerical simulation Section 7 we have examined, in the stochastic SIR model (1.3) the expected values of susceptible, infected and recovered versus time. Also we have examined, in stochastic SIR model (1.3) the variance of susceptible, infected and recovered versus time. We implemented optimal control in the deterministic SIR model (1.2) using a Hamiltonian formulation. A derivation of Hamiltonian, adjoint equations, and optimality conditions are provided. Using numerical method known as the forward - backward sweep, we simulated outcomes of the deterministic SIR model with and without control. It is evident that introducing an optimal control is an effective measure of decreasing the infected in the population. Finally, our results reveal that a certain type of stochastic perturbation may help stabilize the system. Furthermore, we visualized our results numerically.

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