ON APPLICATIONS OF DIFFERENTIAL SUBORDINATION AND SUPERORDINATION

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Abstract. In the present investigation we obtain the sufficient conditions for normalized analytic functions $f$ to satisfy

$$q_1 < \frac{f^2}{z^2 f} < q_2,$$

where $q_1$ and $q_2$ are univalent functions with $q_1(0) = q_2(0) = 1$. Also we obtain the sandwich results involving Carlson-Shaffer linear operator, Salagean derivative and Ruscheweyh derivative.

1. Introduction

Let $A$ be the class of normalized analytic functions $f$ in the open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ satisfying $f(0) = f'(0) - 1 = 0$. For two functions $f$ and $g$ given by

$$f(z) := z + \sum_{n=2}^{\infty} f_n z^n \quad \text{and} \quad g(z) := z + \sum_{n=2}^{\infty} g_n z^n,$$

their Hadamard product or convolution is defined as

$$(f \ast g)(z) := z + \sum_{n=2}^{\infty} f_n g_n z^n.$$

Define the function $\varphi(a, c; z)$ by

$$\varphi(a, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \quad (c \neq 0, -1, -2, \ldots; z \in \Delta),$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n := \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1)(\lambda + 2) \cdots (\lambda + n - 1) & (n = 1, 2, 3 \ldots) \end{cases}. $$

Corresponding to the function $\varphi(a, c; z)$ Carlson-Shaffer [5] introduced an operator $L(a, c)$ for $f \in A$ using Hadamard product as follows:

$$L(a, c)f(z) := \varphi(a, c; z) * f(z)$$

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\[ z = \sum_{n=1}^{\infty} \left( \frac{a}{c} \right)_n \alpha f_{n+1} z^{n+1}. \]

Note that \( L(a, a)f = f; L(2, 1)f = z f' \) and \( L(\delta + 1, 1)f = D^\delta f \), where \( D^\delta f \) is the Ruscheweyh derivative of order \( \delta \) [6].

Sălăgean derivative operator of order \( m \) [7] for \( f \in \mathcal{A} \), denoted by \( \mathcal{D}^m f \), defined as

\[ \mathcal{D}^m f(z) = z + \sum_{n=2}^{\infty} \frac{n^m a_n}{n} z^n. \]

Note that \( \mathcal{D}^0 f = f \) and \( \mathcal{D}^1 f = z f' \).

Let \( \mathcal{H} \) denotes the class of functions analytic in \( \Delta \) and \( \mathcal{H}(a, n) \) denotes the subclass of \( \mathcal{H} \) consisting of functions of the form \( f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \). For two analytic functions \( f, F \in \mathcal{H} \) we say \( F \) is superordinate to \( f \), if \( f \) is subordinate to \( F \). Let \( p, h \in \mathcal{H} \) and let \( \phi(r, s, t; z) : C^3 \times \Delta \rightarrow C \). If \( p \) and \( \phi(p, z p', z^2 p''; z) \) are univalent and if \( p \) satisfies the second order superordination

\[ h < \phi(p, z p', z^2 p''; z), \quad (1.1) \]

then \( p \) is the solution of the differential superordination \((1.1)\). An analytic function \( q \) is called subordinant, if \( q < p \) for all \( p \) satisfying \((1.1)\). A univalent subordinant \( \tilde{q} \) that satisfies \( q < \tilde{q} \) for all subordinates \( q \) of \((1.1)\), is said to be best subordinant. Recently Miller and Mocanu [3] obtained conditions on \( h, q \) and \( \phi(r, s, t; z) \) to satisfy the following:

\[ h < \phi(p, z p', z^2 p''; z) \Rightarrow q < p. \]

Using the results of Miller and Mocanu [3], Bulboacă have considered certain classes of first order differential superordinations [2] as well as superordination preserving integral operators [1].

In the present investigation we give some application of first order differential subordination and superordination to obtain sufficient conditions for normalized analytic functions \( f \) to satisfy

\[ q_1 < \frac{f^2}{z^2 f'} < q_2 \]

where \( q_1 \) and \( q_2 \) are univalent in \( \Delta \). Also applications to Carlson-Shaffer linear operator and Sălăgean derivative are studied.

2. Preliminaries

For the present investigation we need the following definition and results.

**Definition 2.1.** [3, Definition 2, p.817] Denote by \( \mathcal{D} \), the set of all functions \( f \) that are analytic and univalent in \( \overline{\Delta} \setminus E(f) \), where

\[ E(f) := \{ \zeta \in \partial \Delta : \lim_{z \to \zeta} f(z) = \infty \} \]
and are such that \( f'(ζ) \neq 0 \) for \( ζ \in ∂Δ \setminus E(f) \).

Theorem 2.1. (cf. Miller and Mocanu [4, Theorem 3.4h, p.132]) Let \( q \) be univalent in \( Δ \) and \( θ \) and \( φ \) be analytic in a domain \( D \) containing \( q(Δ) \) with \( ϕ(w) \neq 0 \), when \( w ∈ q(Δ) \). Set \( Q = zq'φ(q), h = θ(q) + Q \). Suppose that

i) \( Q \) is starlike univalent in \( Δ \) and

ii) \( \Re \{\frac{zh'}{Q}\} > 0 \) for \( z ∈ Δ \).

If \( p \) is analytic in \( Δ \) with \( p(Δ) \subseteq D \) and

\[
θ(p) + zp'φ(p) ≺ θ(q) + zq'φ(q)
\]

then

\( p < q \)

and \( q \) is the best dominant.

Theorem 2.2. [2] Let \( q \) be univalent in \( Δ \) and \( θ \) and \( φ \) be analytic in domain \( D \) containing \( q(Δ) \). Suppose that

i) \( \Re \{\frac{θ(q)}{φ(q)}\} ≥ 0 \) for \( z ∈ Δ \) and

ii) \( g = zq'φ(q) \) is starlike univalent in \( Δ \).

If \( p ∈ \mathcal{H}[q(0), 1] \cap \mathcal{Ω} \) with \( p(Δ) \subseteq D \) and \( θ(p) + zp'φ(p) \) is univalent in \( Δ \), and

\[
θ(q) + zq'φ(q) ≺ θ(p) + zp'φ(p),
\]

then

\( q < p \)

and \( q \) is the best subordinant.

3. Application to Analytic Functions

Theorem 3.1. Let \( 0 ≠ \alpha ∈ \mathbb{C} \) and \( \Re \{\frac{1}{\alpha}\} > 0 \). Let \( q \) be convex univalent in \( Δ \) with \( q(0) = 1 \). Let

\[
ψ₁ := \frac{2αf}{z} + \frac{f^2}{z^2 f'} \left[ (1 - 2α) - \frac{αzf''}{f'} \right], \quad (3.1)
\]

and \( χ₁ := q + αzq' \). Let \( f ∈ \mathcal{A} \), and \( \frac{f^2}{z^2 f'} ∈ \mathcal{H}[1, 1] \cap \mathcal{Ω} \) and \( ψ₁ \) is univalent in \( Δ \).

i) If \( ψ₁ < χ₁ \) then

\[
\frac{f^2}{z^2 f'} < q
\]

where \( q \) is the best dominant.
(ii) If $\chi_1 \prec \psi_1$ then
\[ q \prec \frac{f^2}{z^2 f'} \]
where $q$ is the best subordinant.

**Proof.** Define the function $p$ by
\[ p := \frac{f^2}{z^2 f'}. \tag{3.2} \]
A computation using (3.2) shows that
\[ \frac{zp'}{p} = \frac{2zf''}{f} - \frac{zf'}{f'} - 2. \tag{3.3} \]
Also we note that an application of (3.3) yields
\[ \psi_1 = \frac{2\alpha f}{z} + \frac{f^2}{z^2 f'} \left[ (1 - 2\alpha) - \frac{\alpha zf''}{f'} \right] \]
\[ = p + \alpha zp', \]
and this can be written as (2.1) when $\theta(w) = w$ and $\phi(w) = \alpha$. Note that $\phi(w) \neq 0$ and $\theta$ and $\phi$ are analytic in $\mathbb{C}$. Set
\[ Q := \alpha zq', \]
\[ h := \theta(q) + Q \]
\[ = q + \alpha zq'. \]
In light of the hypothesis of Theorem 2.1, we see that $Q$ is starlike and
\[ \Re \left\{ \frac{zh'}{Q} \right\} = \Re \left\{ \frac{1}{\alpha} \right\} \geq 0. \]
By an application of Theorem 2.1 we conclude that $p \prec q$ or
\[ \frac{f^2}{z^2 f'} \prec q. \]
Note that
\[ \Re \left\{ \frac{\theta'(q)}{\phi(q)} \right\} = \Re \left\{ \frac{1}{\alpha} \right\} \geq 0. \]
Hence the result (ii) of Theorem 3.1 follows as a similar application of Theorem 2.2.

By making use of Theorem 3.1 we get the following sandwich type result.

**Theorem 3.2.** Let $0 \neq \alpha \in \mathbb{C}$ and $\Re \left\{ \frac{1}{\alpha} \right\} > 0$. Let $q_i$ for $i = 1, 2$ be convex univalent in $\Delta$, with $q_i(0) = 1$. Let $\chi_i = q_i + \alpha z q_i'$ for $i = 1, 2$ and $\psi_1$ as given by (3.1) be univalent in $\Delta$. If $f \in \mathcal{A}$, $\frac{f^2}{z^2 f'} \in \mathcal{H}[1, 1] \cap \mathbb{D}$ and
\[ \chi_1 \prec \psi_1 \prec \chi_2 \]
then
\[ q_1 < \frac{f^2}{z^2 f'} < q_2 \]
where \( q_1 \) and \( q_2 \) are respectively the best subordinant and best dominant.

**Theorem 3.3.** Let \( \alpha, \beta \) and \( \gamma \) be complex numbers and \( \gamma \neq 0 \). Let \( q \) be a convex univalent functions in \( \Delta \) with \( q(0) = 1 \) and \( \frac{\gamma z q'}{q} \) is starlike univalent in \( \Delta \). Let
\[
\psi_2 := (\alpha - 2\gamma) + \frac{2\gamma z f'}{f} + \frac{\beta f^2}{z^2 f'} - \frac{\gamma z f''}{f'}
\]
and \( \chi_2 = \alpha + \beta q + \frac{\gamma z q'}{q} \). Let \( f \in \mathcal{A} \) and \( \frac{f^2}{z f'} \in \mathcal{H}[1,1] \cap \mathcal{Q} \) and \( \psi_2 \) is univalent in \( \Delta \).

(i) If \( q \) satisfies
\[
\Re\left\{\frac{\beta q}{\gamma} - \frac{z q'}{q}\right\} > 0 \quad \text{(3.4)}
\]
then
\[
\psi_2 < \chi_2 \Rightarrow \frac{f^2}{z^2 f'} < q
\]
where \( q \) is the best dominant.

(ii) If \( q \) satisfies
\[
\Re\left\{\frac{\beta q}{\gamma}\right\} > 0 \quad \text{(3.5)}
\]
then
\[
\chi_2 < \psi_2 \Rightarrow q < \frac{f^2}{z^2 f'}
\]
where \( q \) is the best subordinant.

**Proof.** Define the function \( p \) by
\[
p := \frac{f^2}{z^2 f'} \quad \text{(3.6)}
\]
A simple computation using (3.3) shows that
\[
\psi_2 := (\alpha - 2\gamma) + \frac{2\gamma z f'}{f} + \frac{\beta f^2}{z^2 f'} - \frac{\gamma z f''}{f'}
= \alpha + \beta p + \frac{\gamma z q'}{p}
\]
This can be written as (2.1) when \( \theta(w) := \alpha + \beta w \) and \( \phi(w) := \frac{\gamma}{w} \). Note that \( \theta \) and \( \phi \) are analytic in \( \mathbb{C} \setminus \{0\} \). Set
\[
Q := \frac{\gamma z q'}{q}
\]
\[ h := a + \beta q + Q = a + \beta q + \frac{\gamma z q'}{q} \]

In light of hypothesis of Theorem 2.1 we see that \( Q \) is starlike and
\[ \Re \left\{ \frac{zh'}{Q} \right\} = \Re \left\{ \frac{\beta q}{\gamma} - \frac{zq'}{q} + (1 + \frac{zq''}{q}) \right\} > 0. \]

By an application of Theorem 2.1 we conclude that
\[ \frac{f^2}{z^2 f'} < q. \]

The result (ii) of Theorem (3.3) follows as a similar exercise using Theorem 2.2.

### 4. Application to Carlson-Shaffer Operator

**Theorem 4.1.** Let \( 0 \neq \alpha \in \mathbb{C} \) and \( \Re(\frac{1}{\alpha}) > 0 \). Let \( q \) be convex univalent in \( \Delta \) with \( q(0) = 1 \). Let
\[ \psi_3 := \frac{[L(a,c) f]^2}{zL(a+1,c)f} \left[ 1 + a(1-a) - \frac{a(a+1)L(a+2,c)f}{L(a+1,c)f} \right] + \frac{2aa}{z}L(a,c)f \]
and \( \chi_3 := q + \alpha z q' \). Let \( f \in \mathcal{A} \) and \( \frac{[L(a,c) f]^3}{zL(a+1,c)f} \in \mathcal{H}[1,1] \cap \mathcal{B} \) and \( \psi_3 \) is univalent in \( \Delta \).

(i) If \( \psi_3 < \chi_3 \) then
\[ \frac{[L(a,c) f]^2}{zL(a+1,c)f} < q \]
where \( q \) is the best dominant.

(ii) If \( \chi_3 < \psi_3 \) then
\[ q < \frac{[L(a,c) f]^2}{zL(a+1,c)f} \]
where \( q \) is the best subordinant.

**Proof.** Define the function \( p \) by
\[ p := \frac{[L(a,c) f]^2}{zL(a+1,c)f}. \] (4.1)

A simple computation using (4.1) gives
\[ \frac{zp'}{p} = 2z(L(a,c) f)'L(a,c) f - 1 - \frac{z(L(a+1,c) f)'}{L(a+1,c) f}. \] (4.2)

By using the identity
\[ z(L(a,c) f)' = aL(a+1,c) f - (a-1)L(a,c) f \]
in (4.2) we obtain
\[
\frac{zp'}{p} = (1 - a) + \frac{2aL(a + 1, c)f}{L(a, c)f} - (a + 1) \frac{L(a + 2, c)f}{L(a + 1, c)f}.
\]

Note that
\[
\psi_3 := \frac{(L(a, c)f)^2}{zL(a + 1, c)f} \left[ 1 + a(1 - a) - \frac{a(a + 1)L(a + 2, c)f}{L(a + 1, c)f} \right] + \frac{2aa}{z} L(a, c)f
\]
and this can be written as (2.1) when \(\theta(w) = w\) and \(\phi(w) = \alpha\). Hence the result (i) follows as an application of Theorem (2.1). The proof of result (ii) of Theorem 4.1 follows as a similar application of Theorem 2.2.

By taking \(a = \delta + 1\) and \(c = 1\) we get the following result involving Ruscheweyh derivative.

**Corollary 4.2.** Let \(0 \neq \alpha \in \mathbb{C}\) and \(\Re \{\frac{1}{\alpha}\} > 0\). Let \(q\) be convex univalent in \(D\) with \(q(0) = 1\). Let
\[
\psi := \frac{(D^\delta f)^2}{zD^{\delta + 1}f} \left[ 1 + a(1 - a) - \frac{a(a + 1)L(a + 2, c)f}{L(a + 1, c)f} \right] + \frac{2aa}{z} D^\delta f.
\]
and \(\chi := q + azp'\). Let \(f \in \mathcal{A}\) and \(\frac{(D^\delta f)^2}{zD^{\delta + 1}f} \in \mathcal{H}[1, 1] \cap \mathcal{Q}\) and \(\psi\) is univalent in \(D\).

(i) If \(\psi < \chi\) then
\[
\frac{(D^\delta f)^2}{zD^{\delta + 1}f} < q
\]
where \(q\) is the best dominant.

(ii) If \(\chi < \psi\) then
\[
q < \frac{(D^\delta f)^2}{zD^{\delta + 1}f}
\]
where \(q\) is the best subordinant.

**Theorem 4.3.** Let \(\alpha, \beta\) and \(\gamma\) be complex numbers with \(\gamma \neq 0\). Let \(q\) be a convex univalent in \(D\) with \(q(0) = 1\) and \(\frac{\gamma z q'}{q}\) is starlike univalent in \(D\). Let
\[
\psi_4 := \alpha + \gamma(1 - a) + \frac{\beta(L(a, c)f)^2}{zL(a + 1, c)f} + \frac{2ayL(a + 1, c)f}{L(a, c)f} - \frac{\gamma(a + 1)L(a + 2, c)f}{L(a + 1, c)f}
\]
and \(\chi_4 := \alpha + \beta q + \frac{\gamma z q'}{q}\). Let \(f \in \mathcal{A}\) and \(\frac{(L(a, c)f)^2}{z(L(a + 1, c)f)} \in \mathcal{H}[1, 1] \cap \mathcal{Q}\) and \(\psi_4\) is univalent in \(D\).

(i) If \(q\) satisfies (3.4) then
\[
\psi_4 < \chi_4 \Rightarrow \frac{(L(a, c)f)^2}{zL(a + 1, c)f} < q
\]
where \(q\) is the best dominant.
If $q$ satisfies (3.5) then
\[
\chi^4 < \psi^4 \Rightarrow q < \frac{\{L(a, c) f\}^2}{zL(a + 1, c) f}
\]
where $q$ is the best subordinant.

**Proof.** The proof of the Theorem 4.3 is similar to that of Theorem 4.1, where
\[
\theta(w) = \alpha + \beta w
\]
and
\[
\phi(w) = \frac{\gamma}{w}.
\]
By taking $a = \delta + 1$ and $c = 1$ we get the following result involving Ruscheweyh derivative.

**Corollary 4.4.** Let $\alpha, \beta$ and $\gamma$ be complex numbers with $\gamma \neq 0$. Let $q$ be a convex univalent in $\Delta$ with $q(0) = 1$ and $\frac{z f'}{q}$ is starlike univalent in $\Delta$. Let
\[
\psi_5 := \alpha + \gamma(1 - a) + \frac{\beta |D^\delta f|^2}{zD^\delta+1 f} + \frac{2a\gamma D^{\delta+1} f}{D^\delta f} - \frac{\gamma(a + 1)D^{\delta+2} f}{D^\delta+1 f}
\]
and
\[
\chi_5 := \alpha + \beta q + \frac{\gamma z f'}{q}. \text{ Let } f \in \mathcal{A} \text{ and } \frac{|D^\delta f|^2}{zD^\delta+1 f} \in \mathcal{H}[1, 1] \cap \mathcal{H} \text{ and } \psi_5 \text{ is univalent in } \Delta.
\]
(i) If $q$ satisfies (3.4) then
\[
\psi_5 < \chi_5 \Rightarrow \frac{|D^\delta f|^2}{zD^\delta+1 f} < q
\]
where $q$ is the best dominant.

(ii) If $q$ satisfies (3.5) then
\[
\chi_5 < \psi_5 \Rightarrow q < \frac{|D^\delta f|^2}{zD^\delta+1 f}
\]
where $q$ is the best subordinant.

5. Application to Sălăgean Derivative Operator

**Theorem 5.1.** Let $0 \neq \alpha \in \mathbb{C}$ and $\Re \left\{ \frac{1}{\alpha} \right\} > 0$. Let $q$ be convex univalent in $\Delta$ with $q(0) = 1$. Let
\[
\psi_6 := 2\alpha \frac{\mathcal{D}^m f}{z} + \frac{|\mathcal{D}^m f|^2}{z\mathcal{D}^{m+1} f} \left[ 1 - \alpha - \frac{\alpha \mathcal{D}^{m+2} f}{z\mathcal{D}^{m+1} f} \right]
\]
and
\[
\chi_6 := q + \alpha z q'. \text{ Let } f \in \mathcal{A} \text{ and } \frac{|\mathcal{D}^m f|^2}{z\mathcal{D}^{m+1} f} \in \mathcal{H}[1, 1] \cap \mathcal{H} \text{ and } \psi_6 \text{ is univalent in } \Delta.
\]
(i) If $\psi_6 < \chi_6$ then
\[
\frac{|\mathcal{D}^m f|^2}{z\mathcal{D}^{m+1} f} < q
\]
where $q$ is the best dominant.

(ii) If $\chi_6 < \psi_6$ then
\[
q < \frac{|\mathcal{D}^m f|^2}{z\mathcal{D}^{m+1} f}
\]
where $q$ is the best subordinant.
Proof. Define the function $p$ by

$$p := \left(\frac{D_m f}{z D_m^{m+1} f}\right)^2.$$  \hspace{1cm} (5.1)

A simple computation using (5.1) shows that

$$zp' = 2z(D_m f)' - 1 - z(D_m^{m+1} f)'.$$  \hspace{1cm} (5.2)

Using the identity

$$z(D_m f)' = D_m^{m+1} f,$$

in (5.2) we obtain

$$zp' = 2D_m^{m+1} f - 1 - D_m^{m+1} f.$$  \hspace{1cm} (5.3)

Note that

$$\psi_6 := 2\alpha \frac{D_m f}{z} + \left(\frac{D_m f}{z D_m^{m+1} f}\right)^2 \left[1 - \frac{\alpha D_m^{m+1} f}{D_m^{m+1} f}\right]$$

and this can be written as (2.1) when $\theta(w) := w$ and $\phi(w) := \alpha$. Now the result (i) follows as an application of Theorem 2.1. A similar exercise using Theorem (2.2) will give the result (ii).

Theorem 5.2. Let $\alpha, \beta$ and $\gamma$ be complex numbers and $\gamma \neq 0$. Let $q$ be a convex univalent in $\Delta$ with $q(0) = 1$ and $\frac{z q'}{q}$ is starlike univalent in $\Delta$. Let

$$\psi_7 := \alpha - \gamma + \frac{2\gamma D_m^{m-1} f}{z D_m^{m+1} f} - \frac{\gamma D_m^{m+2} f}{z D_m^{m+1} f} + \beta \frac{D_m^2 f}{z D_m f},$$

and $\chi_7 := \alpha + \beta q + \frac{z q'}{q}$. Let $f \in \mathcal{A}$ and $\frac{|D_m f|^2}{z D_m^{m+1} f} \in \mathcal{H}(1,1) \cap \mathcal{D}$ and $\psi_7$ is univalent in $\Delta$. Let

(i) If $q$ satisfies (3.4), then

$$\psi_7 < \chi_7 \Rightarrow \frac{|D_m f|^2}{z D_m^{m+1} f} < q$$

where $q$ is the best dominant.

(ii) If $q$ satisfies (3.5), then

$$\chi_7 < \psi_7 \Rightarrow \frac{|D_m f|^2}{z D_m^{m+1} f} < q$$

where $q$ is the best subordinant.

Proof. The proof follows as an application of Theorem 2.1 and Theorem 2.2 with $\theta(w) = \alpha + \beta w$ and $\phi(w) = \frac{\tau}{w}$.

Sandwich results for the Theorems 3.3–5.2 can be obtained by a similar exercise as we have obtained the sandwich result (Theorem 3.2) of Theorem 3.1, however we omit the details of the proof.
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