TAMKANG JOURNAL OF MATHEMATICS
Volume 39, Number 2, 155-164, Summer 2008

ON APPLICATIONS OF DIFFERENTIAL SUBORDINATION AND SUPERORDINATION

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Abstract. In the present investigation we obtain the sufficient conditions for normalized analytic functions f to satisfy

$$q_1 \prec \frac{f^2}{z^2 f'} \prec q_2,$$

where q_1 and q_2 are univalent functions with $q_1(0) = q_2(0) = 1$. Also we obtain the sandwich results involving Carlson-Shaffer linear operator, Sălăgean derivative and Ruscheweyh derivative.

1. Introduction

Let $\mathscr{A}$ be the class of normalized analytic functions f in the open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ satisfying $f(0) = f'(0) - 1 = 0$. For two functions f and g given by

$$f(z) := z + \sum_{n=2}^{\infty} f_n z^n \quad \text{and} \quad g(z) := z + \sum_{n=2}^{\infty} g_n z^n,$$

their Hadamard product or convolution is defined as

$$(f * g)(z) := z + \sum_{n=2}^{\infty} f_n g_n z^n.$$

Define the function $\varphi(a, c; z)$ by

$$\varphi(a, c; z) := \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \qquad (c \neq 0, -1, -2, \ldots; z \in \Delta),$$

where $(\lambda)_n$ is the Pocchhammer symbol defined by

$$(\lambda)_n := \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1) & (n = 1, 2, 3 \ldots). \end{cases}$$

Corresponding to the function $\varphi(a, c; z)$ Carlson-Shaffer [5] introduced an operator $L(a, c)$ for $f \in \mathscr{A}$ using Hadamard product as follows:

$$L(a, c) f(z) := \varphi(a, c; z) * f(z)$$

Received May 16, 2007; Revised November 5, 2007.
2000 *Mathematics Subject Classification*. Primary30C45, secondary 30C80.
Key words and phrases. Differential subordination, superordination, Carlson-Shaffer linear operator, Sălăgean derivative and Ruscheweyh derivative.

$$= z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} a f_{n+1} z^{n+1}.$$

Note that $L(a,a)f = f$; $L(2,1)f = zf'$ and $L(\delta+1,1)f = D^{\delta} f$, where $D^{\delta} f$ is the Ruscheweyh derivative of order δ [6].
Sălăgean derivative operator of order m [7] for $f \in \mathscr{A}$, denoted by $\mathscr{D}^m f$, defined as

$$\mathscr{D}^m f(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n.$$

Note that $\mathscr{D}^0 f = f$ and $\mathscr{D}^1 f = zf'$.

Let $\mathscr{H}$ denotes the class of functions analytic in Δ and $\mathscr{H}[a,n]$ denotes the subclass of $\mathscr{H}$ consisting of functins of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. For two analytic functions $f, F \in \mathscr{H}$ we say F is *superordinate* to f, if f is subordinate to F. Let $p, h \in \mathscr{H}$ and let $\phi(r,s,t;z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}$. If p and $\phi(p, zp', z^2p''; z)$ are univalent and if p satisfies the second order superordination

$$h \prec \phi(p, zp', z^2 p''; z), \tag{1.1}$$

then p is the solution of the differential superordination (1.1). An analytic function q is called *subordinant*, if $q \prec p$ for all p satisfying (1.1). A univalent subordinant $\tilde{q}$ thatsatisfies $q \prec \tilde{q}$ for all subordinants q of (1.1), is said to be *best subordinant*.Recently Miller and Mocanu [3] obtained conditionson h, q and $\phi(r,s,t;z)$ to satisfy the following:

$$h \prec \phi(p, zp', z^2 p''; z) \Rightarrow q \prec p.$$

Using the results of Miller and Mocanu [3], Bulboacă have considered certain classes of first order differential superordinations [2] as well as superordination preserving integral operators [1].

In the present investigation we give some application of first order differential subordination and superordination to obtain sufficient conditions for normalized analytic functions f to satisfy

$$q_1 \prec \frac{f^2}{z^2 f'} \prec q_2$$

where q_1 and q_2 are univalent in Δ. Also applications to Carlson-Shaffer linear operator and Sălăgean derivative are studied.

2. Preliminaries

For the present investigation we need the following definition and results.

Definition 2.1. [3, Definition 2, p.817] Denote by $\mathscr{Q}$, the set of all functions f that are analytic and univalent in $\overline{\Delta} \setminus E(f)$, where

$$E(f) := \left\{ \zeta \in \partial\Delta : \lim_{z\to\zeta} f(z) = \infty \right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta \setminus E(f)$.

Theorem 2.1. (cf. Miller and Mocanu [4, Theorem 3.4h, p.132]) *Let q be univalent in Δ and θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(w) \neq 0$, when $w \in q(\Delta)$. Set $Q = zq'\phi(q)$, $h = \theta(q) + Q$. Suppose that*

(i) *Q is starlike univalent in Δ and*

(ii) $\Re\left\{\frac{zh'}{Q}\right\} > 0$ *for $z \in \Delta$.*

If p is analytic in Δ with $p(\Delta) \subseteq D$ and

$$\theta(p) + zp'\phi(p) \prec \theta(q) + zq'\phi(q) \tag{2.1}$$

then

$$p \prec q$$

and q is the best dominant.

Theorem 2.2. [2] *Let q be univalent in Δ and θ and ϕ be analytic in domain D containing $q(\Delta)$. Suppose that*

(i) $\Re\left(\frac{\theta'(q)}{\phi(q)}\right) \geq 0$ *for $z \in \Delta$ and*

(ii) *$g = zq'\phi(q)$ is starlike univalent in Δ.*

If $p \in \mathscr{H}[q(0), 1] \cap \mathscr{Q}$ with $p(\Delta) \subseteq D$ and $\theta(p) + zp'\phi(p)$ is univalent in Δ, and

$$\theta(q) + zq'\phi(q) \prec \theta(p) + zp'\phi(p),$$

then

$$q \prec p$$

and q is the best subordinant.

3. Application to Analytic Functions

Theorem 3.1. *Let $0 \neq \alpha \in \mathbb{C}$ and $\Re\left\{\frac{1}{\alpha}\right\} > 0$. Let q be convex univalent in Δ with $q(0) = 1$. Let*

$$\psi_1 := \frac{2\alpha f}{z} + \frac{f^2}{z^2 f'}\left[(1-2\alpha) - \frac{\alpha z f''}{f'}\right], \tag{3.1}$$

and $\chi_1 := q + \alpha z q'$. Let $f \in \mathscr{A}$, and $\frac{f^2}{z^2 f'} \in \mathscr{H}[1,1] \cap \mathscr{Q}$ and ψ_1 is univalent in Δ.

(i) *If $\psi_1 \prec \chi_1$ then*

$$\frac{f^2}{z^2 f'} \prec q$$

where q is the best dominant.

(ii) *If* $\chi_1 \prec \psi_1$ *then*

$$q \prec \frac{f^2}{z^2 f'}$$

where q *is the best subordinant.*

Proof. Define the function p by

$$p := \frac{f^2}{z^2 f'}. \tag{3.2}$$

A computation using (3.2) shows that

$$\frac{zp'}{p} = \frac{2zf'}{f} - \frac{zf''}{f'} - 2. \tag{3.3}$$

Also we note that an application of (3.3) yields

$$\begin{aligned} \psi_1 &= \frac{2\alpha f}{z} + \frac{f^2}{z^2 f'}\Big[(1-2\alpha) - \frac{\alpha z f''}{f'}\Big] \\ &= p + \alpha z p', \end{aligned}$$

and this can be written as (2.1) when $\theta(w) = w$ and $\phi(w) = \alpha$. Note that $\phi(w) \neq 0$ and θ and ϕ are analytic in $\mathbb{C}$. Set

$$\begin{aligned} Q &:= \alpha z q', \\ h &:= \theta(q) + Q \\ &= q + \alpha z q'. \end{aligned}$$

In light of the hypothesis of Theorem 2.1, we see that Q is starlike and

$$\Re\Big\{\frac{zh'}{Q}\Big\} = \Re\Big\{\frac{1}{\alpha} + (1 + \frac{zq''}{q'})\Big\} > 0.$$

By an application of Theorem 2.1 we conclude that $p \prec q$ or

$$\frac{f^2}{z^2 f'} \prec q.$$

Note that

$$\Re\Big\{\frac{\theta'(q)}{\phi(q)}\Big\} = \Re\Big\{\frac{1}{\alpha}\Big\} \geq 0.$$

Hence the result (ii) of Theorem 3.1 follows as a similar application of Theorem 2.2.

By making use of Theorem 3.1 we get the following sandwich type result.

Theorem 3.2. *Let* $0 \neq \alpha \in \mathbb{C}$ *and* $\Re\{\frac{1}{\alpha}\} > 0$. *Let* q_i *for* $i = 1, 2$ *be convex univalent in* Δ, *with* $q_i(0) = 1$. *Let* $\chi_i = q_i + \alpha z q_i'$ *for* $i = 1, 2$ *and* ψ_1 *as given by* (3.1) *be univalent in* Δ. *If* $f \in \mathscr{A}$, $\frac{f^2}{z^2 f'} \in \mathscr{H}[1, 1] \cap \mathscr{Q}$ *and*

$$\chi_1 \prec \psi_1 \prec \chi_2$$

then

$$q_1 \prec \frac{f^2}{z^2 f'} \prec q_2$$

where q_1 and q_2 are respectively the best subordinant and best dominant.

Theorem 3.3. *Let α, β and γ be complex numbers and $\gamma \neq 0$. Let q be a convex univalent functions in Δ with $q(0) = 1$ and $\frac{\gamma z q'}{q}$ is starlike univalent in Δ. Let*

$$\psi_2 := (\alpha - 2\gamma) + \frac{2\gamma z f'}{f} + \frac{\beta f^2}{z^2 f'} - \frac{\gamma z f''}{f'}$$

and $\chi_2 = \alpha + \beta q + \frac{\gamma z q'}{q}$. Let $f \in \mathscr{A}$ and $\frac{f^2}{z^2 f'} \in \mathscr{H}[1,1] \cap \mathscr{Q}$ and ψ_2 is univalent in Δ.

(i) *If q satisfies*

$$\Re\left\{\frac{\beta q}{\gamma} - \frac{z q'}{q}\right\} > 0 \tag{3.4}$$

then

$$\psi_2 \prec \chi_2 \Rightarrow \frac{f^2}{z^2 f'} \prec q$$

where q is the best dominant.

(ii) *If q satisfies*

$$\Re\left\{\frac{\beta q}{\gamma}\right\} > 0 \tag{3.5}$$

then

$$\chi_2 \prec \psi_2 \Rightarrow q \prec \frac{f^2}{z^2 f'}$$

where q is the best subordinant.

Proof. Define the function p by

$$p := \frac{f^2}{z^2 f'}. \tag{3.6}$$

A simple computation using (3.3) shows that

$$\begin{aligned} \psi_2 &:= (\alpha - 2\gamma) + \frac{2\gamma z f'}{f} + \frac{\beta f^2}{z^2 f'} - \frac{\gamma z f''}{f'} \\ &= \alpha + \beta p + \frac{\gamma z p'}{p}. \end{aligned}$$

This can be written as (2.1) when $\theta(w) := \alpha + \beta w$ and $\phi(w) := \frac{\gamma}{w}$. Note that θ and ϕ are analytic in $\mathbb{C} \setminus \{0\}$. Set

$$Q := \frac{\gamma z q'}{q}$$

$$\begin{aligned} h &:= \alpha + \beta q + Q \\ &= \alpha + \beta q + \frac{\gamma z q'}{q} \end{aligned}$$

In light of hypothesis of Theorem 2.1 we see that Q is starlike and

$$\Re\Big\{\frac{zh'}{Q)}\Big\} = \Re\Big\{\frac{\beta q}{\gamma} - \frac{zq'}{q} + (1 + \frac{zq''}{q})\Big\} > 0.$$

By an application of Theorem 2.1 we conclude that

$$\frac{f^2}{z^2 f'} \prec q.$$

The result (ii) of Theorem (3.3) follows as a similar exercise using Theorem 2.2.

4. Application to Carlson-Shaffer Operator

Theorem 4.1. *Let* $0 \neq \alpha \in \mathbb{C}$ *and* $\Re\{\frac{1}{\alpha} t\} > 0$. *Let* q *be convex univalent in* Δ *with* $q(0) = 1$. *Let*

$$\psi_3 := \frac{\{L(a,c)f\}^2}{zL(a+1,c)f}\Big[1 + \alpha(1-a) - \frac{\alpha(a+1)L(a+2,c)f}{L(a+1,c)f}\Big] + \frac{2a\alpha}{z} L(a,c)f$$

and $\chi_3 := q + \alpha z q'$. *Let* $f \in \mathscr{A}$ *and* $\frac{\{L(a,c)f\}^2}{zL(a+1,c)f} \in \mathscr{H}[1,1] \cap \mathscr{Q}$ *and* ψ_3 *is univalent in* Δ.

(i) *If* $\psi_3 \prec \chi_3$ *then*

$$\frac{\{L(a,c)f\}^2}{zL(a+1,c)f} \prec q$$

where q *is the best dominant.*

(ii) *If* $\chi_3 \prec \psi_3$ *then*

$$q \prec \frac{\{L(a,c)f\}^2}{zL(a+1,c)f}$$

where q *is the best subordinant.*

Proof. Define the function p by

$$p := \frac{\{L(a,c)f\}^2}{zL(a+1,c)f}. \tag{4.1}$$

A simple computation using (4.1) gives

$$\frac{zp'}{p} = \frac{2z(L(a,c)f)'}{L(a,c)f} - 1 - \frac{z(L(a+1,c)f)'}{L(a+1,c)f}. \tag{4.2}$$

By using the identity

$$z(L(a,c)f)' = aL(a+1,c)f - (a-1)L(a,c)f$$

in (4.2) we obtain

$$\frac{zp'}{p} = (1-a) + \frac{2aL(a+1,c)f}{L(a,c)f} - (a+1)\frac{L(a+2,c)f}{L(a+1,c)f}.$$

Note that

$$\begin{aligned}\psi_3 &:= \frac{\{L(a,c)f\}^2}{zL(a+1,c)f}\left[1+\alpha(1-a) - \frac{\alpha(a+1)L(a+2,c)f}{L(a+1,c)f}\right] + \frac{2a\alpha}{z}L(a,c)f \\ &= p + \alpha z p'\end{aligned}$$

and this can be written as (2.1) when $\theta(w) = w$ and $\phi(w) = \alpha$. Hence the result (i) follows as an application of Theorem (2.1). The proof of result (ii) of Theorem 4.1 follows as a similar application of Theorem 2.2.

By taking $a = \delta + 1$ and $c = 1$ we get the following result involving Ruscheweyh derivative.

Corollary 4.2. *Let $0 \neq \alpha \in \mathbb{C}$ and $\Re\left\{\frac{1}{\alpha}\right\} > 0$. Let q be convex univalent in Δ with $q(0) = 1$. Let*

$$\psi := \frac{\{D^\delta f\}^2}{zD^{\delta+1}f}\left[1+\alpha(1-a) - \alpha(a+1)\frac{D^{\delta+2}f}{D^{\delta+1}f}\right] + \frac{2a\alpha}{z}D^\delta f.$$

and $\chi := q + \alpha z q'$. Let $f \in \mathscr{A}$ and $\frac{\{D^\delta f\}^2}{zD^{\delta+1}f} \in \mathscr{H}[1,1] \cap \mathscr{Q}$ and ψ is univalent in Δ.

(i) *If $\psi \prec \chi$ then*

$$\frac{\{D^\delta f\}^2}{zD^{\delta+1}f} \prec q$$

where q is the best dominant.

(ii) *If $\chi \prec \psi$ then*

$$q \prec \frac{\{D^\delta f\}^2}{zD^{\delta+1}f}$$

where q is the best subordinant.

Theorem 4.3. *Let α, β and γ be complex numbers with $\gamma \neq 0$. Let q be a convex univalent in Δ with $q(0) = 1$ and $\frac{\gamma z q'}{q}$ is starlike univalent in Δ. Let*

$$\psi_4 := \alpha + \gamma(1-a) + \frac{\beta\{L(a,c)f\}^2}{zL(a+1,c)f} + \frac{2a\gamma L(a+1,c)f}{L(a,c)f} - \frac{\gamma(a+1)L(a+2,c)f}{L(a+1,c)f}$$

and $\chi_4 := \alpha + \beta q + \frac{\gamma z q'}{q}$. Let $f \in \mathscr{A}$ and $\frac{\{L(a,c)f\}^2}{zL(a+1,c)f} \in \mathscr{H}[1,1] \cap \mathscr{Q}$ and ψ_4 is univalent in Δ.

(i) *If q satisfies (3.4) then*

$$\psi_4 \prec \chi_4 \Rightarrow \frac{\{L(a,c)f\}^2}{zL(a+1,c)f} \prec q$$

where q is the best dominant.

(ii) *If q satisfies* (3.5) *then*

$$\chi_4 \prec \psi_4 \Rightarrow q \prec \frac{\{L(a,c)f\}^2}{zL(a+1,c)f}$$

where q is the best subordinant.

Proof. The proof of the Theorem 4.3 is similar to that of Theorem 4.1, where $\theta(w) = \alpha + \beta w$ and $\phi(w) = \frac{\gamma}{w}$.

By taking $a = \delta + 1$ and $c = 1$ we get the following result involving Ruscheweyh derivative.

Corollary 4.4. *Let α, β and γ be complex numbers with $\gamma \neq 0$. Let q be a convex univalent in Δ with $q(0) = 1$ and $\frac{\gamma z q'}{q}$ is starlike univalent in Δ. Let*

$$\psi_5 := \alpha + \gamma(1-a) + \frac{\beta\{D^\delta f\}^2}{zD^{\delta+1}f} + \frac{2a\gamma D^{\delta+1}f}{D^\delta f} - \frac{\gamma(a+1)D^{\delta+2}f}{D^{\delta+1}f}$$

and $\chi_5 := \alpha + \beta q + \frac{\gamma z q'}{q}$. Let $f \in \mathscr{A}$ and $\frac{\{D^\delta f\}^2}{zD^{\delta+1}f} \in \mathscr{H}[1,1] \cap \mathscr{Q}$ and ψ_5 is univalent in Δ.

(i) *If q satisfies* (3.4) *then*

$$\psi_5 \prec \chi_5 \Rightarrow \frac{\{D^\delta f\}^2}{zD^{\delta+1}f} \prec q$$

where q is the best dominant.

(ii) *If q satisfies* (3.5) *then*

$$\chi_5 \prec \psi_5 \Rightarrow q \prec \frac{\{D^\delta f\}^2}{zD^{\delta+1}f}$$

where q is the best subordinant.

5. Application to Sălăgean Derivative Operator

Theorem 5.1. *Let $0 \neq \alpha \in \mathbb{C}$ and $\Re\left\{\frac{1}{\alpha}\right\} > 0$. Let q be convex univalent in Δ with $q(0) = 1$. Let*

$$\psi_6 := 2\alpha\frac{\mathscr{D}^m f}{z} + \frac{\{\mathscr{D}^m f\}^2}{z\mathscr{D}^{m+1}f}\left[1 - \alpha - \frac{\alpha\mathscr{D}^{m+2}f}{\mathscr{D}^{m+1}f}\right]$$

and $\chi_6 := q + \alpha z q'$. Let $f \in \mathscr{A}$ and $\frac{\{\mathscr{D}^m f\}^2}{z\mathscr{D}^{m+1}f} \in \mathscr{H}[1,1] \cap \mathscr{Q}$ and ψ_6 is univalent in Δ.

(i) *If $\psi_6 \prec \chi_6$ then*

$$\frac{\{\mathscr{D}^m f\}^2}{z\mathscr{D}^{m+1}f} \prec q$$

where q is the best dominant.

(ii) *If $\chi_6 \prec \psi_6$ then*

$$q \prec \frac{\{\mathscr{D}^m f\}^2}{z\mathscr{D}^{m+1}f}$$

where q is the best subordinant.

Proof. Define the function p by

$$p := \frac{\{\mathscr{D}^m f\}^2}{z\mathscr{D}^{m+1} f}. \tag{5.1}$$

A simple computation using (5.1) shows that

$$\frac{zp'}{p} = \frac{2z(\mathscr{D}^m f)'}{\mathscr{D}^m f} - 1 - \frac{z(\mathscr{D}^{m+1} f)'}{\mathscr{D}^{m+1} f}. \tag{5.2}$$

Using the identity

$$z(\mathscr{D}^m f)' = \mathscr{D}^{m+1} f,$$

in (5.2) we obtain

$$\frac{zp'}{p} = \frac{2\mathscr{D}^{m+1} f}{\mathscr{D}^m f} - 1 - \frac{\mathscr{D}^{m+2} f}{\mathscr{D}^{m+1} f}.$$

Note that

$$\begin{aligned}\psi_6 &:= 2\alpha\frac{\mathscr{D}^m f}{z} + \frac{\{\mathscr{D}^m f\}^2}{z\mathscr{D}^{m+1} f}\left[1 - \alpha - \frac{\alpha\mathscr{D}^{m+2} f}{\mathscr{D}^{m+1} f}\right] \\ &= p + \alpha z p'\end{aligned}$$

and this can be written as (2.1) when $\theta(w) := w$ and $\phi(w) := \alpha$. Now the result (i) follows as an application of Theorem 2.1. A similar exercise using Theorem (2.2) will give the result(ii).

Theorem 5.2. *Let α, β and γ be complex numbers and $\gamma \neq 0$. Let q be a convex univalent in Δ with $q(0) = 1$ and $\frac{\gamma z q'}{q}$ is starlike univalent in Δ. Let*

$$\psi_7 := \alpha - \gamma + \frac{2\gamma\mathscr{D}^{m-1} f}{\mathscr{D}^m f} - \frac{\mathscr{D}^{m+2} f}{\mathscr{D}^{m+1} f} + \frac{\beta\{\mathscr{D}^m f\}^2}{z\mathscr{D}^{m+1} f}$$

and $\chi_7 := \alpha + \beta q + \frac{\gamma z q'}{q}$. Let $f \in \mathscr{A}$ and $\frac{\{\mathscr{D}^m f\}^2}{z\mathscr{D}^{m+1} f} \in \mathscr{H}[1,1] \cap \mathscr{Q}$ and ψ_7 is univalent in Δ.

(i) *If q satisfies (3.4), then*

$$\psi_7 \prec \chi_7 \Rightarrow \frac{\{\mathscr{D}^m f\}^2}{z\mathscr{D}^{m+1} f} \prec q$$

where q is the best dominant.

(ii) *If q satisfies (3.5), then*

$$\chi_7 \prec \psi_7 \Rightarrow q \prec \frac{\{\mathscr{D}^m f\}^2}{z\mathscr{D}^{m+1} f}$$

where q is the best subordinant.

Proof. The proof follows as an application of Theorem 2.1 and Theorem 2.2 with $\theta(w) = \alpha + \beta w$ and $\phi(w) = \frac{\gamma}{w}$.

Sandwich results for the Theorems 3.3–5.2 can be obtained by a similar exercise as we have obtained the sandwich result(Theorem 3.2) of Theorem 3.1, however we omit the details of the proof.

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