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ON THE STUDY OF CONJUGATE SERIES OF A FOURIER SERIES BY K^λ-SUMMABILITY METHODS

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Abstract. Vuĉkoviĉ (1965) and Kathal (1969) have studied the K^λ-summability of Fourier series. In this paper, generalizing an earlier result of Kathal, a theorem on K^λ-summability of conjugate series of a Fourier series has been established.

1. Introduction

The method K^λ-was first introduced by Karamata (1935), Lotosky (1963) reintroduced the special case $\lambda = 1$. Only after the paper of Agnew (1957), an intensive study of these and similar method took place. Vuĉkoviĉ (1965) applied this method for summability of Fourier series, Kathal (1969) extended Vuĉkoviĉ result. Working in the same direction Ojha (1982), Tripathi and Lal (1984), Lal (1996), Lal and Pratap (1999) have studied K^λ-summability of Fourier series under different conditions. But till now nothing seems to have been done so for on the study of conjugte series of a Fourier series by K^λ-summability method. In an attempt to make an advance study in this direction, in this paper, a new theorem on K^λ-summability of conjugate series of a Fourier series has been established under very general conditions.

2. Definitions and Notations

Let us define, for $n = 0, 1, 2, 3, \dots$, the numbers $\begin{bmatrix} n \\ m \end{bmatrix}$, for $0 \le m \le n$, by

$$\prod_{v=0}^{n-1}(x+v) = \sum_{k=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} x^m \tag{2.1}$$

where $\prod_{v=0}^{n-1}(x+v) = \frac{\Gamma(x+v)}{\Gamma x} = x(x+1)(x+2)\cdots(x+n-1)$

The numbers $\begin{bmatrix} n \\ m \end{bmatrix}$ are known as the absolute values of Stirling numbers of the first kind.

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Let $\{S_n\}$ be the sequence of partial sums of an infinite series $\sum a_n$ and let us write.

$$S_n^\lambda = \frac{\Gamma\lambda}{\Gamma(\lambda+n)} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \lambda^m S_m \tag{2.2}$$

to denote the n^{th} K^λ-mean of order $\lambda > 0$. If $S_n^\lambda \to S$ as $n \to \infty$ where S, If a fixed finite quantity then the sequence $\{S_n\}$ or the series $\sum a_n$ is said to be summable by Karamata method K^λ of order $\lambda > 0$ to the sum S and we write.

$$S_n^\lambda \to S(K^\lambda) \quad \text{as} \quad n \to \infty \tag{2.3}$$

The method K^λ is regular for $\lambda > 0$ and this case will be supposed throught this paper.

Let $f(t)$ be the 2π-periodic and Lebesgue integrable function of t over the interval $(-\pi, \pi)$.

Let the Fourier series of function $f(t)$ be given by

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t) \tag{2.4}$$

and then

$$f(t) \sim \sum_{n=1}^{\infty}(a_n \sin nt - b_n \cos nt) = \sum_{n=0}^{\infty} B_n(t) \tag{2.5}$$

is known as conjugate series of Fourier series (2.4), we write

$$\begin{aligned}
\phi(t) &= f(x+t) + f(x-t) - 2f(x) \\
\psi(t) &= f(x+t) + f(x-t) \\
\phi(t) &= \int_0^t |\phi(u)| du \\
\psi(t) &= \int_0^t |\psi(u)| du \\
k_n(t) &= \frac{\Gamma\lambda \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}}{2\pi\Gamma(\lambda+m)} \frac{\lambda^m \cos(m+\frac{1}{2})t}{\sin(\frac{t}{2})} \\
\tau &= [1/t] = \text{Integral part of } 1/t
\end{aligned}$$

3. Known Theorem

Vuĉkoviĉ (1965) has establish the following theorem:

Theorem A. *If*

$$\phi(t) = o\left[\frac{1}{\log(1/t)}\right], \quad \textit{as} \quad (t \to +0) \tag{3.1}$$

then the Fourier series is summable $K^{\lambda}(\lambda>0)$ *to the sum* $f(x)$ *at the point* $t=x$*,*

Kathal (1969) prove the following theorem:

Theorem B. *If*

$$\phi(t)=\int_0^t|\phi(u)|du=o\left[\frac{t}{\log(1/t)}\right],\quad as\ (t\to+0) \tag{3.2}$$

then the Fourier series (2.4) *is summable* $K^{\lambda}(\lambda>0)$ *to the sum* $f(x)$ *at the point* $t=x$*,*

4. Main Theorem

Here in this paper, the above theorem has been generalized for conjugate series of a Fourier series in the following form:

Theorem. *Let* $\{p_n\}$ *be a sequence monotonic decreasing sequence of real constant such that*

$$p_n=\sum_{v=0}^{n}p_v\to\infty,\quad as\quad n\to\infty$$

If

$$\psi(t)=\int_0^t|\psi(u)|du=o\left[\frac{\alpha(\frac{1}{t})t}{P_t}\right],\quad as\quad t\to+0 \tag{4.1}$$

Provide $\alpha(t)$ *is a positive monotonic decreasing function of* t*, such that*

$$\alpha(n)\log n=O(P_n),\quad as\quad n\to\infty$$

then the conjugate series of Fourier series (2.5) *is* K^{λ} *summable to*

$$-\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cot(\frac{1}{2}t)dt$$

at the every point x *where this integral exists in Lebesgue sense.*

5. Proof of the Theorem

Let $S_m(x)$ denote the nth partial sum of series (2.5) at $t=x$, Then

$$\begin{aligned}\overline{S}_m(x)&=\sum_{k=1}^{m}(a_k\sin(kx)-b_k\cos(kx))\\&=\frac{1}{n}\sum_{k=1}^{m}(\sin(kx))\int_{-\pi}^{\pi}f(t)\cos(kt).dt-\cos(kx)\int_{-\pi}^{\pi}f(t)\sin(kt).dt\\&=-\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\left(\sum_{k=1}^{m}\sin k(t-x)\right)dt\end{aligned}$$

i.e.,

$$= -\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin\left(\frac{(m+1)(t-x)}{2}\right)\sin\left(\frac{m(t-x)}{2}\right)}{\sin\left(\frac{(t-x)}{2}\right)}.dt$$
$$= -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\frac{\cos(\frac{1}{2}t)-\cos(m+\frac{1}{2})t}{2\sin(\frac{1}{2}t)}dt$$
$$= -\frac{1}{2\pi}\int_{0}^{\pi}\{f(x+t)-f(x-t)\}\cot\frac{1}{2}t.dt$$
$$= +\frac{1}{2\pi}\int_{0}^{\pi}\{f(x+t)-f(x-t)\}\frac{\cos(m+\frac{1}{2})t}{\sin\frac{1}{2}t}.dt$$

Hence

$$\overline{S}_m(x) - \left(-\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cot\frac{1}{2}t.dt\right) = \int_0^{\pi}\frac{1}{2\pi}\psi(t)\frac{\cos(m+\frac{1}{2})t}{\sin(\frac{1}{2})t}.dt$$

Therefore,

$$\frac{\Gamma\lambda}{\Gamma(\lambda+n)}\sum_{m=0}^{n}\begin{bmatrix} n \\ m\end{bmatrix}\lambda^m\left\{S_m(x) - \left(-\frac{1}{2\pi}\int_0^{\pi}\psi(t)\frac{\cos(\frac{1}{2}t)}{\sin(\frac{1}{2}t)}\right)\right\}$$
$$= \frac{1}{2\pi}\int_0^{\pi}\psi(t)\frac{\Gamma\lambda}{\Gamma(\lambda+n)}\sum_{m=0}^{n}\begin{bmatrix} n \\ m\end{bmatrix}\lambda^m\frac{\cos(m+\frac{1}{2})t}{\sin(\frac{1}{2}t)}dt$$

i.e.,

$$S_n^{-\lambda}(x) - \left(-\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cot\frac{1}{2}t.dt\right) = \int_0^{\pi}\psi(t)K_n(t).dt$$
$$= \left[\left\{\int_0^{1/n} + \int_{1/n}^{\pi}\right\}\Big|\psi(t)\Big|\Big|K_n(t)\Big|.dt\right]$$
$$= I_1 + I_2, \quad \text{say} \tag{5.1}$$

Since the conjugate function exists therefore,

$$\frac{\Gamma\lambda}{\Gamma(\lambda+n)}\sum_{m=0}^{n}\begin{bmatrix} n \\ m\end{bmatrix}\frac{\lambda^m}{2\pi}\int_0^{1/n}\psi(t)\cot\frac{1}{2}t.dt = o(1) \quad \text{as} \quad n\to\infty$$

Hence,

$$\frac{\Gamma\lambda}{\Gamma(\lambda+n)}\sum_{m=0}^{n}\begin{bmatrix} n \\ m\end{bmatrix}\lambda^m\cdot\frac{1}{2\pi}\int_0^{1/n}\psi(t)\frac{1}{2}t.dt - I_1$$
$$= \frac{1}{2\pi}\int_0^{1/n}\psi(t)\left[\cot\frac{1}{2}t - \frac{\Gamma\lambda}{\Gamma\lambda+n}\sum_{m=0}^{n}\begin{bmatrix} n \\ m\end{bmatrix}\lambda^m\frac{\cos\left(m+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right]dt$$

$$= \frac{1}{2\pi}\int_0^{1/n}\psi(t)\left[\frac{\Gamma\lambda}{\Gamma\lambda+n}\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\lambda^m.\frac{\left(\cos(\frac{t}{2})-\cos(m+\frac{1}{2})t\right)}{\sin\frac{t}{2}}\right].$$

$$= \frac{1}{2\pi}\int_0^{1/n}\psi(t)\frac{\Gamma\lambda}{\Gamma\lambda+m}\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\lambda^m\left[\sum_{p=0}^{m}2\sin pt\right]dt$$

$$= \frac{1}{2\pi}\int_0^{1/n}\psi(t)\frac{\Gamma\lambda}{\Gamma\lambda+m}\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\lambda^m m.dt$$

i.e.,

$$\le O(n)\int_0^{1/n}|\psi(t)|dt$$

$$= O\int_0^{1/n}|\psi(t)|dt$$

$$= O(n).o\left(\frac{1}{n}\frac{\alpha(n)}{P_n}\right)$$

$$= .o\left(\frac{\alpha(n)}{P_n}\right)$$

$$= o(1) \quad \text{as} \quad n\to\infty \quad \text{, by the hypothesis of the theorem.}$$

Therefore $$I_1 = o(1) \quad \text{as} \quad n\to\infty \tag{5.2}$$

Now by (2.1)

$$K_n(t) = \frac{\mathrm{Re}\left\{e^{it/2}\frac{\Gamma(\lambda e^{it}+n)}{\Gamma(\lambda e^{it})}\right\}}{\Gamma(\lambda+n).\sin(\frac{t}{2})}$$

$$= o\left|\frac{\mathrm{Re}\left\{e^{it/2}\frac{\Gamma(\lambda e^{it}+n)}{\Gamma(\lambda e^{it})}\right\}}{\Gamma(\lambda+n).\sin(\frac{t}{2})}\right|$$

$$= o\left[\frac{\mathrm{Re}\Gamma(\lambda e^{it}+n)}{\Gamma(\lambda+n)\sin(\frac{t}{2})}\right] + o\left[\frac{\mathrm{lm}\Gamma(\lambda e^{it}+n)}{\Gamma(\lambda+n)}\right]$$

$$= o\left[\frac{\Gamma(\lambda\cos t+n)}{\Gamma(\lambda+n)\sin(\frac{t}{2})}\right] + o\left[\frac{\Gamma(\lambda\cos t+n)}{\Gamma(\lambda+n)}\cdot\frac{\mathrm{lm}\Gamma(\lambda e^{it}+n)}{\Gamma(\lambda\cos t+n)}\right]$$

For $1/n < t < \pi$,

$$K_n(t) = \left(\frac{1}{\Gamma(\lambda+n)\sin(\frac{1}{2n})}\right) = o(1) \quad \text{as} \quad n\to\infty$$

Lastly, let us consider I_2
since $\psi(t)$ is bounded for $1/n < t < \pi$,
therefore,

$$I_2 = o(1)\int_{1/n}^{\pi}|\psi(t)|.dt = o(1), \quad \text{as} \quad n\to\infty \tag{5.3}$$

From (5.1), (5.2) and (5.3) we get,

$$S_n^{-\lambda}(x) - \left(-\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cot\frac{t}{2}.dt\right) = o(1) \quad \text{as} \quad n \to \infty,$$

This is completes proof of theorem.

6. Corollaries

Following corollaries can be derived from our theorem:

Corollary 6.1. *If* $\psi(t) = \int_0^t |\psi(u)|du = o(t)$, *as* $t \to +0$ *then the conjugate series of Fourier series i.e.* (2.5) *is* K^{λ}*-summable to,*

$$-\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cot\left(\frac{1}{2}t\right)dt$$

Corollary 6.2. *If* $\psi(t) = o\left(\frac{t}{\log\frac{1}{t}}\right)$, *as* $t \to +0$ *then the conjugate series of Fourier series i.e.* (2.5) *is* K^{λ}*-summable to.*

$$-\frac{1}{2\pi}\int_0^{\pi}\psi(t)\cot\left(\frac{1}{2}t\right)dt$$

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