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SOME NEW INVERSE TYPE HILBERT-PACHPATTE INEQUALITIES

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Abstract. In this paper, some new inverse type Hilbert-Pachpatte inequalities are given.

1. Introduction

In [1, p.253] the following extension of Hilber's double-series theorem is given.

Theorem A. *Let $p>1$, $q>1$, $1/p+1/q\geq 1$, $0<\lambda=2-1/p-1/q=1/p+1/q\geq 1$. Then*

$$\sum_1^\infty\sum_1^\infty \frac{a_m b_n}{(m+n)^\lambda}\leq K\Big(\sum_1^\infty a_m^p\Big)^{1/p}\Big(\sum_1^\infty b_n^q\Big)^{1/q}$$

where $K=K(p,q)$ depends on p and q only.

The following intergral analogue of Theorem A is also given in [1, p.254].

Theorem B. *Under the same conditions as in Theorem A we have*

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda}dxdy\leq K\left(\int_0^\infty f^p dx\right)^{1/p}\left(\int_0^\infty g^q dy\right)^{1/q}$$

where $K=K(p,q)$ depends on p and q only.

In [2] some new inequalities similar to the inequalities given in Theorem A and Theorem B were established.

Theorem C. *Let $p>1, 1/p+1/q=1$. Let $a(s):N_m\to R, b(t):N_n\to R$, and $a(0)=b(0)=0$. Then*

$$\sum_{s=1}^m\sum_{t=1}^n \frac{|a(s)||b(t)|}{qs^{p-1}+pt^{q-1}}$$
$$\leq M(p,q,m,n)\Big(\sum_{s=1}^m (m-s+1)\mid \nabla a(s)\mid^p\Big)^{1/p}\Big(\sum_{t=1}^n (n-t+1)\mid \nabla b(t)\mid^q\Big)^{1/q}$$

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where $M(p,q,m,n) = \frac{1}{pq} m^{(p-1)/p} n^{(q-1)/q}$.

Theorem D. *Let* $p > 1, 1/p + 1/q = 1$ *Let* $f(s)$ *and* $g(t)$ *be real-valued continuous functions defined on* I_x *and* I_y*, respectively, and let* $f(0) = g(0) = 0$*. Then*

$$\int_0^x \int_0^y \frac{| f(s) || g(t) |}{qs^{p-1} + pt^{q-1}} dsdt \leq K(p,q,x,y)\Big(\int_0^x (x-s) \mid f'(s) \mid^p ds\Big)^{1/p} \Big(\int_0^y (y-t) \mid g'(t) \mid^q dt\Big)^{1/q}$$

where $K(p,q,x,y) = \frac{1}{pq} x^{(p-1)/p} y^{(q-1)/q}$.

Theorem E. *Let* $p > 1, 1/p+1/q = 1$*. Let* $a(s,t) : N_x \times N_y \to R, b(k,r) : N_z \times N_w \to R$*, and* $a(0,t) = b(0,t) = 0$*,* $b(s,0) = b(s,0) = 0$ *Then*

$$\sum_{s=1}^{x}\sum_{t=1}^{y}\Big(\sum_{k=1}^{z}\sum_{r=1}^{w} \frac{| a(s,t) || b(k,r) |}{q(st)^{p-1} + p(kr)^{q-1}}\Big) \leq L(p,q,x,y,z,w)\Big(\sum_{s=1}^{x}\sum_{t=1}^{y}(x-s+1)(y-t+1) \mid \nabla_2\nabla_1 a(s,t) \mid^p \Big)^{1/p} \times\Big(\sum_{k=1}^{z}\sum_{r=1}^{w}(z-k+1)(w-r+1) \mid \nabla_2\nabla_1 b(k,r) \mid^q \Big)^{1/q}$$

where $L(p,q,x,y,z,w) = \frac{1}{pq}(xy)^{(p-1)/p}(zw)^{(q-1)/q}$.

Theorem F. *Let* $p > 1, 1/p+1/q = 1$*. Let* $f(s,t)$ *and* $g(k,r)$ *be real-valued continuous functions defined on* $I_x \times I_y$ *and* $I_z \times I_w$*, respectively, and let* $f(0,t) = g(0,t) = 0$*,* $f(s,0) = g(s,0) = 0$*. Then*

$$\int_0^x \int_0^y \left(\int_0^z \int_0^w \frac{| f(s,t) || g(k,r) |}{q(st)^{p-1} + p(kr)^{q-1}} dkdr\right) dsdt \leq C(p,q,x,y,z,w) \left(\int_0^x \int_0^y (x-s)(y-t) \mid D_2D_1 f(s,t) \mid^p dsdt\right)^{1/p} \times \left(\int_0^z \int_0^w (z-k)(w-r) \mid D_2D_1 g(k,r) \mid^q dkdr\right)^{1/q}$$

where $C(p,q,x,y,z,w) = \frac{1}{pq}(xy)^{(p-1)/p}(zw)^{(q-1)/q}$.

In this paper,we show some new inverse type inequalities on above Theorem C, D, E, F.

2. Statement of Results

In what follows we denote by R the set of real numbers. Let $N = \{1, 2, \ldots\}$, $N_0 = \{0, 1, 2, \ldots\}$, $N_\lambda = \{0, 1, 2, \ldots, \lambda\}$, $\lambda \in N$. We define the operator ∇ by $\nabla u(t) = u(t) -$

$u(t-1)$ for any function u defined on N_0. For a function $v(s,t) : N_0 \times N_0 \to R$, we define the operators $\nabla_1 v(s,t) = v(s,t) - v(s-1,t)$, $\nabla_2 v(s,t) = v(s,t) - v(s,t-1)$, and $\nabla_2\nabla_1 v(s,t) = \nabla_2(\nabla_1 v(s,t)) = \nabla_1(\nabla_2 v(s,t))$. Let $I = [0,\infty)$, $I_0 = (0,\infty)$, $I_\beta = [0,\beta)$, $\beta \in I_0$, denote the subintervals of R. For any function $u : I \to R$, we denote by u' the derivatives of u, and for the function $u(s,t) : I \times I \to R$, we denote the partial derivatives $(\partial/\partial s)u(s,t)$, $(\partial/\partial t)u(s,t)$, and $(\partial^2/\partial s\partial t)u(s,t)$ by $D_1u(s,t)$, $D_2u(s,t)$, and $D_2D_1u(s,t) = D_1D_2u(s,t)$, respectively.

Our main result is given in the following theorem.

Theorem 1. *Let $0 < p < 1$ or $p < 0, 1/p + 1/q = 1$. Let $a(s) : N_m \to R$, $b(t) : N_n \to R$, and $\nabla a(s) > 0, \nabla b(t) > 0$, $a(0) = b(0) = 0$. Then*

$$\sum_{s=1}^{m}\sum_{t=1}^{n}\frac{a(s)b(t)}{s^{1/p}t^{1/q}} \geq m^{1/q}n^{1/p}\left(\sum_{s=1}^{m}(m-s+1)(\nabla a(s))^p\right)^{1/p}\left(\sum_{t=1}^{n}(n-t+1)(\nabla b(t))^q\right)^{1/q} \quad (1)$$

for $m, n \in N$.

Theorem 2. *Let $0 < p < 1$ or $p < 0$, $1/p + 1/q = 1$. Let $f(s)$ and $g(t)$ be real-valued continuous functions defined on I_x and I_y, respectively, $f'(s) > 0$ and $g'(t) > 0$, and let $f(0) = g(0) = 0$. Then*

$$\int_0^x\int_0^y \frac{f(s)g(t)}{s^{1/p}t^{1/q}}dsdt \geq x^{1/q}y^{1/p}\left(\int_0^x (x-s)f'(s)^p ds\right)^{1/p}\left(\int_0^y (y-t)g'(t)^q dt\right)^{1/q} \quad (2)$$

for x, $y \in I_0$.

Theorem 3. *Let $0 < p < 1$ or $p < 0$, $1/p + 1/q = 1$. Let $a(s,t) : N_x \times N_y \to R$, $b(k,r) : N_z \times N_w \to R$, and $\nabla_1 a(s,t) > 0$, $\nabla_2 a(s,t) > 0$, $\nabla_1 b(s,t) > 0$, $\nabla_2 b(s,t) > 0$, $a(0,t) = b(0,t) = 0$, $a(s,0) = b(s,0) = 0$ Then*

$$\begin{aligned}\sum_{s=1}^{x}\sum_{t=1}^{y}&\left(\sum_{k=1}^{z}\sum_{r=1}^{w}\frac{a(s,t)b(k,r)}{(st)^{1/q}(kr)^{1/p}}\right)\\ &\geq (xy)^{1/q}(zw)^{1/p}\left(\sum_{s=1}^{x}\sum_{t=1}^{y}(x-s+1)(y-t+1)(\nabla_2\nabla_1 a(s,t)^p\right)^{1/p}\\ &\quad\times\left(\sum_{k=1}^{z}\sum_{r=1}^{w}(z-k+1)(w-r+1)(\nabla_2\nabla_1 b(k,r))^q\right)^{1/q} \quad (3)\end{aligned}$$

for x, y, z, $w \in N$.

Theorem 4. *Let $0 < p < 1$ or $p < 0, 1/p + 1/q = 1$. Let $f(s,t)$ and $g(k,r)$ be real-valued continuous functions defined on $I_x \times I_y$ and $I_z \times I_w$, respectively, and let $D_1 f(s,t) > 0$, $D_2 f(s,t) > 0)$, $D_1 g(s,t) > 0$, $D_2 g(s,t) > 0$, $f(0,t) = g(0,t) = 0$,*

$f(s,0)=g(s,0)=0$. Then

$$\int_0^x\int_0^y\Big(\int_0^z\int_0^w \frac{f(s,t)g(k,r)}{(st)^{1/q}(kr)^{1/p}}dkdr\Big)dsdt$$
$$\geq (xy)^{1/q}(zw)^{1/p}\Big(\int_0^x\int_0^y (x-s)(y-t)(D_2D_1f(s,t))^p dsdt\Big)^{1/p}$$
$$\times\Big(\int_0^z\int_0^w (z-k)(w-r)(D_2D_1g(k,r))^q dkdr\Big)^{1/q} \tag{4}$$

for $x,\ y,\ z,\ w\in I_0$.

3. Proofs of Theorems 1 and 2

From the hypotheses of Theorem 1, it is easy to note that

$$a(s)=\sum_{\tau=1}^{s}\nabla a(\tau),\quad b(t)=\sum_{\delta=1}^{t}\nabla b(\delta) \tag{5}$$

From (5) and in view of the special case of the Hölder inequality, we have

$$a(s)\geq s^{1/q}\Big(\sum_{\tau=1}^{s}(\nabla a(\tau))^p\Big)^{1/p}, \tag{6}$$
$$b(t)\geq t^{1/p}\Big(\sum_{\delta=1}^{t}(\nabla b(\delta))^q\Big)^{1/q} \tag{7}$$

for $s\in N_m$, $t\in N_n$. By (6) and (7) it follows that

$$\frac{a(s)b(t)}{s^{1/q}t^{1/p}}\geq\Big(\sum_{\tau=1}^{s}(\nabla a(\tau))^p\Big)^{1/p}\Big(\sum_{\delta=1}^{t}(\nabla b(\delta))^q\Big)^{1/q} \tag{8}$$

for $s\in N_m$, $t\in N_n$. Taking the sumon both sides of (8) over t from 1 to n first and taking the sum on both sides of the resulting inequality over s from 1 to m and using the special case of the Hölder inequality, we obtain

$$\sum_{s=1}^{m}\sum_{t=1}^{n}\frac{a(s)b(t)}{s^{1/q}t^{1/p}}\geq\Big\{\sum_{s=1}^{m}\Big(\sum_{\tau=1}^{s}(\nabla a(\tau))^p\Big)^{1/p}\Big\}\Big\{\sum_{t=1}^{n}\Big(\sum_{\delta=1}^{t}(\nabla b(\delta))^q\Big)^{1/q}\Big\}$$
$$\geq m^{1/q}\Big\{\sum_{s=1}^{m}\Big(\sum_{\tau=1}^{s}(\nabla a(\tau))^p\Big)\Big\}^{1/p} n^{1/p}\Big\{\sum_{t=1}^{n}\Big(\sum_{\delta=1}^{t}(\nabla b(\delta))^q\Big)\Big\}^{1/q}$$
$$=m^{1/q}n^{1/p}\Big(\sum_{s=1}^{m}(m-s+1)(\nabla a(s))^p\Big)^{1/p}\Big(\sum_{t=1}^{n}(n-t+1)(\nabla b(t))^q\Big)^{1/q} \tag{9}$$

The proof is complete.

From the hypotheses of Theorem 2, it is easy to note that

$$f(s)=\int_0^s f'(\tau)d\tau,\quad g(t)=\int_0^t g'(\delta)d\delta \tag{10}$$

From (10) and in view of the special case of the Hölder integeal inequality, we have

$$f(s)\geq s^{1/q}\left(\int_0^s (f'(\tau))^p d\tau\right)^{1/p}, \tag{11}$$

$$g(t)\geq t^{1/p}\left(\int_0^t (g'(\delta))^q d\delta\right)^{1/q} \tag{12}$$

for $s\in I_x$, $t\in I_y$. By (11) and (12) it follows that

$$\frac{f(s)g(t)}{s^{1/q}t^{1/p}}\geq\left(\int_0^s (f'(\tau))^p d\tau\right)^{1/p}\left(\int_0^t (g'(\delta))^q d\delta\right)^{1/q} \tag{13}$$

for $s\in I_x$, $t\in I_y$. Integrating over t from 0 to y first and integrating the resulting inequality over s from 0 to x and using the special case of the Hölder integral inequality, we obtain

$$\begin{aligned}\int_0^x\int_0^y \frac{f(s)g(t)}{s^{1/q}t^{1/p}}dsdt &\geq\left\{\int_0^x\left(\int_0^s (f'(\tau))^p d\tau\right)^{1/p}ds\right\}\left\{\int_0^y\left(\int_0^t (g'(\delta))^q d\delta\right)^{1/q}dt\right\}\\ &\geq x^{1/q}\left\{\int_0^x\left(\int_0^s (f'(\tau))^p d\tau\right)ds\right\}^{1/p}y^{1/p}\left\{\int_0^y\left(\int_0^t (g'(\delta))^q d\delta\right)ds\right\}^{1/q}\\ &= x^{1/q}y^{1/p}\left(\int_0^x (x-s)(f'(s))^p ds\right)^{1/p}\left(\int_0^y (y-t)(g'(t))^q dt\right)^{1/q}\end{aligned} \tag{14}$$

The proof is complete.

4. Proofs of Theorems 3 and 4

From the hypotheses of Theorem 3, it is easy to observe that

$$a(s,t)=\sum_{\xi=1}^{s}\sum_{\eta=1}^{t}\nabla_2\nabla_1 a(\xi,\eta),\quad b(k,r)=\sum_{\delta=1}^{k}\sum_{\tau=1}^{r}\nabla_2\nabla_1 b(\delta,\tau) \tag{15}$$

for $(s,t)\in N_x\times N_y$, $(k,r)\in N_z\times N_w$. From (15) and in view of the special case of the Hölder inequality, we have

$$a(s,t)\geq (st)^{1/q}\left(\sum_{\xi=1}^{s}\sum_{\eta=1}^{t}(\nabla_2\nabla_1 a(\xi,\eta))^p\right)^{1/p}, \tag{16}$$

$$b(k,r) \geq (kr)^{1/p}\left(\sum_{\delta=1}^{k}\sum_{\tau=1}^{r}(\nabla_2\nabla_1 b(\delta,\tau))^q\right)^{1/q} \tag{17}$$

for $(s,t) \in N_x \times N_y$, $(k,r) \in N_z \times N_w$. By (16) and (17) it follows that

$$\frac{a(s,t)b(k,r)}{(st)^{1/q}(kr)^{1/p}} \geq \left(\sum_{\xi=1}^{s}\sum_{\eta=1}^{t}(\nabla_2\nabla_1 a(\xi,\eta))^p\right)^{1/p}\left(\sum_{\delta=1}^{k}\sum_{\tau=1}^{r}(\nabla_2\nabla_1 b(\delta,\tau))^q\right)^{1/q} \tag{18}$$

for $(s,t) \in N_x \times N_y$, $(k,r) \in N_z \times N_w$. Taking the sum on both sides of (18) first over r from 1 to w and then over k from 1 to z and taking the sum on both sides of the resulting inequality first over t from 1 to y and then over s from 1 to x and using the special case of the Hölder inequality and interchanging the order of the summations, we obtain

$$\begin{aligned}
&\sum_{s=1}^{x}\sum_{t=1}^{y}\left(\sum_{k=1}^{z}\sum_{r=1}^{w}\frac{a(s,t)b(k,r)}{(st)^{1/q}(kr)^{1/p}}\right)\\
&\geq \left\{\sum_{s=1}^{x}\sum_{t=1}^{y}\left(\sum_{\xi=1}^{s}\sum_{\eta=1}^{t}(\nabla_2\nabla_1 a(\xi,\eta))^p\right)^{1/p}\right\}\left\{\sum_{k=1}^{z}\sum_{r=1}^{w}\left(\sum_{\delta=1}^{k}\sum_{\tau=1}^{r}(\nabla_2\nabla_1 b(\delta,\tau))^q\right)^{1/q}\right\}\\
&\geq (xy)^{1/q}\left\{\sum_{s=1}^{x}\sum_{t=1}^{y}\left(\sum_{\xi=1}^{s}\sum_{\eta=1}^{t}(\nabla_2\nabla_1 a(\xi,\eta))^p\right)\right\}^{1/p}\\
&\quad\times(zw)^{1/p}\left\{\sum_{k=1}^{z}\sum_{r=1}^{w}\left(\sum_{\delta=1}^{k}\sum_{\tau=1}^{r}(\nabla_2\nabla_1 b(\delta,\tau))^q\right)\right\}^{1/q}\\
&= (xy)^{1/q}(zw)^{1/p}\left(\sum_{s=1}^{x}\sum_{t=1}^{y}(x-s+1)(y-t+1)(\nabla_2\nabla_1 a(s,t))^p\right)^{1/p}\\
&\quad\times\left(\sum_{k=1}^{z}\sum_{r=1}^{w}(z-k+1)(w-r+1)(\nabla_2\nabla_1 b(k,r))^q\right)^{1/q}
\end{aligned} \tag{19}$$

The proof is complete.

From the hypotheses of Theorem 4,we have the following identities

$$f(s,t) = \int_0^s\int_0^t D_2D_1 f(\xi,\eta)d\xi d\eta,\ g(k,r) = \int_0^k\int_0^r D_2D_1 g(\delta,\tau)d\delta d\tau \tag{20}$$

for $(s,t) \in I_s \times I_t$, $(k,r) \in I_z \times I_w$. From (20) and in view of the special case of the Hölder integeal inequality, we have

$$f(s,t) \geq (st)^{1/q}\left(\int_0^s\int_0^t (D_2D_1 f(\xi,\tau))^p d\tau\right)^{1/p}, \tag{21}$$

$$g(k,r) \geq (kr)^{1/p}\left(\int_0^k\int_0^r (D_2D_1 g(\delta,\tau))^q d\delta d\tau\right)^{1/q} \tag{22}$$

for $(s,t) \in I_s \times I_t$, $(k,r) \in I_z \times I_w$. By (21) and (22) it follows that

$$\frac{f(s,t)g(k,r)}{(st)^{1/q}(kr)^{1/p}} \geq \left(\int_0^s \int_0^t (D_2D_1f(\xi,\eta))^p d\xi d\eta\right)^{1/p} \left(\int_0^k \int_0^r (D_2D_1g(\delta,\tau))^q d\delta d\tau\right)^{1/q} \tag{23}$$

for $(s,t) \in I_s \times I_t$, $(k,r) \in I_z \times I_w$. Integrating both sides of (23) first over r from 0 to w and then over k from 0 to z and integrating both sides of the resulting inequality over t from 0 to y and over s from 0 to x and using the special case of the Hölder integral inequality and Fubini's theorem, we obtain

$$\begin{aligned}
&\int_0^x \int_0^y \left(\int_0^z \int_0^w \frac{f(s,t)g(k,r)}{(st)^{1/q}(kr)^{1/p}} dkdr\right) dsdt \\
&\quad \geq \left\{\int_0^x \int_0^y \left(\int_0^s \int_0^t (D_2D_1f(\xi,\eta))^p d\xi d\eta\right)^{1/p} dsdt\right\} \\
&\qquad \times \left\{\int_0^z \int_0^w \left(\int_0^k \int_0^r (D_2D_1g(\delta,\tau))^q d\delta d\tau\right)^{1/q} dkdr\right\} \\
&\quad \geq (xy)^{1/q} \left\{\int_0^x \int_0^y \left(\int_0^s \int_0^t (D_2D_1f(\xi,\eta))^p d\xi d\eta\right) dsdt\right\}^{1/p} \\
&\qquad \times (zw)^{1/p} \left\{\int_0^z \int_0^w \left(\int_0^k \int_0^r (D_2D_1g(\delta,\tau))^q d\delta d\tau\right) dkdr\right\}^{1/q} \\
&\quad = (xy)^{1/q}(zw)^{1/p} \left(\int_0^x \int_0^y (x-s)(y-t)(D_2D_1f(s,t))^p dsdt\right)^{1/p} \\
&\qquad \times \left(\int_0^z \int_0^w (z-k)(w-r)(D_2D_1g(k,r))^q dkdr\right)^{1/q}
\end{aligned} \tag{24}$$

The proof is complete.

References

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