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A NOTE ON VARIANTS OF CERTAIN INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. The present note deals with some variants of the integral inequalities recently established by Dragomir for convex functions.

1. Introduction

Recently, in [1-3] S. S. Dragomir has established some integral inequalities related to the well known Hadamard's inequalities:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}, \tag{1}$$

which hold for all convex functions $f : [a,b] \to R$. The main results proved in [2] can be stated as follows:

Theorem D. *Let $f : [a,b] \to R$ be a differentiable convex function. Then*

$$\begin{aligned} 0 &\le \frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{b-a}\int_a^b f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right)dx \\ &\le (1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right], \end{aligned} \tag{2}$$

$$\begin{aligned} 0 &\le \frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)y)dxdy \\ &\le t\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right], \end{aligned} \tag{3}$$

for all t in $[0,1]$.

The inequalities established in (2) and (3) are interesting in their own right and the inequality (3), in one sense, an improvement of the right inequality in (1), see [2]. The

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main purpose of this note is to point out some variants of the inequalities obtained in (2) and (3) which we believe are of independent interest. The analysis used in the proof is fairly elementary and based on some basic properties of convex functions.

2. Statement of Results

Let $f : [a, b] \to R$ be a convex function. For $x,\ y$ two elements in $[a, b]$, we shall define the following mapping (see [3]):

$$F(x, y)(t) = \frac{1}{2}[f(tx + (1-t)y) + f((1-t)x + ty)]. \tag{4}$$

Furthermore, for a given convex function $f : [a, b] \to R$, we shall define the mappings G, $H : [0, 1]$ into R by

$$G(t) = \frac{1}{b-a}\int_a^b F\left(x, \frac{a+b}{2}\right)(t)dx, \tag{5}$$

$$H(t) = \frac{1}{(b-a)^2}\int_a^b \int_a^b F(x, y)(t)dxdy, \tag{6}$$

where F is defined as in (4). For similar definitions, see [1].

Our main results are given in the following theorems.

Theorem 1. *Let $f : [a, b]$ into R be a differentiable convex function and F be as defined in* (4)*. Then*

$$\int_a^b F\left(x, \frac{a+b}{2}\right)(t)dx \le \int_a^b f(x)dx, \tag{7}$$

$$\frac{3}{2}\frac{1}{(b-a)}\int_a^b f(x)dx \le \frac{1}{(b-a)}\int_a^b F\left(x, \frac{a+b}{2}\right)(t)dx + \frac{f(a)+f(b)}{4}, \tag{8}$$

for all t in $[0, 1]$.

Theorem 2. *Let f, F be as in Theorem* 1*. Then*

$$\frac{1}{(b-a)}\int_a^b \int_a^b F(x, y)(t)dxdy \le \int_a^b f(x)dx, \tag{9}$$

$$\frac{3}{2}\frac{1}{(b-a)}\int_a^b f(x)dx \le \frac{1}{(b-a)^2}\int_a^b \int_a^b F(x, y)(t)dxdy + \frac{f(a)+f(b)}{4}. \tag{10}$$

for all t in [0, 1].

Remark 1. It is easy to observe that, by rewritting the inequalities (7), (8) and (9), (10), we get

$$0 \le \frac{1}{(b-a)}\int_a^b f(x)dx - \frac{1}{(b-a)}\int_a^b F\left(x, \frac{a+b}{2}\right)(t)dx$$

$$\leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(x)dx\right], \tag{11}$$

$$0 \leq \frac{1}{(b-a)}\int_a^b f(x)dx - \frac{1}{(b-a)^2}\int_a^b\int_a^b F(x,y)(t)dxdy$$
$$\leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(x)dx\right], \tag{12}$$

for all t in [0, 1]. The inequalities (11) and (12) are the variants of the inequalities established by Dragomir [2] in Theorem D. The main difference between the inequalities (2), (3) and those obtained in (11), (12) is that the bounds obtained on the right hand side in (11) and (12) are independent of the factors involving t.

Theorem 3. *Let $f : [a,b] \to R$ be a convex function and F, G, H be as defined in* (4), (5), (6)*. Then*
(i) *G and H are convex on* $[0,1]$*;*
(ii) *$G(t) \leq H(t)$ for all t in* $[0,1]$.

Remark 2. We note that the results obtained in Theorem 3 are the variants of the results given by Dragomir in [1]. For various other properties of the slight variants of the mappings of the forms defined in (5) and (6), see [1].

3. Proof of Theorem 1

Since f is convex on $[a,b]$, it is easy to observe that

$$F\left(x,\frac{a+b}{2}\right)(t) \leq \frac{1}{2}\left[f(x)+f\left(\frac{a+b}{2}\right)\right]. \tag{13}$$

Integrating (13) on $[a,b]$ and using the left half of the Hadamard's inequality given in (1) we observe that

$$\int_a^b F\left(x,\frac{a+b}{2}\right)(t)dx \leq \frac{1}{2}\int_a^b f(x)dx + \frac{1}{2}(b-a)f\left(\frac{a+b}{2}\right)$$
$$\leq \frac{1}{2}\int_a^b f(x)dx + \frac{1}{2}\int_a^b f(x)dx = \int_a^b f(x)dx. \tag{14}$$

This is the required inequality in (7).

Since f is convex and differentiable on $[a,b]$ we have

$$f\left(tx+(1-t)\left(\frac{a+b}{2}\right)\right) \geq f(x)+(1-t)\left(\frac{a+b}{2}-x\right)f'(x), \tag{15}$$

$$f\left((1-t)x+t\left(\frac{a+b}{2}\right)\right) \geq f(x)+t\left(\frac{a+b}{2}-x\right)f'(x), \tag{16}$$

for x in $[a,b]$ and t in [0, 1]. From (15) and (16) we observe that

$$F\left(x,\frac{a+b}{2}\right)(t)\ge f(x)+\frac{1}{2}\left(\frac{a+b}{2}-x\right)f'(x). \tag{17}$$

Integrating (17) on $[a,b]$ we get

$$\int_a^b F\left(x,\frac{a+b}{2}\right)(t)dx\ge\frac{3}{2}\int_a^b f(x)dx-\frac{1}{4}(b-a)[f(a)+f(b)]. \tag{18}$$

Dividing both sides of (18) by $(b-a)$ and rewriting we get the required inequality in (8). The proof is finished.

4. Proof of Theorem 2

Since f is convex on $[a,b]$, it is easy to observe that

$$F(x,y)(t)\le\frac{1}{2}[f(x)+f(y)]. \tag{19}$$

Integrating (19) on $[a,b]\times[a,b]$, we get

$$\int_a^b\int_a^b F(x,y)(t)dxdy\le(b-a)\int_a^b f(x)dx \tag{20}$$

Dividing both sides of (20) by $(b-a)$ we get the desired inequality in (9).

Since f is convex and differentiable on $[a,b]$, as in the proof of Theorem 1, it is easy to observe that

$$F(x,y)(t)\ge f(x)+\frac{1}{2}(y-x)f'(x), \tag{21}$$

for $x,\ y$ in $[a,b]$ and t in $[0,1]$. Integrating (21) on $[a,b]\times[a,b]$ we get

$$\int_a^b\int_a^b F(x,y)(t)dxdy\ge\frac{3}{2}(b-a)\int_a^b f(x)dx-\frac{1}{4}(b-a)^2[f(a)+f(b)]. \tag{22}$$

Dividing both sides of (22) by $(b-a)^2$ and rewriting we get the desired inequality in (10). The proof is complete.

5. Proof of Theorem 3

(i) The proofs that $G(t)$ and $H(t)$ are convex on $[0,1]$ follows by the same arguments as in the proof of similar results given in [1] with suitable modifications. Here we omit the details.

(ii) It is easy to observe that

$$G(t)=\frac{1}{b-a}\int_a^b\frac{1}{2}\left[f\left(\frac{1}{b-a}\int_a^b[tx+(1-t)y]dy\right)+f\left(\frac{1}{b-a}\int_a^b[(1-t)x+ty]dy\right)\right]dx. \tag{23}$$

Now an application of Jensen's inequality on the right side of (23) and using the definitions of $F(x,y)(t)$ and $H(t)$ given in (4) and (6) we have

$$G(t) \le \frac{1}{(b-a)^2}\int_a^b\int_a^b F(x,y)(t)dxdy = H(t). \tag{24}$$

The proof is complete.

Remark 3. We note that the inequalities established in this paper can very easily be extended involving concave functions. For some more inequalities and their applications, see [1-4].

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