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NEW OSTROWSKI AND OSTROWSKI–GRÜSS TYPE INEQUALITIES FOR DOUBLE INTEGRALS ON TIME SCALES INVOLVING A COMBINATION OF Δ-INTEGRAL MEANS

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Abstract. In 2014, some Ostrowski type inequalities for functions of a single variable were obtained in [Y. Jiang, H. Rüzgar, W. J. Liu and A. Tuna: Some new generalizations of Ostrowski type inequalities on time scales involving combination of Δ-integral means, J. Nonlinear Sci. Appl., **7** (2014), 311–324]. In this paper, we extend some of the inequalities obtained in the above paper for double integrals. One of our results generalizes a result in the article [W. J. Liu, Q. A. Ngô and W. Chen: On new Ostrowski type inequalities for double integrals on time scales, Dyn. Syst. Appl., **19** (2010), 189–198]. We also apply our results to the continuous, discrete and quantum time scales to obtain some interesting inequalities.

1. Introduction

The following result, obtained by Alexander Ostrowski [20] in 1938, provides a bound for the deviation of a function from its integral mean.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable in (a,b) and its derivative $f':(a,b)\to\mathbb{R}$ is bounded in (a,b). If $|f'(t)|\leq M$ for all $t\in[a,b]$, then we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\left(\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a)M \tag{1}$$

for all $x\in[a,b]$. The inequality is sharp in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

Inequality (1) is known in the literature as Ostrowski inequality. This inequality has been studied and generalized by many researchers over the past years, see [8] and [17] for example. The

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German mathematician Stefan Hilger [9] in 1988 introduced the theory of time scales to unify the continuous and discrete calculus in a consistent manner. In 2008, Bohner and Mathews [6] extended Theorem 1 to an arbitrary time scale $\mathbb{T}$ as follows:

Theorem 2. *Let $a, b, s, t \in \mathbb{T}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function. Then*

$$\left| f(t) - \frac{1}{b-a}\int_a^b f(\sigma(s))\Delta s \right| \leq \frac{M}{b-a}\Big[h_2(t,a) + h_2(t,b)\Big], \tag{2}$$

where $h_2(\cdot,\cdot)$ is defined by Definition 14 in Section 2 and $M = \sup_{a<t<b} |f^{\Delta}(t)| < \infty$. Inequality (2) is sharp in the sense that the right-hand side cannot be replaced by a smaller one.

Over the years many authors have studied different generalizations of Theorem 2 for functions of a single variable (see [13, 17, 18, 19, 23] and the references therein) as well as for functions of two independent variables (see [10, 11, 15, 16, 21, 22] and the references therein). Recently, Jiang et al. [12] obtained some new Ostrowski type inequalities on time scales involving a combination of Δ-integral means. Specifically, they proved the following two results amongst others.

Theorem 3. *Let $a, b, x \in \mathbb{T}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function. Then for all $x \in [a, b]$, we have*

$$\begin{aligned} &\left| f(x) - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t + \frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right]\right| \\ &\qquad \leq \frac{M}{\alpha+\beta}\left[\frac{\alpha}{x-a}h_2(x,a) + \frac{\beta}{b-x}h_2(x,b)\right], \end{aligned} \tag{3}$$

where $M = \sup_{a<t<b} |f^{\Delta}(t)| < \infty$.

Theorem 4. *Let $a, b, x \in \mathbb{T}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function such that there exists constants $\gamma, \Gamma \in \mathbb{R}$ with $\gamma \leq f^{\Delta}(x) \leq \Gamma$ for all $x \in [a, b]$. Then for all $x \in [a, b]$, we have*

$$\begin{aligned} &\left| f(x) - \frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t + \frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right] \right.\\ &\qquad \left. - \frac{\gamma+\Gamma}{2(\alpha+\beta)}\left[\frac{\alpha}{x-a}h_2(x,a) - \frac{\beta}{b-x}h_2(x,b)\right]\right| \\ &\qquad \leq \frac{\Gamma-\gamma}{2(\alpha+\beta)}\left[\frac{\alpha}{x-a}h_2(x,a) + \frac{\beta}{b-x}h_2(x,b)\right]. \end{aligned} \tag{4}$$

Motivated by the above works, our goal is to extend Theorems 3 and 4 to functions of two independent variables. Our results generalize some results in the literature as we will point out later.

This paper is arranged in the following order: first, we present some time scale essentials in Section 2. In Section 3, our results are then formulated and proved. Finally, we consider some applications of our results in Section 4.

2. Preliminaries

In this section, we briefly recall some fundamental facts about the time scale theory. For further details and proofs we invite the interested reader to Hilger's Ph.D. thesis [9], the books [2, 3, 14], and the survey [1].

Definition 5. A time scale is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.

Throughout this work we assume $\mathbb{T}$ is a time scale and $\mathbb{T}$ has the topology that is inherited from the standard topology on $\mathbb{R}$. It is also assumed throughout that in $\mathbb{T}$ the interval $[a,b]$ means the set $\{t \in \mathbb{T} \colon a \le t \le b\}$ for the points $a < b$ in $\mathbb{T}$. Since a time scale may not be connected, we need the following concept of jump operators.

Definition 6. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\}$, respectively.

The jump operators σ and ρ allow the classification of points in $\mathbb{T}$ as follows.

Definition 7. If $\sigma(t) > t$, then we say that t is right-scattered, while if $\rho(t) < t$, then we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(t) = t$, then t is called right-dense, and if $\rho(t) = t$, then t is called left-dense. Points that are both right-dense and left-dense are called dense.

Definition 8. The graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) = \sigma(t) - t$ for $t \in \mathbb{T}$. The set $\mathbb{T}^{\kappa}$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$; otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$.

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$, and when $\mathbb{T} = \mathbb{Z}$, we have $\mu(t) = 1$.

Definition 9. Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that for any given $\varepsilon > 0$ there exists a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)\,[\sigma(t) - s] \right| \le \varepsilon\,|\sigma(t) - s|, \quad \forall s \in U.$$

We call $f^{\Delta}(t)$ the delta derivative of f at t. Moreover, we say that f is delta differentiable (or in short: differentiable) on $\mathbb{T}^{\kappa}$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$. The function $f^{\Delta} : \mathbb{T}^{\kappa} \to \mathbb{R}$ is then called the delta derivative of f on $\mathbb{T}^{\kappa}$.

In the case $\mathbb{T}=\mathbb{R}$, $f^{\Delta}(t)=\frac{df(t)}{dt}$. In the case $\mathbb{T}=\mathbb{Z}$, $f^{\Delta}(t)=\Delta f(t)=f(t+1)-f(t)$, which is the usual forward difference operator. If $\mathbb{T}=q^{\mathbb{N}_0}$, $q>1$ and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, then $f^{\Delta}(t)=\frac{f(qt)-f(t)}{(q-1)t}$

Theorem 10. *Assume $f,g:\mathbb{T}\to\mathbb{R}$ are differentiable at $t\in\mathbb{T}^{\kappa}$. Then the product $fg:\mathbb{T}\to\mathbb{R}$ is differentiable at t with*

$$(fg)^{\Delta}(t)=f^{\Delta}(t)g(t)+f(\sigma(t))g^{\Delta}(t).$$

Definition 11. The function $f:\mathbb{T}\to\mathbb{R}$ is said to be rd-continuous on $\mathbb{T}$ provided it is continuous at all right-dense points $t\in\mathbb{T}$ and its left-sided limits exist at all left-dense points $t\in\mathbb{T}$. The set of all rd-continuous function $f:\mathbb{T}\to\mathbb{R}$ is denoted by $C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$. Also, the set of functions $f:\mathbb{T}\to\mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $C^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$.

It follows from [2, Theorem 1.74] that every rd-continuous function has an anti-derivative.

Definition 12. Let $F:\mathbb{T}\to\mathbb{R}$ be a function. Then $F:\mathbb{T}\to\mathbb{R}$ is called the anti-derivative of f on $\mathbb{T}$ if it satisfies $F^{\Delta}(t)=f(t)$ for any $t\in\mathbb{T}^{\kappa}$. In this case, the Cauchy integral is defined by

$$\int_a^b f(t)\Delta t=F(b)-F(a),\quad a,b\in\mathbb{T}.$$

Theorem 13. *If $a,b,c\in\mathbb{T}$ with $a<c<b$, $\alpha\in\mathbb{R}$ and $f,g\in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$, then*

(i) $\int_a^b[f(t)+g(t)]\Delta t=\int_a^b f(t)\Delta t+\int_a^b g(t)\Delta t.$

(ii) $\int_a^b \alpha f(t)\Delta t=\alpha\int_a^b f(t)\Delta t.$

(iii) $\int_a^b f(t)\Delta t=-\int_b^a f(t)\Delta t.$

(iv) $\int_a^b f(t)\Delta t=\int_a^c f(t)\Delta t+\int_c^b f(t)\Delta t.$

(v) $\left|\int_a^b f(t)\Delta t\right|\le\int_a^b|f(t)|\Delta t.$

(vi) $\int_a^b f(t)g^{\Delta}(t)\Delta t=(fg)(b)-(fg)(a)-\int_a^b f^{\Delta}(t)g^{\sigma}(t)\Delta t.$

Definition 14. Let $h_k:\mathbb{T}^2\to\mathbb{R}$, $k\in\mathbb{N}_0$ be defined by $h_0(t,s)=1$, for all $s,t\in\mathbb{T}$ and then recursively by $h_{k+1}(t,s)=\int_s^t h_k(\tau,s)\Delta\tau$, for all $s,t\in\mathbb{T}$.

If $\mathbb{T}=\mathbb{R}$, then $h_k(t,s)=\frac{(t-s)^k}{k!}$, for all $s,t\in\mathbb{R}$. If $\mathbb{T}=\mathbb{Z}$, then $h_k(t,s)=\binom{t-s}{k}$, for all $s,t\in\mathbb{Z}$. If $\mathbb{T}=q^{\mathbb{N}_0}$, $q>1$, then $h_k(t,s)=\prod_{\nu=0}^{k-1}\frac{t-q^{\nu}s}{\sum_{\mu=0}^{\nu}q^{\mu}}$ for all $s,t\in q^{\mathbb{N}_0}$.

3. Main results

To prove our theorems, we need the following lemmas. The first lemma was first provided in [7] for the real case and later extended to an arbitrary time scale in [12].

Lemma 15 ([12])**.** *Let $a,b,x,t\in\mathbb{T}, a<b$, and $f:[a,b]\to\mathbb{R}$ be a differentiable function. Then for all $x\in[a,b]$, we have*

$$\int_a^b K(x,t)f^{\Delta}(t)\Delta t = f(x)-\frac{1}{\alpha+\beta}\Big[\frac{\alpha}{x-a}\int_a^x f(\sigma(t)\Delta t+\frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\Big], \tag{5}$$

where

$$K(x,t)=\begin{cases}\frac{\alpha}{\alpha+\beta}\left(\frac{t-a}{x-a}\right), & t\in[a,x),\\ \frac{-\beta}{\alpha+\beta}\left(\frac{b-t}{b-x}\right), & t\in[x,b],\end{cases}$$

where $\alpha,\beta\in\mathbb{R}$ are nonnegative and α and β are not both zero.

In what follows, we will let $\mathbb{T}_1$ and $\mathbb{T}_2$ denote two arbitrary time scales, with forward jump orperators σ_1 and σ_2 respectively. For an interval $[a,b]$, $[a,b]_{\mathbb{T}_i}:=[a,b]\cap\mathbb{T}_i, i=1,2$. For $a<b$ and $c<d$, we define the rectangle $[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}$ as follows: $[a,b]_{\mathbb{T}_1}\times[c,d]_{\mathbb{T}_2}=\{(x,y):x\in[a,b]_{\mathbb{T}_1}, y\in[c,d]_{\mathbb{T}_2}\}$. For more on the two-variable time scale calculus, we invite the interested reader to the papers [4, 5] and the references therein.

The following lemma is the 2-dimensional version of Lemma 15.

Lemma 16. *Let $a,b,x,s\in\mathbb{T}_1, a<b$, $c,d,y,t\in\mathbb{T}_2, c<d$ and $f:[a,b]\times[c,d]\to\mathbb{R}$ be a function such that the partial derivative $\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}$ exist and is continuous on $[a,b]\times[c,d]$. Then we have*

$$\begin{aligned}
&\int_a^b\int_c^d P(x,y,s,t)\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}\Delta_2 t\Delta_1 s\\
&=f(x,y)-\frac{1}{\alpha_1+\beta_1}\left(\frac{\alpha_1}{x-a}\int_a^x f(\sigma_1(s),y)\Delta_1 s+\frac{\beta_1}{b-x}\int_x^b f(\sigma_1(s),y)\Delta_1 s\right)\\
&\quad-\frac{1}{\alpha_2+\beta_2}\left(\frac{\alpha_2}{y-c}\int_c^y f(x,\sigma_2(t))\Delta_2 t+\frac{\beta_2}{d-y}\int_y^d f(x,\sigma_2(t))\Delta_2 t\right)\\
&\quad+\frac{1}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)}\left(\frac{\alpha_2\alpha_1}{(y-c)(x-a)}\int_c^y\int_a^x f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Delta_2 t\right.\\
&\quad\left.+\frac{\alpha_2\beta_1}{(y-c)(b-x)}\int_c^y\int_x^b f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Delta_2 t\right)\\
&\quad+\frac{1}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)}\left(\frac{\beta_2\alpha_1}{(d-y)(x-a)}\int_y^d\int_a^x f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Delta_2 t\right.\\
&\quad\left.+\frac{\beta_2\beta_1}{(d-y)(b-x)}\int_y^d\int_x^b f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Delta_2 t\right)
\end{aligned} \tag{6}$$

for all $x\in[a,b]$ and $y\in[c,d]$, where $P(x,y,s,t)=P_1(x,s)P_2(y,t)$ with

$$P_1(x,s)=\begin{cases}\frac{\alpha_1}{\alpha_1+\beta_1}\left(\frac{s-a}{x-a}\right), & s\in[a,x),\\ \frac{-\beta_1}{\alpha_1+\beta_1}\left(\frac{b-s}{b-x}\right), & s\in[x,b],\end{cases}$$

$$P_2(y,t)=\begin{cases}\frac{\alpha_2}{\alpha_2+\beta_2}\left(\frac{t-c}{y-c}\right), & t\in[c,y),\\ \frac{-\beta_2}{\alpha_2+\beta_2}\left(\frac{d-t}{d-y}\right), & t\in[y,d],\end{cases}$$

$\alpha_1,\beta_1,\alpha_2,\beta_2\in\mathbb{R}$ nonnegative and α_1 and β_1 are not both zero. The same applies to α_2 and β_2.

Proof. By applying Lemma 15 to the partial maps $\frac{\partial f(s,\cdot)}{\Delta_1 s}$ for $s\in[a,b]$ and $f(\cdot,t)$ for $t\in[c,d]$, we have

$$\begin{aligned}
&\int_a^b\int_c^d P_1(x,s)P_2(y,t)\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}\Delta_2 t\Delta_1 s\\
&=\int_a^b P_1(x,s)\Big[\int_c^d P_2(y,t)\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}\Delta_2 t\Big]\Delta_1 s\\
&=\int_a^b P_1(x,s)\Big[\frac{\partial f(s,y)}{\Delta_1 s}-\frac{1}{\alpha_2+\beta_2}\Big(\frac{\alpha_2}{y-c}\int_c^y\frac{\partial f(s,\sigma_2(t))}{\Delta_1 s}\Delta_2 t+\frac{\beta_2}{d-y}\int_y^d\frac{\partial f(s,\sigma_2(t))}{\Delta_1 s}\Delta_2 t\Big)\Big]\Delta_1 s\\
&=\int_a^b P_1(x,s)\frac{\partial f(s,y)}{\Delta_1 s}\Delta_1 s-\frac{1}{\alpha_2+\beta_2}\frac{\alpha_2}{y-c}\int_c^y\int_a^b P_1(x,s)\frac{\partial f(s,\sigma_2(t))}{\Delta_1 s}\Delta_1 s\Delta_2 t\\
&\qquad-\frac{1}{\alpha_2+\beta_2}\frac{\beta_2}{d-y}\int_y^d\int_a^b P_1(x,s)\frac{\partial f(s,\sigma_2(t))}{\Delta_1 s}\Delta_1 s\Delta_2 t\\
&=f(x,y)-\frac{1}{\alpha_1+\beta_1}\Big(\frac{\alpha_1}{x-a}\int_a^x f(\sigma_1(s),y)\Delta_1 s+\frac{\beta_1}{b-x}\int_x^b f(\sigma_1(s),y)\Delta_1 s\Big)\\
&\quad-\frac{1}{\alpha_2+\beta_2}\frac{\alpha_2}{y-c}\int_c^y\Big(f(x,\sigma_2(t))-\frac{1}{\alpha_1+\beta_1}\Big[\frac{\alpha_1}{x-a}\int_a^x f(\sigma_1(s),\sigma_2(t))\Delta_1 s\\
&\qquad+\frac{\beta_1}{b-x}\int_x^b f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Big]\Big)\Delta_2 t\\
&\quad-\frac{1}{\alpha_2+\beta_2}\frac{\beta_2}{d-y}\int_y^d\Big(f(x,\sigma_2(t))-\frac{1}{\alpha_1+\beta_1}\Big[\frac{\alpha_1}{x-a}\int_a^x f(\sigma_1(s),\sigma_2(t))\Delta_1 s\\
&\qquad+\frac{\beta_1}{b-x}\int_x^b f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Big]\Big)\Delta_2 t\\
&=f(x,y)-\frac{1}{\alpha_1+\beta_1}\Big(\frac{\alpha_1}{x-a}\int_a^x f(\sigma_1(s),y)\Delta_1 s+\frac{\beta_1}{b-x}\int_x^b f(\sigma_1(s),y)\Delta_1 s\Big)\\
&\qquad-\frac{1}{\alpha_2+\beta_2}\Big(\frac{\alpha_2}{y-c}\int_c^y f(x,\sigma_2(t))\Delta_2 t+\frac{\beta_2}{d-y}\int_y^d f(x,\sigma_2(t))\Delta_2 t\Big)\\
&\quad+\frac{\alpha_2}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)(y-c)}\Big(\frac{\alpha_1}{x-a}\int_c^y\int_a^x f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Delta_2 t\\
&\qquad+\frac{\beta_1}{b-x}\int_c^y\int_x^b f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Delta_2 t\Big)\\
&\quad+\frac{\beta_2}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)(d-y)}\Big(\frac{\alpha_1}{x-a}\int_y^d\int_a^x f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Delta_2 t\\
&\qquad+\frac{\beta_1}{b-x}\int_y^d\int_x^b f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Delta_2 t\Big).
\end{aligned}$$

Hence, we have the identity:

$$\begin{aligned}
&\int_a^b \int_c^d P(x,y,s,t)\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}\Delta_2 t\Delta_1 s \\
&= f(x,y) - \frac{1}{\alpha_1+\beta_1}\Big(\frac{\alpha_1}{x-a}\int_a^x f(\sigma_1(s),y)\Delta_1 s + \frac{\beta_1}{b-x}\int_x^b f(\sigma_1(s),y)\Delta_1 s\Big) \\
&\quad - \frac{1}{\alpha_2+\beta_2}\Big(\frac{\alpha_2}{y-c}\int_c^y f(x,\sigma_2(t))\Delta_2 t + \frac{\beta_2}{d-y}\int_y^d f(x,\sigma_2(t))\Delta_2 t\Big) \\
&\quad + \frac{1}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)}\Big(\frac{\alpha_2\alpha_1}{(y-c)(x-a)}\int_c^y\int_a^x f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Delta_2 t \\
&\qquad\qquad + \frac{\alpha_2\beta_1}{(y-c)(b-x)}\int_c^y\int_x^b f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Delta_2 t\Big) \\
&\quad + \frac{1}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)}\Big(\frac{\beta_2\alpha_1}{(d-y)(x-a)}\int_y^d\int_a^x f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Delta_2 t \\
&\qquad\qquad + \frac{\beta_2\beta_1}{(d-y)(b-x)}\int_y^d\int_x^b f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Delta_2 t\Big). \qquad \square
\end{aligned}$$

We are now in a position to state and prove our first theorem. This is an extension of Theorem 3 to the 2-dimensional case.

Theorem 17. *Under the conditions of Lemma 16, we have*

$$\begin{aligned}
&\Big| f(x,y) - \frac{1}{\alpha_1+\beta_1}\Big(\frac{\alpha_1}{x-a}\int_a^x f(\sigma_1(s),y)\Delta_1 s + \frac{\beta_1}{b-x}\int_x^b f(\sigma_1(s),y)\Delta_1 s\Big) \\
&\quad - \frac{1}{\alpha_2+\beta_2}\Big(\frac{\alpha_2}{y-c}\int_c^y f(x,\sigma_2(t))\Delta_2 t + \frac{\beta_2}{d-y}\int_y^d f(x,\sigma_2(t))\Delta_2 t\Big) \\
&\quad + \frac{1}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)}\Big(\frac{\alpha_2\alpha_1}{(y-c)(x-a)}\int_c^y\int_a^x f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Delta_2 t \\
&\quad + \frac{\alpha_2\beta_1}{(y-c)(b-x)}\int_c^y\int_x^b f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Delta_2 t\Big) \\
&\quad + \frac{1}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)}\Big(\frac{\beta_2\alpha_1}{(d-y)(x-a)}\int_y^d\int_a^x f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Delta_2 t \\
&\quad + \frac{\beta_2\beta_1}{(d-y)(b-x)}\int_y^d\int_x^b f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Delta_2 t\Big)\Big| \\
&\le \frac{M}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\Big[\frac{\alpha_1}{x-a}h_2(x,a) + \frac{\beta_1}{b-x}h_2(x,b)\Big]\Big[\frac{\alpha_2}{y-c}h_2(y,c) + \frac{\beta_2}{d-y}h_2(y,d)\Big] \qquad (7)
\end{aligned}$$

for all $(x,y)\in[a,b]\times[c,d]$, *where*

$$M = \sup_{a<s<b,c<t<d}\left|\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}\right| < \infty.$$

Proof. We first observe that,

$$\int_a^b |P_1(x,s)|\Delta_1 s = \frac{1}{\alpha_1+\beta_1}\Big[\frac{\alpha_1}{x-a}h_2(x,a)+\frac{\beta_1}{b-x}h_2(x,b)\Big] \tag{8}$$

and similarly,

$$\int_c^d |P_2(y,t)|\Delta_2 t = \frac{1}{\alpha_2+\beta_2}\Big[\frac{\alpha_2}{y-c}h_2(y,c)+\frac{\beta_2}{d-y}h_2(y,d)\Big]. \tag{9}$$

Hence,

$$\begin{aligned}\int_a^b\int_c^d |P(x,y,s,t)|\Delta_2 t\Delta_1 s &= \int_a^b |P_1(x,s)|\Delta_1 s\int_c^d |P_2(y,t)|\Delta_2 t\\ &= \frac{1}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\Big[\frac{\alpha_1}{x-a}h_2(x,a)+\frac{\beta_1}{b-x}h_2(x,b)\Big]\Big[\frac{\alpha_2}{y-c}h_2(y,c)+\frac{\beta_2}{d-y}h_2(y,d)\Big].\end{aligned} \tag{10}$$

By applying item (v) of Theorem 13, we have

$$\Big|\int_a^b\int_c^d P(x,y,s,t)\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}\Delta_2 t\Delta_1 s\Big| \le M\int_a^b\int_c^d |P(x,y,s,t)|\Delta_2 t\Delta_1 s. \tag{11}$$

The desired inequality is obtained by using the inequality (11) and the identities in (6) and (10). ∎

Remark 18. If we take $\alpha_1 = x-a$, $\beta_1 = b-x$, $\alpha_2 = y-c$ and $\beta_2 = d-y$, then Theorem 17 reduces to [16, Theorem 2.1].

Theorem 19. *Under the conditions of Lemma 16, and suppose there exists constants* $\gamma,\ \Gamma\in\mathbb{R}$ *such that* $\gamma \le \frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s} \le \Gamma$ *for all* $s\in[a,b]$ *and* $t\in[c,d]$, *then we have the inequality*

$$\begin{aligned}&\Big|f(x,y)-\frac{1}{\alpha_1+\beta_1}\Big(\frac{\alpha_1}{x-a}\int_a^x f(\sigma_1(s),y)\Delta_1 s+\frac{\beta_1}{b-x}\int_x^b f(\sigma_1(s),y)\Delta_1 s\Big)\\
&-\frac{1}{\alpha_2+\beta_2}\Big(\frac{\alpha_2}{y-c}\int_c^y f(x,\sigma_2(t))\Delta_2 t+\frac{\beta_2}{d-y}\int_y^d f(x,\sigma_2(t))\Delta_2 t\Big)\\
&+\frac{1}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)}\Big(\frac{\alpha_2\alpha_1}{(y-c)(x-a)}\int_c^y\int_a^x f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Delta_2 t\\
&+\frac{\alpha_2\beta_1}{(y-c)(b-x)}\int_c^y\int_x^b f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Delta_2 t\Big) \qquad (12)\\
&+\frac{1}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)}\Big(\frac{\beta_2\alpha_1}{(d-y)(x-a)}\int_y^d\int_a^x f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Delta_2 t\\
&+\frac{\beta_2\beta_1}{(d-y)(b-x)}\int_y^d\int_x^b f(\sigma_1(s),\sigma_2(t))\Delta_1 s\Delta_2 t\Big) \qquad (13)\\
&-\frac{\gamma+\Gamma}{2(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\Big[\frac{\alpha_1}{x-a}h_2(x,a)-\frac{\beta_1}{b-x}h_2(x,b)\Big]\Big[\frac{\alpha_2}{y-c}h_2(y,c)-\frac{\beta_2}{d-y}h_2(y,d)\Big]\Big|\\
&\le\frac{\Gamma-\gamma}{2(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\Big[\frac{\alpha_1}{x-a}h_2(x,a)+\frac{\beta_1}{b-x}h_2(x,b)\Big]\Big[\frac{\alpha_2}{y-c}h_2(y,c)+\frac{\beta_2}{d-y}h_2(y,d)\Big].\end{aligned}$$

Proof. From the assumption that $\gamma \le \frac{\partial^2 f(s,t)}{\Delta_2 t \Delta_1 s} \le \Gamma$ for all $(s,t) \in [a,b] \times [c,d]$, it follows that

$$\sup_{a<s<b,c<t<d} \left| \frac{\partial^2 f(s,t)}{\Delta_2 t \Delta_1 s} - \frac{\Gamma+\gamma}{2} \right| \le \frac{\Gamma-\gamma}{2}. \tag{14}$$

On the other hand,

$$\begin{aligned} \int_a^b \int_c^d P(x,y,s,t) \frac{\partial^2 f(s,t)}{\Delta_2 t \Delta_1 s} \Delta_2 t \Delta_1 s - \frac{\Gamma+\gamma}{2} \int_a^b \int_c^d P(x,y,s,t) \Delta_2 t \Delta_1 s \\ = \int_a^b \int_c^d P(x,y,s,t) \Big(\frac{\partial^2 f(s,t)}{\Delta_2 t \Delta_1 s} - \frac{\Gamma+\gamma}{2} \Big) \Delta_2 t \Delta_1 s. \end{aligned} \tag{15}$$

We observe that,

$$\int_a^b P_1(x,s)\Delta_1 s = \frac{1}{(\alpha_1+\beta_1)} \Big[\frac{\alpha_1}{x-a} h_2(x,a) - \frac{\beta_1}{b-x} h_2(x,b) \Big]$$

and

$$\int_c^d P_2(y,t)\Delta_2 t = \frac{1}{(\alpha_2+\beta_2)} \Big[\frac{\alpha_2}{y-c} h_2(y,c) - \frac{\beta_2}{d-y} h_2(y,d) \Big].$$

Hence,

$$\begin{aligned} &\int_a^b \int_c^d P(x,y,s,t) \Delta_2 t \Delta_1 s \\ &= \frac{1}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)} \Big[\frac{\alpha_1}{x-a} h_2(x,a) - \frac{\beta_1}{b-x} h_2(x,b) \Big] \Big[\frac{\alpha_2}{y-c} h_2(y,c) - \frac{\beta_2}{d-y} h_2(y,d) \Big]. \end{aligned} \tag{16}$$

By applying item (v) of Theorem 13 to (15) and using (14), we have

$$\begin{aligned} \Big| \int_a^b \int_c^d P(x,y,s,t,) \frac{\partial^2 f(s,t)}{\Delta_2 t \Delta_1 s} \Delta_2 t \Delta_1 s - \frac{\Gamma+\gamma}{2} \int_a^b \int_c^d P(x,y,s,t) \Delta_2 t \Delta_1 s \Big| \\ \le \frac{\Gamma-\gamma}{2} \int_a^b \int_c^d |P(x,y,s,t)| \Delta_2 t \Delta_1 s. \end{aligned} \tag{17}$$

The desired inequality is obtained by using (6), (10), (16) and (17). □

Remark 20. Theorem 19 extends Theorem 4 to the 2D case.

Corollary 21. *Under the conditions of Lemma 16, and suppose that there exists constants* $\gamma, \Gamma \in \mathbb{R}$ *such that* $\gamma \le \frac{\partial^2 f(s,t)}{\Delta_2 t \Delta_1 s} \le \Gamma$ *for all* $s \in [a,b]$ *and* $t \in [c,d]$, *then we have the inequality*

$$\begin{aligned} &\Big| f(x,y) - \frac{1}{b-a} \int_a^b f(\sigma_1(s),y)\Delta_1 s - \frac{1}{d-c} \int_c^d f(x,\sigma_2(t))\Delta_2 t \\ &\quad + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(\sigma_1(s),\sigma_2(t)) \Delta_2 t \Delta_1 s \\ &\quad - \frac{\gamma+\Gamma}{2(b-a)(d-c)} \Big[h_2(x,a) - h_2(x,b) \Big] \Big[h_2(y,c) - h_2(y,d) \Big] \Big| \\ &\le \frac{\Gamma-\gamma}{2(b-a)(d-c)} \Big[h_2(x,a) + h_2(x,b) \Big] \Big[h_2(y,c) + h_2(y,d) \Big]. \end{aligned} \tag{18}$$

Proof. The proof follows directly from Theorem 19 by taking $\alpha_1 = x-a,\ \beta_1 = b-x,\ \alpha_2 = y-c$ and $\beta_2 = d-y$. □

4. Applications

In this section, we apply our theorems to the continuous, discrete, and quantum calculus.

Corollary 22. *If we let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ in Theorem 17, then we have the inequality*

$$\begin{aligned}
&\Big| f(x,y) - \frac{1}{\alpha_1+\beta_1}\Big(\frac{\alpha_1}{x-a}\int_a^x f(s,y)ds + \frac{\beta_1}{b-x}\int_x^b f(s,y)ds\Big) \\
&- \frac{1}{\alpha_2+\beta_2}\Big(\frac{\alpha_2}{y-c}\int_c^y f(x,t)dt + \frac{\beta_2}{d-y}\int_y^d f(x,t)dt\Big) \\
&+ \frac{1}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)}\Big(\frac{\alpha_2\alpha_1}{(y-c)(x-a)}\int_c^y\int_a^x f(s,t)dsdt + \frac{\alpha_2\beta_1}{(y-c)(b-x)}\int_c^y\int_x^b f(s,t)dsdt\Big) \\
&+ \frac{1}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)}\Big(\frac{\beta_2\alpha_1}{(d-y)(x-a)}\int_y^d\int_a^x f(s,t)dsdt + \frac{\beta_2\beta_1}{(d-y)(b-x)}\int_y^d\int_x^b f(s,t)dsdt\Big)\Big| \\
&\le \frac{M}{4(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\Big[\alpha_1(x-a)+\beta_1(b-x)\Big]\Big[\alpha_2(y-c)+\beta_2(d-y)\Big] \qquad (19)
\end{aligned}$$

for all $(x,y)\in[a,b]\times[c,d]$, where

$$M = \sup_{a<s<b,c<t<d}\left|\frac{\partial^2 f(s,t)}{\partial t\partial s}\right| < \infty.$$

Corollary 23. *If we let $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ in Theorem 17, then we have the inequality:*

$$\begin{aligned}
&\Big| f(x,y) - \frac{1}{\alpha_1+\beta_1}\Big(\frac{\alpha_1}{x-a}\sum_{s=a}^{x-1} f(s+1,y) + \frac{\beta_1}{b-x}\sum_{s=x}^{b-1} f(s+1,y)\Big) \\
&\quad - \frac{1}{\alpha_2+\beta_2}\Big(\frac{\alpha_2}{y-c}\sum_{t=c}^{y-1} f(x,t+1) + \frac{\beta_2}{d-y}\sum_{t=y}^{d-1} f(x,t+1)\Big) \\
&\quad + \frac{1}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)}\Big(\frac{\alpha_2\alpha_1}{(y-c)(x-a)}\sum_{t=c}^{y-1}\sum_{s=a}^{x-1} f(s+1,t+1) \\
&\qquad\qquad + \frac{\alpha_2\beta_1}{(y-c)(b-x)}\sum_{t=c}^{y-1}\sum_{s=x}^{b-1} f(s+1,t+1)\Big) \\
&\quad + \frac{1}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)}\Big(\frac{\beta_2\alpha_1}{(d-y)(x-a)}\sum_{t=y}^{d-1}\sum_{s=a}^{x-1} f(s+1,t+1) \\
&\qquad\qquad + \frac{\beta_2\beta_1}{(d-y)(b-x)}\sum_{t=y}^{d-1}\sum_{s=x}^{b-1} f(s+1,t+1)\Big)\Big| \\
&\le \frac{M}{4(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\Big[\alpha_1(x-a-1)+\beta_1(b-x+1)\Big]\Big[\alpha_2(y-c-1)+\beta_2(d-y+1)\Big], \qquad (20)
\end{aligned}$$

for all $(x,y) \in \{a, a+1, \cdots, b-1, b\} \times \{c, c+1, \cdots, d-1, d\}$, *where*

$$M = \sup_{a<s<b, c<t<d} \Big| f(s+1,t+1) - f(s+1,t) - f(s,t+1) + f(s,t) \Big| < \infty.$$

Corollary 24. *Let* $\mathbb{T}_1 = q_1^{\mathbb{N}_0}$, $q_1 > 1$ *and* $\mathbb{T}_2 = q_2^{\mathbb{N}_0}$, $q_2 > 1$ *in Theorem 17. Then we have the inequality:*

$$\begin{aligned}
&\Big| f(x,y) - \frac{1}{\alpha_1+\beta_1}\Big(\frac{\alpha_1}{x-a}\int_a^x f(q_1 s, y) d_{q_1} s + \frac{\beta_1}{b-x}\int_x^b f(q_1 s, y) d_{q_1} s\Big) \\
&\quad - \frac{1}{\alpha_2+\beta_2}\Big(\frac{\alpha_2}{y-c}\int_c^y f(x, q_2 t) d_{q_2} t + \frac{\beta_2}{d-y}\int_y^d f(x, q_2 t) d_{q_2} t\Big) \\
&\quad + \frac{1}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)}\Big(\frac{\alpha_2\alpha_1}{(y-c)(x-a)}\int_c^y\int_a^x f(q_1 s, q_2 t) d_{q_1} s d_{q_2} t \\
&\qquad\qquad + \frac{\alpha_2\beta_1}{(y-c)(b-x)}\int_c^y\int_x^b f(q_1 s, q_2 t)\, d_{q_1} s d_{q_2} t\Big) \\
&\quad + \frac{1}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)}\Big(\frac{\beta_2\alpha_1}{(d-y)(x-a)}\int_y^d\int_a^x f(q_1 s, q_2 t) d_{q_1} s d_{q_2} t \\
&\qquad\qquad + \frac{\beta_2\beta_1}{(d-y)(b-x)}\int_y^d\int_x^b f(q_1 s, q_2 t)\, d_{q_1} s d_{q_2} t\Big)\Big| \\
&\leq \frac{M}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\Big[\frac{\alpha_1(x-q_1 a)+\beta_1(q_1 b - x)}{1+q_1}\Big]\Big[\frac{\alpha_2(y-q_2 c)+\beta_2(q_2 d - y)}{1+q_2}\Big],
\end{aligned} \tag{21}$$

where

$$M = \sup_{a<s<b, c<t<d} \Big| \frac{f(q_1 s, q_2 t) - f(q_1 s, t) - f(s, q_2 t) + f(s,t)}{(q_1-1)(q_2-1)st} \Big| < \infty.$$

Corollary 25. *If we let* $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{R}$ *in Theorem 19, then we have the inequality:*

$$\begin{aligned}
&\Big| f(x,y) - \frac{1}{\alpha_1+\beta_1}\Big(\frac{\alpha_1}{x-a}\int_a^x f(s,y)\, ds + \frac{\beta_1}{b-x}\int_x^b f(s,y)\, ds\Big) \\
&\quad - \frac{1}{\alpha_2+\beta_2}\Big(\frac{\alpha_2}{y-c}\int_c^y f(x,t)\, dt + \frac{\beta_2}{d-y}\int_y^d f(x,t)\, dt\Big) \\
&\quad + \frac{1}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)}\Big(\frac{\alpha_2\alpha_1}{(y-c)(x-a)}\int_c^y\int_a^x f(s,t)\, ds dt + \frac{\alpha_2\beta_1}{(y-c)(b-x)}\int_c^y\int_x^b f(s,t)\, ds dt\Big) \\
&\quad + \frac{1}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)}\Big(\frac{\beta_2\alpha_1}{(d-y)(x-a)}\int_y^d\int_a^x f(s,t)\, ds dt + \frac{\beta_2\beta_1}{(d-y)(b-x)}\int_y^d\int_x^b f(s,t)\, ds dt\Big) \\
&\quad - \frac{\gamma+\Gamma}{8(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\Big[\alpha_1(x-a)-\beta_1(b-x)\Big]\Big[\alpha_2(y-c)-\beta_2(d-y)\Big]\Big| \\
&\leq \frac{\Gamma-\gamma}{8(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\Big[\alpha_1(x-a)+\beta_1(b-x)\Big]\Big[\alpha_2(y-c)+\beta_2(d-y)\Big]
\end{aligned} \tag{22}$$

for all $(x,y) \in [a,b] \times [c,d]$.

Corollary 26. *If we let* $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ *in Theorem 19, then we have that for all* $(x,y) \in \{a, a+1, \cdots, b-1, b\} \times \{c, c+1, \ldots, d-1, d\}$, *the following inequality holds:*

$$\begin{aligned}
&\Big| f(x,y) - \frac{1}{\alpha_1+\beta_1}\Big(\frac{\alpha_1}{x-a}\sum_{s=a}^{x-1} f(s+1,y) + \frac{\beta_1}{b-x}\sum_{s=x}^{b-1} f(s+1,y)\Big) \\
&- \frac{1}{\alpha_2+\beta_2}\Big(\frac{\alpha_2}{y-c}\sum_{t=c}^{y-1} f(x,t+1) + \frac{\beta_2}{d-y}\sum_{t=y}^{d-1} f(x,t+1)\Big) \\
&+ \frac{1}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)}\Big(\frac{\alpha_2\alpha_1}{(y-c)(x-a)}\sum_{t=c}^{y-1}\sum_{s=a}^{x-1} f(s+1,t+1) + \frac{\alpha_2\beta_1}{(y-c)(b-x)}\sum_{t=c}^{y-1}\sum_{s=x}^{b-1} f(s+1,t+1)\Big) \\
&+ \frac{1}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)}\Big(\frac{\beta_2\alpha_1}{(d-y)(x-a)}\sum_{t=y}^{d-1}\sum_{s=a}^{x-1} f(s+1,t+1) + \frac{\beta_2\beta_1}{(d-y)(b-x)}\sum_{t=y}^{d-1}\sum_{s=x}^{b-1} f(s+1,t+1)\Big) \\
&- \frac{\gamma+\Gamma}{8(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\Big[\alpha_1(x-a-1) - \beta_1(b-x+1)\Big]\Big[\alpha_2(y-c-1) - \beta_2(d-y+1)\Big]\Big| \\
\le& \frac{\Gamma-\gamma}{8(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\Big[\alpha_1(x-a-1) + \beta_1(b-x+1)\Big]\Big[\alpha_2(y-c-1) + \beta_2(d-y+1)\Big]. \qquad (23)
\end{aligned}$$

Corollary 27. *Let* $\mathbb{T}_1 = q_1^{\mathbb{N}_0}$, $q_1 > 1$ *and* $\mathbb{T}_2 = q_2^{\mathbb{N}_0}$, $q_2 > 1$ *in Theorem 19. Then we have the inequality*

$$\begin{aligned}
&\Big| f(x,y) - \frac{1}{\alpha_1+\beta_1}\Big(\frac{\alpha_1}{x-a}\int_a^x f(q_1 s, y)d_{q_1}s + \frac{\beta_1}{b-x}\int_x^b f(q_1 s, y)d_{q_1}s\Big) \\
&\quad - \frac{1}{\alpha_2+\beta_2}\Big(\frac{\alpha_2}{y-c}\int_c^y f(x, q_2 t)d_{q_2}t + \frac{\beta_2}{d-y}\int_y^d f(x, q_2 t)d_{q_2}t\Big) \\
&\quad + \frac{1}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)}\Big(\frac{\alpha_2\alpha_1}{(y-c)(x-a)}\int_c^y\int_a^x f(q_1 s, q_2 t)d_{q_1}s d_{q_2}t \\
&\qquad\qquad + \frac{\alpha_2\beta_1}{(y-c)(b-x)}\int_c^y\int_x^b f(q_1 s, q_2 t)d_{q_1}s d_{q_2}t\Big) \\
&\quad + \frac{1}{(\alpha_2+\beta_2)(\alpha_1+\beta_1)}\Big(\frac{\beta_2\alpha_1}{(d-y)(x-a)}\int_y^d\int_a^x f(q_1 s, q_2 t)d_{q_1}s d_{q_2}t \\
&\qquad\qquad + \frac{\beta_2\beta_1}{(d-y)(b-x)}\int_y^d\int_x^b f(q_1 s, q_2 t)d_{q_1}s d_{q_2}t\Big) \\
&\quad - \frac{\gamma+\Gamma}{2(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\Big[\frac{\alpha_1(x-q_1 a) - \beta_1(q_1 b - x)}{1+q_1}\Big]\Big[\frac{\alpha_2(y-q_2 c) - \beta_2(q_2 d - y)}{1+q_2}\Big]\Big| \\
\le& \frac{\Gamma-\gamma}{2(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\Big[\frac{\alpha_1(x-q_1 a) + \beta_1(q_1 b - x)}{1+q_1}\Big]\Big[\frac{\alpha_2(y-q_2 c) + \beta_2(q_2 d - y)}{1+q_2}\Big]. \qquad (24)
\end{aligned}$$

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