



DUAL WAVELETS ASSOCIATED WITH NONUNIFORM MRA

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Abstract. A generalization of Mallat's classical multiresolution analysis, based on the theory of spectral pairs, was considered in two articles by Gabardo and Nashed. In this setting, the associated translation set is no longer a discrete subgroup of \mathbb{R} but a spectrum associated with a certain one-dimensional spectral pair and the associated dilation is an even positive integer related to the given spectral pair. In this paper, we construct dual wavelets which are associated with Nonuniform Multiresolution Analysis. We show that if the translates of the scaling functions of two multiresolution analyses are biorthogonal, then the associated wavelet families are also biorthogonal. Under mild assumptions on the scaling functions and the wavelets, we also show that the wavelets generate Riesz bases.

1. Introduction

Multiresolution analysis (MRA) is an important mathematical tool since it provides a natural framework for understanding and constructing discrete wavelet systems. A multiresolution analysis is an increasing family of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ such that $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$, $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and which satisfies $f \in V_j$ if and only if $f(2 \cdot) \in V_{j+1}$. Furthermore, there exists an element $\varphi \in V_0$ such that the collection of integer translates of function φ , $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ represents a complete orthonormal system for V_0 . The function φ is called the *scaling function* or the *father wavelet*. The concept of multiresolution analysis has been extended in various ways in recent years. These concepts are generalized to $L^2(\mathbb{R}^d)$, to lattices different from \mathbb{Z}^d , allowing the subspaces of multiresolution analysis to be generated by Riesz basis instead of orthonormal basis, admitting a finite number of scaling functions, replacing the dilation factor 2 by an integer $M \geq 2$ or by an expansive matrix $A \in GL_d(\mathbb{R})$ as long as $A \subset A\mathbb{Z}^d$. But in all these cases, the translation set is always a group. Recently, Gabardo and Nashed in [9] defined a multiresolution analysis associated with a translation set $\{0, r/N\} + 2\mathbb{Z}$, where $N \geq 1$ is an integer, $1 \leq r \leq 2N - 1$, r is an odd integer and r, N are relatively prime, a

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discrete set which is not necessarily a group. They call this an *NUMRA*. As, the case $N = 1$ reduces to the standard definition of *MRA* with dyadic dilation. *NUMRA* with multiplicity D , is called *NUMRA-D* that generalizes a particular case of a result of Calogero and Garrigos [3] on biorthogonal *MRA*'s of multiplicity D in nonstandard setup. A study with respect to *NUMRA* has been done by many authors in the references [8, 9, 10, 11, 12, 15, 16, 17].

The concept of biorthogonal wavelets plays an important role in applications. We refer to [1, 2, 4, 5, 13] for various aspects of this theory on \mathbb{R} . For the higher dimensional situation on \mathbb{R}^n , we refer to the articles [4, 5, 6, 7, 14].

In this article we construct dual wavelets which are associated with Nonuniform Multiresolution Analysis. We show that if φ and $\tilde{\varphi}$ are the scaling functions of two multiresolution analyses (MRAs) such that their translates are biorthogonal, then the associated families of wavelets are also biorthogonal. Under mild decay conditions on the scaling functions and the wavelets, we also show that the wavelets generate Riesz bases for $L^2(\mathbb{R})$.

The article is organized as follows. In Section 2, we give a brief introduction about nonuniform wavelets on \mathbb{R} . In Section 3, we find necessary and sufficient conditions for the translates of a function to form a Riesz basis for its closed linear span. In the last section, we prove that the wavelets associated with dual MRAs are biorthogonal and generate Riesz bases for $L^2(\mathbb{R})$.

2. Preliminaries

Definition 2.1. A multiresolution analysis (MRA) of $L^2(\mathbb{R})$ is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ satisfying the following properties:

- (a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
- (b) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$;
- (c) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;
- (d) $f(x) \in V_j$ if and only if $f(2x) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
- (e) There is a function $\varphi \in V_0$, called the *scaling function*, such that $\{\varphi(x - k) : k \in \mathbb{Z}\}$ forms an orthonormal basis for V_0 .

According to the standard scheme for construction of MRA-based wavelets, for each j , we define a space W_j (*wavelet space*) as the orthogonal complement of V_j in V_{j+1} , i.e., $V_{j+1} = V_j \oplus W_j$, $j \in \mathbb{Z}$, where $W_j \perp V_j$, $j \in \mathbb{Z}$. It is not difficult to see that

$$f(x) \in W_j \quad \text{if and only if} \quad f(2x) \in W_{j+1}, \quad j \in \mathbb{Z}. \quad (2.1)$$

Moreover, they are mutually orthogonal, and we have the following orthogonal decompositions:

$$L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j = V_0 \oplus \left(\bigoplus_{j \geq 0} W_j \right). \quad (2.2)$$

For an integer $N \geq 1$ and an odd integer r with $1 \leq r \leq 2N - 1$ such that r and N are relatively prime, we define

$$\Lambda = \left\{0, \frac{r}{N}\right\} + 2\mathbb{Z} = \left\{\frac{rk}{N} + 2n : n \in \mathbb{Z}, k = 0, 1\right\}. \quad (2.3)$$

It is easy to verify that Λ is not necessarily a group nor a uniform discrete set, but is the union of \mathbb{Z} and a translate of \mathbb{Z} . Moreover, the set Λ is the spectrum for the spectral set $\Gamma = \left[0, \frac{1}{2}\right) \cup \left[\frac{N}{2}, \frac{N+1}{2}\right)$ and the pair (Λ, Γ) is called a *spectral pair* [9, 10].

Definition 2.2. For an integer $N \geq 1$ and an odd integer r with $1 \leq r \leq 2N - 1$ such that r and N are relatively prime, an associated nonuniform multiresolution analysis is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ such that the following properties hold:

- (a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;
- (b) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$;
- (c) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;
- (d) $f(x) \in V_j$ if and only if $f(2Nx) \in V_{j+1}$ for all $j \in \mathbb{Z}$;
- (e) There exists a function φ in V_0 such that $\{\varphi(x - \lambda) : \lambda \in \Lambda\}$, is a complete orthonormal basis for V_0 .

It is worth noticing that, when $N = 1$, one recovers from the definition above the standard definition of one dimensional multiresolution analysis with dyadic dilation. When, $N > 1$, the dilation factor of $2N$ ensures that $2N\Lambda \subset \mathbb{Z} \subset \Lambda$.

For every $j \in \mathbb{Z}$, define W_j to be the orthogonal complement of V_j in V_{j+1} . Then we have

$$V_{j+1} = V_j \oplus W_j \quad \text{and} \quad W_\ell \perp W_{\ell'} \quad \text{if } \ell \neq \ell'. \quad (2.4)$$

It follows that for $j > J$,

$$V_j = V_J \oplus \bigoplus_{\ell=0}^{j-J} W_{j-\ell}, \quad (2.5)$$

where all these subspaces are orthogonal. By virtue of condition (b) in the Definition 2.2, this implies

$$L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j, \quad (2.6)$$

a decomposition of $L^2(\mathbb{R})$ into mutually orthogonal subspaces.

As in the standard case, one expects the existence of $2N - 1$ number of functions so that their translation by elements of Λ and dilations by the integral powers of $2N$ form an orthonormal basis for $L^2(\mathbb{R})$.

Definition 2.3. A set of functions $\{\psi_1, \psi_1, \dots, \psi_{2N-1}\}$ in $L^2(\mathbb{R})$ is said to be a *set of basic wavelets* associated with the nonuniform multiresolution analysis $\{V_j : j \in \mathbb{Z}\}$ if the family of functions $\{\psi_\ell(x - \lambda) : 1 \leq \ell \leq 2N - 1, \lambda \in \Lambda\}$ forms an orthonormal basis for W_0 .

3. Riesz Bases of Translates

Lemma 3.1. Let $\varphi, \tilde{\varphi} \in L^2(\mathbb{R})$ be given. Then $\{\varphi(x - \lambda) : \lambda \in \Lambda\}$ is biorthogonal to $\{\tilde{\varphi}(x - \lambda) : \lambda \in \Lambda\}$ if and only if

$$\sum_{\lambda \in \Lambda} \hat{\varphi}(\xi + \lambda) \overline{\hat{\tilde{\varphi}}(\xi + \lambda)} = 1 \quad \text{a.e } \xi \in \mathbb{R}.$$

Proof. For $\lambda, \sigma \in \Lambda$, it follows that $\langle \varphi(x - \lambda), \tilde{\varphi}(x - \sigma) \rangle = \delta_{\lambda, \sigma} \Leftrightarrow \langle \varphi, \tilde{\varphi}(x - \sigma) \rangle = \delta_{0, \sigma}$. Moreover,

$$\begin{aligned} \langle \varphi, \tilde{\varphi}(x - \sigma) \rangle &= \langle \hat{\varphi}, \hat{\tilde{\varphi}}(x - \sigma) \rangle \\ &= \int_{\mathbb{R}} \hat{\varphi}(\xi) \overline{\hat{\tilde{\varphi}}(\xi)} e^{-2\pi i \sigma \xi} d\xi \\ &= \int_0^{1/2} \left\{ \sum_{p \in \mathbb{Z}} \hat{\varphi}\left(\xi + \frac{p}{2}\right) \overline{\hat{\tilde{\varphi}}\left(\xi + \frac{p}{2}\right)} e^{\pi i \sigma p} \right\} e^{-2\pi i \sigma \xi} d\xi, \end{aligned}$$

and using the fact that $\{e^{-2\pi i \sigma \xi} : \sigma \in \Lambda\}$ is an orthonormal basis of $L^2[0, \frac{1}{2})$, we obtain the desired result. \square

We now provide a sufficient condition for the translates of a function to be linearly independent.

Lemma 3.2. Let $\varphi \in L^2(\mathbb{R})$. Suppose there exists two constants $A, B > 0$ such that

$$A \leq \sum_{\lambda \in \Lambda} |\hat{\varphi}(\xi + \lambda)|^2 \leq B \quad \text{for a.e } \xi \in \mathbb{R}. \quad (3.1)$$

Then $\{\varphi(x - \lambda) : \lambda \in \Lambda\}$ is linearly independent.

Proof. For the proof of the Lemma, it is sufficient to find another function say $\tilde{\varphi}$ whose translates are biorthogonal to φ . Let us define the function $\tilde{\varphi}$ by

$$\hat{\tilde{\varphi}}(\xi) = \frac{\hat{\varphi}(\xi)}{\sum_{\lambda \in \Lambda} |\hat{\varphi}(\xi + \lambda)|^2}.$$

By equation (3.1), $\tilde{\varphi}$ is well defined. Now

$$\sum_{\sigma \in \Lambda} \hat{\varphi}(\xi + \sigma) \overline{\hat{\tilde{\varphi}}(\xi + \sigma)} = \sum_{\sigma \in \Lambda} \hat{\varphi}(\xi + \sigma) \frac{\overline{\hat{\varphi}(\xi + \sigma)}}{\sum_{\lambda \in \Lambda} |\hat{\varphi}(\xi + \lambda + \sigma)|^2}$$

$$\begin{aligned}
 & \sum_{\sigma \in \Lambda} |\hat{\varphi}(\xi + \sigma)|^2 \\
 &= \frac{\sum_{\sigma \in \Lambda} |\hat{\varphi}(\xi + \sigma)|^2}{\sum_{\nu \in \Lambda} |\hat{\varphi}(\xi + \nu)|^2} \\
 &= 1.
 \end{aligned}$$

Applying Lemma 3.1, it follows that $\{\varphi(x - \lambda) : \lambda \in \Lambda\}$ is linearly independent. This completes the proof of the Lemma. \square

Lemma 3.3. *Suppose that the scaling function φ satisfies inequality (3.1). Also let $f = \sum_{\lambda \in \Lambda} h_\lambda \varphi(x - \lambda)$, where $f \in \text{span}\{\varphi(x - \lambda) : \lambda \in \Lambda\}$ and $\{h_\lambda\}$ is a finite sequence. Define the Fourier transform of h by $\hat{h}(\xi) = \sum_{\lambda \in \Lambda} h_\lambda e^{-2\pi i \lambda \xi}$. Then*

$$A \int_0^{1/2} |\hat{h}(\xi)|^2 d\xi \leq \|f\|_2^2 \leq B \int_0^{1/2} |\hat{h}(\xi)|^2 d\xi.$$

Proof. Using Plancherel's theorem, we have

$$\begin{aligned}
 \int_{\mathbb{R}} |f(x)|^2 dx &= \int_{\mathbb{R}} \left| \sum_{\lambda \in \Lambda} h_\lambda \varphi(x - \lambda) \right|^2 dx \\
 &= \int_{\mathbb{R}} \left| \sum_{\lambda \in \Lambda} h_\lambda \hat{\varphi}(\xi) e^{-2\pi i \lambda \xi} \right|^2 d\xi \\
 &= \int_{\mathbb{R}} |\hat{\varphi}(\xi)|^2 \left| \sum_{\lambda \in \Lambda} h_\lambda e^{-2\pi i \lambda \xi} \right|^2 d\xi \\
 &= \int_{\mathbb{R}} |\hat{\varphi}(\xi)|^2 |\hat{h}(\xi)|^2 d\xi \\
 &= \int_0^{1/2} \sum_{p \in \mathbb{Z}} \left| \hat{\varphi}\left(\xi + \frac{p}{2}\right) \right|^2 |\hat{h}(\xi)|^2 d\xi.
 \end{aligned}$$

Hence, using identity (3.1), we get the desired result. \square

Theorem 3.4. *Let $\{\varphi(x - \lambda) : \lambda \in \Lambda\}$ be a Riesz basis for its closed linear span. Assume that there exists a function $\{\tilde{\varphi}(x - \lambda) : \lambda \in \Lambda\}$ which is biorthogonal to $\{\varphi(x - \lambda) : \lambda \in \Lambda\}$. Then for every $f \in \overline{\text{span}}\{\varphi(x - \lambda) : \lambda \in \Lambda\}$, we have*

$$f = \sum_{\lambda \in \Lambda} \langle f, \tilde{\varphi}(x - \lambda) \rangle \varphi(x - \lambda); \quad (3.2)$$

and there exists constants $A, B > 0$ such that

$$A \|f\|_2^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \hat{\varphi}(\xi - \lambda) \rangle|^2 \leq B \|f\|_2^2. \quad (3.3)$$

Proof. We first prove (3.2) and (3.3) for any $f \in \text{span}\{\varphi(x - \lambda) : \lambda \in \Lambda\}$ and then generalize it to $\overline{\text{span}}\{\varphi(x - \lambda) : \lambda \in \Lambda\}$. Assume that $f \in \text{span}\{\varphi(x - \lambda) : \lambda \in \Lambda\}$, then there exists a finite sequence $\{h_\lambda : \lambda \in \Lambda\}$ such that $f = \sum_{\lambda \in \Lambda} h_\lambda \varphi(x - \lambda)$. Using biorthogonality, we obtain

$$\begin{aligned} \langle f, \tilde{\varphi}(x - \sigma) \rangle &= \left\langle \sum_{\lambda \in \Lambda} h_\lambda \varphi(x - \lambda), \tilde{\varphi}(x - \sigma) \right\rangle \\ &= \sum_{\lambda \in \Lambda} h_\lambda \langle \varphi(x - \lambda), \tilde{\varphi}(x - \sigma) \rangle \\ &= h_\lambda. \end{aligned}$$

This proves (3.2). In order to prove (3.3), we make use of Lemma 3.3 to get

$$B^{-1} \|f\|_2^2 \leq \int_0^{1/2} |\hat{h}(\xi)|^2 d\xi \leq A^{-1} \|f\|_2^2.$$

Therefore, using the Plancherel formula for Fourier series and the fact that $h_\lambda = \langle f, \tilde{\varphi}(x - \lambda) \rangle$, we have

$$\int_0^{1/2} |\hat{h}(\xi)|^2 d\xi = \sum_{\lambda \in \Lambda} |h_\lambda|^2 = \sum_{\lambda \in \Lambda} |\langle f, \tilde{\varphi}(x - \lambda) \rangle|^2.$$

This proves (3.3). We now generalize the results to $\overline{\text{span}}\{\varphi(x - \lambda) : \lambda \in \Lambda\}$. Let us first prove (3.3). For $f \in \overline{\text{span}}\{\tilde{\varphi}(x - \lambda) : \lambda \in \Lambda\}$, there exists a sequence $\{f_m : m \in \mathbb{Z}\}$ in $\text{span}\{\tilde{\varphi}(x - \lambda) : \lambda \in \Lambda\}$ such that $\|f_m - f\|_2 \rightarrow 0$ as $m \rightarrow \infty$. Thus for each $\lambda \in \Lambda$, we have

$$\langle f_m, \tilde{\varphi}(x - \lambda) \rangle \rightarrow \langle f, \tilde{\varphi}(x - \lambda) \rangle \quad \text{as } m \rightarrow \infty.$$

So the result holds for each f_m . Therefore,

$$\begin{aligned} \sum_{\lambda \in \Lambda} |\langle f, \tilde{\varphi}(x - \lambda) \rangle|^2 &= \sum_{\lambda \in \Lambda} \lim_{m \rightarrow \infty} |\langle f_m, \tilde{\varphi}(x - \lambda) \rangle|^2 \\ &= \lim_{m \rightarrow \infty} \sum_{\lambda \in \Lambda} |\langle f_m, \tilde{\varphi}(x - \lambda) \rangle|^2 \\ &\leq B \lim_{m \rightarrow \infty} \|f_m\|_2^2 \\ &= B \|f\|_2^2. \end{aligned}$$

Thus, the upper bound holds in (3.3). Further, we have

$$\left\{ \sum_{\lambda \in \Lambda} |\langle f_m, \tilde{\varphi}(x - \lambda) \rangle|^2 \right\}^{1/2} \leq \left\{ \sum_{\lambda \in \Lambda} |\langle f_m - f, \tilde{\varphi}(x - \lambda) \rangle|^2 \right\}^{1/2} + \left\{ \sum_{\lambda \in \Lambda} |\langle f, \tilde{\varphi}(x - \lambda) \rangle|^2 \right\}^{1/2}.$$

As the upper bound in (3.3) holds for $f_m - f$ and lower bound for each f_m , we get

$$A^{1/2} \|f_m\|_2 \leq B^{1/2} \|f_m - f\|_2 + \left(\sum_{\lambda \in \Lambda} |\langle f_m, \tilde{\varphi}(x - \lambda) \rangle|^2 \right)^{1/2},$$

from which, it follows that

$$A\|f\|_2^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \tilde{\varphi}(x - \lambda) \rangle|^2.$$

This proves (3.3). Similarly, we can prove (3.2) for $f \in \overline{\text{span}}\{\varphi(x - \lambda) : \lambda \in \Lambda\}$ and the proof of the theorem is complete. \square

4. Biorthogonal Properties of Nonuniform Wavelets

Let $\{V_j : j \in \mathbb{Z}\}$ and $\{\tilde{V}_j : j \in \mathbb{Z}\}$ be biorthogonal NUMRA's with scaling functions φ and $\tilde{\varphi}$. Then there exists integral periodic functions m_0 and \tilde{m}_0 such that $\hat{\varphi}(\xi) = m_0(\xi/2N)\hat{\varphi}(\xi/2N)$ and $\hat{\tilde{\varphi}}(\xi) = \tilde{m}_0(\xi/2N)\hat{\tilde{\varphi}}(\xi/2N)$. Suppose there exists integral periodic functions m_ℓ and \tilde{m}_ℓ , $1 \leq \ell \leq 2N - 1$ such that

$$\mathcal{M}(\xi)\overline{\tilde{\mathcal{M}}(\xi)} = I, \quad (4.1)$$

where

$$\mathcal{M}(\xi) = \begin{pmatrix} m_0\left(\frac{\xi}{2N}\right) & m_0\left(\frac{\xi}{2N} + \frac{1}{4N}\right) & \dots & m_0\left(\frac{\xi}{2N} + \frac{2N-1}{4N}\right) \\ m_1\left(\frac{\xi}{2N}\right) & m_1\left(\frac{\xi}{2N} + \frac{1}{4N}\right) & \dots & m_1\left(\frac{\xi}{2N} + \frac{2N-1}{4N}\right) \\ \vdots & \vdots & \ddots & \vdots \\ m_{2N-1}\left(\frac{\xi}{2N}\right) & m_{2N-1}\left(\frac{\xi}{2N} + \frac{1}{4N}\right) & \dots & m_{2N-1}\left(\frac{\xi}{2N} + \frac{2N-1}{4N}\right) \end{pmatrix}$$

and

$$\tilde{\mathcal{M}}(\xi) = \begin{pmatrix} \tilde{m}_0\left(\frac{\xi}{2N}\right) & \tilde{m}_0\left(\frac{\xi}{2N} + \frac{1}{4N}\right) & \dots & \tilde{m}_0\left(\frac{\xi}{2N} + \frac{2N-1}{4N}\right) \\ \tilde{m}_1\left(\frac{\xi}{2N}\right) & \tilde{m}_1\left(\frac{\xi}{2N} + \frac{1}{4N}\right) & \dots & \tilde{m}_1\left(\frac{\xi}{2N} + \frac{2N-1}{4N}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{m}_{2N-1}\left(\frac{\xi}{2N}\right) & \tilde{m}_{2N-1}\left(\frac{\xi}{2N} + \frac{1}{4N}\right) & \dots & \tilde{m}_{2N-1}\left(\frac{\xi}{2N} + \frac{2N-1}{4N}\right) \end{pmatrix}.$$

For $1 \leq \ell \leq 2N - 1$, define the associated wavelets as ψ_ℓ and $\tilde{\psi}_\ell$ by

$$\hat{\psi}_\ell(\xi) = m_\ell(\xi/2N)\hat{\varphi}(\xi/2N) \quad \text{and} \quad \hat{\tilde{\psi}}_\ell(\xi) = \tilde{m}_\ell(\xi/2N)\hat{\tilde{\varphi}}(\xi/2N).$$

Definition 4.1. A pair of NUMRA's $\{V_j : j \in \mathbb{Z}\}$ and $\{\tilde{V}_j : j \in \mathbb{Z}\}$ with scaling functions φ and $\tilde{\varphi}$ respectively are said to be dual to each other if $\{\varphi(\cdot - \lambda) : \lambda \in \Lambda\}$ and $\{\tilde{\varphi}(\cdot - \lambda) : \lambda \in \Lambda\}$ are biorthogonal.

Definition 4.2. Let φ and $\tilde{\varphi}$ be scaling functions for dual NUMRA's. For each $j \in \mathbb{Z}$, define the operators P_j and \tilde{P}_j on $L^2(\mathbb{R})$ by

$$P_j f = \sum_{\lambda \in \Lambda} \langle f, \tilde{\varphi}_{j,\lambda} \rangle \varphi_{j,\lambda} \quad \text{and} \quad \tilde{P}_j f = \sum_{\lambda \in \Lambda} \langle f, \varphi_{j,\lambda} \rangle \tilde{\varphi}_{j,\lambda},$$

respectively. Here $\varphi_{j,\lambda} = \delta_j \varphi(\cdot - \lambda)$ and $\tilde{\varphi}_{j,\lambda} = \delta_j \tilde{\varphi}(\cdot - \lambda)$. Same is the case with $\psi_{j,\lambda}$ and $\tilde{\psi}_{j,\lambda}$. It is easy to verify that these operators are uniformly bounded on $L^2(\mathbb{R})$ and both the series are convergent in $L^2(\mathbb{R})$.

Remark 4.3. The operators P_j and \tilde{P}_j satisfy the following properties.

- (a) $P_j f = f$ if and only if $f \in V_j$ and $\tilde{P}_j f = f$ if and only if $f \in \tilde{V}_j$.
 (b) $\lim_{j \rightarrow \infty} \|P_j f - f\|_2 = 0$ and $\lim_{j \rightarrow -\infty} \|\tilde{P}_j f - f\|_2 = 0$ for every $f \in L^2(\mathbb{R})$.

Theorem 4.4. Let φ and $\tilde{\varphi}$ be the scaling functions for dual NUMRA's and ψ_ℓ and $\tilde{\psi}_\ell$, $1 \leq \ell \leq 2N - 1$ be the associated wavelets satisfying (4.1). Then, we have the following

- (a) $\{\psi_{\ell,0,\lambda} : \lambda \in \Lambda\}$ is biorthogonal to $\{\tilde{\psi}_{\ell,0,\sigma} : \sigma \in \Lambda\}$,
 (b) $\langle \psi_{\ell,0,\lambda}, \tilde{\varphi}_{0,\sigma} \rangle = \langle \tilde{\psi}_{\ell,0,\lambda}, \varphi_{0,\sigma} \rangle$, for all $\lambda, \sigma \in \Lambda$.

Proof. we have

$$\begin{aligned} & \sum_{p \in \mathbb{Z}} \hat{\psi}_\ell \left(\xi + \frac{p}{2} \right) \overline{\hat{\psi}_\ell \left(\xi + \frac{p}{2} \right)} \\ &= \sum_{p \in \mathbb{Z}} \left\{ m_\ell \left(\frac{\xi}{2N} + \frac{p}{4N} \right) \hat{\varphi} \left(\frac{\xi}{2N} + \frac{p}{4N} \right) \overline{\tilde{m}_\ell \left(\frac{\xi}{2N} + \frac{p}{4N} \right) \hat{\varphi} \left(\frac{\xi}{2N} + \frac{p}{4N} \right)} \right\} \\ &= \sum_{s=0}^{2N-1} \sum_{p \in \mathbb{Z}} \left\{ m_\ell \left(\frac{\xi}{2N} + \frac{p}{2} + \frac{s}{4N} \right) \hat{\varphi} \left(\frac{\xi}{2N} + \frac{p}{2} + \frac{s}{4N} \right) \overline{\tilde{m}_\ell \left(\frac{\xi}{2N} + \frac{p}{2} + \frac{s}{4N} \right) \hat{\varphi} \left(\frac{\xi}{2N} + \frac{p}{2} + \frac{s}{4N} \right)} \right\} \\ &= \sum_{s=0}^{2N-1} \left\{ m_\ell \left(\frac{\xi}{2N} + \frac{s}{4N} \right) \overline{\tilde{m}_\ell \left(\frac{\xi}{2N} + \frac{s}{4N} \right)} \right\} \\ &= 1. \end{aligned}$$

Hence, by Lemma 3.1, $\{\psi_{\ell,0,\lambda} : \lambda \in \Lambda\}$ is biorthogonal to $\{\tilde{\psi}_{\ell,0,\lambda} : \lambda \in \Lambda\}$. This proves part (a).

To prove part (b), we have for, $\lambda, \sigma \in \Lambda$

$$\begin{aligned} \langle \psi_{\ell,0,\lambda}, \tilde{\varphi}_{0,\sigma} \rangle &= \langle \psi_\ell(x - \lambda), \tilde{\varphi}(x - \sigma) \rangle \\ &= \left\langle \hat{\psi}_\ell e^{-2\pi i \lambda}, \hat{\varphi} e^{-2\pi i \sigma} \right\rangle \\ &= \int_{\mathbb{R}} m_\ell \left(\frac{\xi}{2N} \right) \hat{\varphi} \left(\frac{\xi}{2N} \right) e^{-2\pi i \lambda \xi} \overline{\tilde{m}_0 \left(\frac{\xi}{2N} \right) \hat{\varphi} \left(\frac{\xi}{2N} \right)} e^{2\pi i \sigma \xi} d\xi \\ &= \int_0^{1/2} \sum_{p \in \mathbb{Z}} \left\{ m_\ell \left(\frac{\xi}{2N} + \frac{p}{4N} \right) \hat{\varphi} \left(\frac{\xi}{2N} + \frac{p}{4N} \right) \right. \\ &\quad \left. \times \overline{\tilde{m}_0 \left(\frac{\xi}{2N} + \frac{p}{4N} \right) \hat{\varphi} \left(\frac{\xi}{2N} + \frac{p}{4N} \right)} \right\} e^{2\pi i (\sigma - \lambda) \xi} d\xi \\ &= \int_0^{1/2} \sum_{s=0}^{2N-1} \sum_{p \in \mathbb{Z}} \left\{ m_\ell \left(\frac{\xi}{2N} + \frac{p}{2} + \frac{s}{4N} \right) \hat{\varphi} \left(\frac{\xi}{2N} + \frac{p}{2} + \frac{s}{4N} \right) \right\} \end{aligned}$$

$$\begin{aligned}
& \times \overline{\tilde{m}_0\left(\frac{\xi}{2N} + \frac{p}{2} + \frac{s}{4N}\right)} \overline{\hat{\varphi}\left(\frac{\xi}{2N} + \frac{p}{2} + \frac{s}{4N}\right)} \Big\} e^{2\pi i(\sigma-\lambda)} d\xi \\
& = \int_0^{1/2} \sum_{s=0}^{2N-1} \left\{ m_\ell\left(\frac{\xi}{2N} + \frac{s}{4N}\right) \overline{\tilde{m}_0\left(\frac{\xi}{2N} + \frac{s}{4N}\right)} \right\} e^{2\pi i(\sigma-\lambda)} d\xi \\
& = 0.
\end{aligned}$$

The dual one can also be shown equal to zero in a similar manner. This proves part (b) and hence completes the proof of the theorem. \square

Theorem 4.5. *Let φ and $\tilde{\varphi}$ and ψ_ℓ and $\tilde{\psi}_\ell$, $1 \leq \ell \leq 2N-1$ be as in Theorem 4.4. Let $\psi_0 = \varphi$ and $\tilde{\psi}_0 = \tilde{\varphi}$. Then for every $f \in L^2(\mathbb{R})$, we have*

$$P_1 f = P_0 f + \sum_{\ell=1}^{2N-1} \sum_{\lambda \in \Lambda} \langle f, \tilde{\psi}_{\ell,0,\lambda} \rangle \psi_{\ell,0,\lambda} \quad (4.2)$$

and

$$\tilde{P}_1 f = \tilde{P}_0 f + \sum_{\ell=1}^{2N-1} \sum_{\lambda \in \Lambda} \langle f, \psi_{\ell,0,\lambda} \rangle \tilde{\psi}_{\ell,0,\lambda}. \quad (4.3)$$

where the series in (4.2) and (4.3) converges in $L^2(\mathbb{R})$.

Proof. For the sake of convenience, we will only prove (4.2), as (4.3) is an easy consequence. In particular, we will prove it in the weak sense only. For this, let $f, g \in L^2(\mathbb{R})$. Then, we have

$$\begin{aligned}
& \sum_{\ell=0}^{2N-1} \sum_{\lambda \in \Lambda} \langle f, \tilde{\psi}_{\ell,0,\lambda} \rangle \overline{\langle g, \psi_{\ell,0,\lambda} \rangle} \\
& = \sum_{\ell=0}^{2N-1} \sum_{\lambda \in \Lambda} \left\{ \int_{\mathbb{R}} \hat{f}(\xi) \overline{\hat{\psi}_\ell(\xi)} e^{2\pi i \lambda \xi} d\xi \right\} \left\{ \int_{\mathbb{R}} \overline{\hat{g}(\xi)} \hat{\psi}_\ell(\xi) e^{-2\pi i \lambda \xi} d\xi \right\} \\
& = \sum_{\ell=0}^{2N-1} \sum_{\lambda \in \Lambda} \left\{ \int_0^{1/2} \sum_{p \in \mathbb{Z}} \hat{f}\left(\xi + \frac{p}{2}\right) \overline{\hat{\psi}_\ell\left(\xi + \frac{p}{2}\right)} e^{2\pi i \lambda \xi} d\xi \right\} \\
& \quad \times \left\{ \int_0^{1/2} \sum_{q \in \mathbb{Z}} \hat{g}\left(\xi + \frac{q}{2}\right) \hat{\psi}_\ell\left(\xi + \frac{q}{2}\right) e^{-2\pi i \lambda \xi} d\xi \right\} \\
& = \sum_{\ell=0}^{2N-1} \int_0^{1/2} \left\{ \sum_{p \in \mathbb{Z}} \hat{f}\left(\xi + \frac{p}{2}\right) \overline{\hat{\psi}_\ell\left(\xi + \frac{p}{2}\right)} \right\} \left\{ \sum_{q \in \mathbb{Z}} \overline{\hat{g}\left(\xi + \frac{q}{2}\right)} \hat{\psi}_\ell\left(\xi + \frac{q}{2}\right) \right\} d\xi \\
& = \int_0^{1/2} \sum_{\ell=0}^{2N-1} \left\{ \sum_{p \in \mathbb{Z}} \hat{f}\left(\xi + \frac{p}{2}\right) \overline{\tilde{m}_\ell\left(\frac{\xi}{2N} + \frac{p}{4N}\right)} \overline{\hat{\varphi}\left(\frac{\xi}{2N} + \frac{p}{4N}\right)} \right. \\
& \quad \times \left. \sum_{q \in \mathbb{Z}} \overline{\hat{g}\left(\xi + \frac{q}{2}\right)} m_\ell\left(\frac{\xi}{2N} + \frac{q}{4N}\right) \hat{\varphi}\left(\frac{\xi}{2N} + \frac{q}{4N}\right) \right\} d\xi \\
& = \int_0^{1/2} \sum_{\ell=0}^{2N-1} \left\{ \sum_{r=0}^{2N-1} \sum_{p' \in \mathbb{Z}} \hat{f}\left(\xi + \frac{p'}{2} N + \frac{r}{2}\right) \overline{\tilde{m}_\ell\left(\frac{\xi}{2N} + \frac{r}{4N} + \frac{p'}{2}\right)} \overline{\hat{\varphi}\left(\frac{\xi}{2N} + \frac{r}{4N} + \frac{p'}{2}\right)} \right.
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{s=0}^{2N-1} \sum_{q' \in \mathbb{N}_0} \overline{\hat{g}\left(\xi + \frac{q'}{2}N + \frac{s}{2}\right)} m_\ell\left(\frac{\xi}{2N} + \frac{s}{4N} + \frac{q'}{2}\right) \hat{\varphi}\left(\frac{\xi}{2N} + \frac{s}{4N} + \frac{q'}{2}\right) \Big\} d\xi \\
& = \int_0^{1/2} \sum_{r=0}^{2N-1} \sum_{p' \in \mathbb{Z}} \sum_{s=0}^{2N-1} \sum_{q' \in \mathbb{N}_0} \left\{ \sum_{\ell=0}^{2N-1} \overline{\hat{m}_\ell\left(\frac{\xi}{2N} + \frac{r}{4N}\right)} m_\ell\left(\frac{\xi}{2N} + \frac{s}{4N}\right) \right\} \\
& \quad \times \hat{f}\left(\xi + \frac{p'}{2}N + \frac{r}{2}\right) \overline{\hat{\varphi}\left(\frac{\xi}{2N} + \frac{r}{4N} + \frac{p'}{2}\right)} \hat{g}\left(\xi + \frac{q'}{2}N + \frac{s}{2}\right) \hat{\varphi}\left(\frac{\xi}{2N} + \frac{s}{4N} + \frac{q'}{2}\right) d\xi \\
& = \int_0^{1/2} \sum_{p' \in \mathbb{Z}} \sum_{q' \in \mathbb{N}_0} \sum_{s=0}^{2N-1} \hat{f}\left(\xi + \frac{p'}{2}N + \frac{s}{2}\right) \overline{\hat{\varphi}\left(\frac{\xi}{2N} + \frac{s}{4N} + \frac{p'}{2}\right)} \\
& \quad \times \overline{\hat{g}\left(\xi + \frac{q'}{2}N + \frac{s}{2}\right)} \hat{\varphi}\left(\frac{\xi}{2N} + \frac{s}{4N} + \frac{p'}{2}\right) d\xi \\
& = \sum_{s=0}^{2N-1} \int_0^{s+1/2} \sum_{p' \in \mathbb{Z}} \sum_{q' \in \mathbb{N}_0} \hat{f}\left(\xi + \frac{p'}{2}N\right) \overline{\hat{\varphi}\left(\frac{\xi}{2N} + \frac{p'}{2}\right)} \hat{g}\left(\xi + \frac{q'}{2}N\right) \hat{\varphi}\left(\frac{\xi}{2N} + \frac{p'}{2}\right) d\xi. \quad (4.4)
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\sum_{\lambda \in \Lambda} \langle f, \tilde{\varphi}_{1,\lambda} \rangle \overline{\langle g, \varphi_{1,\lambda} \rangle} &= \sum_{\lambda \in \Lambda} \left\{ \int_{\mathbb{R}} \hat{f}(\xi) \overline{\hat{\varphi}\left(\frac{\xi}{2N}\right)} e^{2\pi i \xi / 2N} d\xi \right\} \left\{ \int_{\mathbb{R}} \overline{\hat{g}(\xi)} \hat{\varphi}\left(\frac{\xi}{2N}\right) e^{-2\pi i \xi / 2N} d\xi \right\} \\
&= \int_0^{1/2} \sum_{p \in \mathbb{Z}} \hat{f}\left(\xi + \frac{p}{2}N\right) \overline{\hat{\varphi}\left(\frac{\xi}{2N} + \frac{p}{2}\right)} d\xi \int_0^{1/2} \sum_{q \in \mathbb{Z}} \overline{\hat{g}\left(\xi + \frac{q}{2}N\right)} \hat{\varphi}\left(\frac{\xi}{2N} + \frac{q}{2}\right) d\xi \\
&= \int_0^{1/2} \sum_{p \in \mathbb{Z}} \hat{f}\left(\xi + \frac{p}{2}N\right) \overline{\hat{\varphi}\left(\frac{\xi}{2N} + \frac{p}{2}\right)} d\xi \int_0^{1/2} \sum_{q \in \mathbb{Z}} \overline{\hat{g}\left(\xi + \frac{q}{2}N\right)} \hat{\varphi}\left(\frac{\xi}{2N} + \frac{q}{2}\right) d\xi \\
&= \int_0^{1/2} \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}} \hat{f}\left(\xi + \frac{p}{2}N\right) \overline{\hat{\varphi}\left(\frac{\xi}{2N} + \frac{p}{2}\right)} \overline{\hat{g}\left(\xi + \frac{q}{2}N\right)} \hat{\varphi}\left(\frac{\xi}{2N} + \frac{q}{2}\right) d\xi. \quad (4.5)
\end{aligned}$$

Combing (4.4) and (4.5), we get the desired result. \square

Theorem 4.6. Let φ and $\tilde{\varphi}$ and ψ_ℓ and $\tilde{\psi}_\ell, 1 \leq \ell \leq 2N - 1$ be as in Theorem ???. Then, for every $f \in L^2(\mathbb{R})$, we have

$$f = \sum_{\ell=1}^{2N-1} \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda} \langle f, \tilde{\psi}_{\ell,j,\lambda} \rangle \psi_{\ell,j,\lambda} = \sum_{\ell=1}^{2N-1} \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda} \langle f, \psi_{\ell,j,\lambda} \rangle \tilde{\psi}_{\ell,j,\lambda}, \quad (4.6)$$

where the series converges in $L^2(\mathbb{R})$.

Proof. Using Remark 4.3 and Theorem 4.5, proof of Theorem 4.6 follows. \square

Theorem 4.7. Let φ and $\tilde{\varphi}$ be the scaling functions for dual NUMRA's and ψ_ℓ and $\tilde{\psi}_\ell, 1 \leq \ell \leq 2N - 1$ be the associated wavelets satisfying the matrix condition (4.1). Then, the collection $\{\psi_{\ell,j,\lambda} : 1 \leq \ell \leq 2N - 1, j \in \mathbb{Z}, \lambda \in \Lambda\}$ and $\{\tilde{\psi}_{\ell,j,\lambda} : 1 \leq \ell \leq 2N - 1, j \in \mathbb{Z}, \lambda \in \Lambda\}$ are biorthogonal. Moreover, if

$$|\hat{\varphi}(\xi)| \leq C(1 + |\xi|)^{-\frac{1}{2}-\epsilon}, |\hat{\tilde{\varphi}}(\xi)| \leq C(1 + |\xi|)^{-\frac{1}{2}-\epsilon}, |\hat{\psi}_\ell(\xi)| \leq C|\xi| \text{ and } |\hat{\tilde{\psi}}_\ell(\xi)| \leq C|\xi|, \quad (4.7)$$

for some constant $C > 0$, $\epsilon > 0$ and for a.e. $\xi \in \mathbb{R}$, then $\{\psi_{\ell,j,\lambda} : 1 \leq \ell \leq 2N-1, j \in \mathbb{Z}, \lambda \in \Lambda\}$ and $\{\tilde{\psi}_{\ell,j,\lambda} : 1 \leq \ell \leq 2N-1, j \in \mathbb{Z}, \lambda \in \Lambda\}$ form Riesz bases for $L^2(\mathbb{R})$.

Proof. First we show that $\{\psi_{\ell,j,\lambda} : 1 \leq \ell \leq 2N-1, j \in \mathbb{Z}, \lambda \in \Lambda\}$ and $\{\tilde{\psi}_{\ell,j,\lambda} : 1 \leq \ell \leq 2N-1, j \in \mathbb{Z}, \lambda \in \Lambda\}$ are biorthogonal to each other. For this, we will show that for each ℓ , $1 \leq \ell \leq 2N-1$ and $j \in \mathbb{Z}$,

$$\langle \psi_{\ell,j,\lambda}, \tilde{\psi}_{\ell,j,\sigma} \rangle = \delta_{\lambda,\sigma}. \quad (4.8)$$

We have in fact already proved (4.8) for $j = 0$. For $j \neq 0$, we have

$$\langle \psi_{\ell,j,\lambda}, \tilde{\psi}_{\ell,j,\sigma} \rangle = \langle P_{-j}\psi_{\ell,0,\lambda}, P_{-j}\tilde{\psi}_{\ell,0,\sigma} \rangle = \langle \psi_{\ell,0,\lambda}, \tilde{\psi}_{\ell,0,\sigma} \rangle = \delta_{\lambda,\sigma}.$$

For fixed $\lambda, \sigma \in \Lambda$ and $j, j' \in \mathbb{Z}$ with $j < j'$, we claim that

$$\langle \psi_{\ell,j,\lambda}, \tilde{\psi}_{\ell',j',\sigma} \rangle = 0. \quad (4.9)$$

As $\psi_{\ell,0,\lambda} \in V_1$, hence $\psi_{\ell,j,\lambda} = P_{-j}\psi_{\ell,0,\lambda} \in V_{j+1} \subseteq V_{j'}$. Therefore, it is enough to show that $\tilde{\psi}_{\ell',j',\sigma}$ is orthogonal to every element of $V_{j'}$. Let $f \in V_{j'}$. Since $\{\varphi_{j',\lambda} : \lambda \in \Lambda\}$ is a Riesz basis for $V_{j'}$, hence there exists an l^2 -sequence $\{d_\lambda : \lambda \in \Lambda\}$ such that $f = \sum_{\lambda \in \Lambda} d_\lambda \varphi_{j',\lambda}$ in $L^2(\mathbb{R})$. Using part (b) of Lemma 4.1, we have

$$\langle \tilde{\psi}_{\ell',j',\sigma}, \varphi_{j',\lambda} \rangle = \langle P_{-j'}\tilde{\psi}_{\ell',0,\sigma}, P_{-j'}\varphi_{0,\lambda} \rangle = 0.$$

Therefore,

$$\langle \tilde{\psi}_{\ell',j',\sigma}, f \rangle = \left\langle \tilde{\psi}_{\ell',j',\sigma}, \sum_{\lambda \in \Lambda} d_\lambda \varphi_{j',\lambda} \right\rangle = \sum_{\lambda \in \Lambda} d_\lambda \langle \tilde{\psi}_{\ell',j',\sigma}, \varphi_{j',\lambda} \rangle = 0.$$

We now show that these two collections form Riesz bases for $L^2(\mathbb{R})$. Linear independence is clear from the fact that these collections are biorthogonal to each other. So, we have to check the frame inequalities i.e., there must exist constants $A, \tilde{A}, B, \tilde{B} > 0$ such that

$$A\|f\|_2^2 \leq \sum_{\ell=1}^{2N-1} \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda} |\langle f, \psi_{\ell,j,\lambda} \rangle|^2 \leq B\|f\|_2^2, \quad \text{for all } f \in L^2(\mathbb{R}), \quad (4.10)$$

and

$$\tilde{A}\|f\|_2^2 \leq \sum_{\ell=1}^{2N-1} \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda} |\langle f, \tilde{\psi}_{\ell,j,\lambda} \rangle|^2 \leq \tilde{B}\|f\|_2^2, \quad \text{for all } f \in L^2(\mathbb{R}). \quad (4.11)$$

Let us first check the existence of the upper bounds in (4.10) and (4.11). For this, we have

$$\begin{aligned} \sum_{\lambda \in \Lambda} |\langle f, \tilde{\psi}_{\ell,j,\lambda} \rangle|^2 &= \sum_{\lambda \in \Lambda} \left| \int_{\mathbb{R}} \hat{f}(\xi) (2N)^{-j/2} \overline{\hat{\psi}_\ell((2N)^{-j}\xi)} e^{2\pi i \lambda (2N)^{-j}\xi} d\xi \right|^2 \\ &= (2N)^{-j} \sum_{\lambda \in \Lambda} \left| \int_0^{1/2} \sum_{p \in \mathbb{Z}} \hat{f}\left(\xi + (2N)^j \frac{p}{2}\right) \overline{\hat{\psi}_\ell\left((2N)^{-j}\xi + \frac{p}{2}\right)} e^{2\pi i \lambda (2N)^{-j}\xi} d\xi \right|^2 \end{aligned}$$

$$\begin{aligned}
&= \int_0^{1/2} \left| \sum_{p \in \mathbb{Z}} \hat{f} \left(\xi + (2N)^j \frac{p}{2} \right) \overline{\hat{\psi}_\ell \left((2N)^{-j} \xi + \frac{p}{2} \right)} \right|^2 d\xi \\
&= \int_0^{1/2} \left\{ \sum_{p \in \mathbb{Z}} \left| \hat{f} \left(\xi + (2N)^j \frac{p}{2} \right) \right|^2 \left| \hat{\psi}_\ell \left((2N)^{-j} \xi + \frac{p}{2} \right) \right|^{2\delta} \right\} \\
&\quad \times \left\{ \sum_{q \in \mathbb{Z}} \left| \hat{\psi}_\ell \left((2N)^{-j} \xi + \frac{q}{2} \right) \right|^{2(1-\delta)} \right\} d\xi \\
&= \int_{\mathbb{R}} |\hat{f}(\xi)|^2 |\hat{\psi}_\ell((2N)^{-j}\xi)|^{2\delta} \sum_{q \in \mathbb{Z}} \left| \hat{\psi}_\ell \left((2N)^{-j} \xi + \frac{q}{2} \right) \right|^{2(1-\delta)} d\xi.
\end{aligned}$$

By our assumption (4.7), we have $|\hat{\psi}_\ell(\xi)| \leq C(1 + |(2N)^{-1}\xi|)^{-1/2-\epsilon}$ and therefore it follows that $\sum_{q \in \mathbb{Z}} |\hat{\psi}_\ell((2N)^{-j}\xi + q/2)|^{2(1-\delta)}$ is uniformly bounded if $\delta < 2\epsilon(1 + 2\epsilon)^{-1}$. Thus, there exists a constant $C > 0$ such that

$$\begin{aligned}
\sum_{\lambda \in \Lambda} |\langle f, \tilde{\psi}_{\ell, j, \lambda} \rangle|^2 &\leq C \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \sum_{j \in \mathbb{Z}} |\hat{\psi}_\ell((2N)^{-j}\xi)|^{2\delta} d\xi \\
&\leq C \sup \left\{ \sum_{j \in \mathbb{Z}} |\hat{\psi}_\ell((2N)^{-j}\xi)|^{2\delta} : \xi \in [1, 2N] \right\} \|f\|_2^2.
\end{aligned}$$

Moreover for $\xi \in [1, 2N]$, we have

$$\begin{aligned}
\sum_{j=-\infty}^0 |\hat{\psi}_\ell((2N)^{-j}\xi)|^{2\delta} &\leq \sum_{j=-\infty}^0 \frac{C^{2\delta}}{(1 + |(2N)^{j-1}\xi|)^{\delta(1+2\epsilon)}} \leq \sum_{j=-\infty}^0 \frac{C^{2\delta}}{(2N)^{(j-1)\delta(1+2\epsilon)}} \\
&\leq C^{2\delta} \frac{q^{\delta(1+2\epsilon)}}{1 - (2N)^{-\delta(1+2\epsilon)}}.
\end{aligned}$$

Furthermore,

$$\sum_{j=1}^{\infty} |\hat{\psi}_\ell((2N)^{-j}\xi)|^{2\delta} \leq \sum_{j=1}^{\infty} (C(2N)^{-j}|\xi|)^{2\delta} \leq C^{2\delta} \sum_{j=1}^{\infty} (2N)^{(-j+1)2\delta} = C^{2\delta} \frac{1}{1 - (2N)^{-2\delta}}.$$

Therefore, it follows that $\sup \{ \sum_{j \in \mathbb{Z}} |\hat{\psi}_\ell((2N)^{-j}\xi)|^{2\delta} : \xi \in [1, 2N] \}$ is finite. Hence, there exist $B > 0$ such that of (4.10) holds. Similarly, we can show for dual one also. The existence of lower bounds for both the cases can be shown in similar fashion. Using Theorem 4.6, it follows that if $f \in L^2(\mathbb{R})$, then (4.6) holds. Thus, we have

$$\begin{aligned}
\|f\|_2^2 &= \langle f, f \rangle \\
&= \left\langle \sum_{\ell=1}^{2N-1} \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda} \langle f, \tilde{\psi}_{\ell, j, \lambda} \rangle \psi_{\ell, j, \lambda}, f \right\rangle \\
&= \sum_{\ell=1}^{2N-1} \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda} \langle f, \tilde{\psi}_{\ell, j, \lambda} \rangle \langle \psi_{\ell, j, \lambda}, f \rangle
\end{aligned}$$

$$\begin{aligned} &\leq \left(\sum_{\ell=1}^{2N-1} \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda} |\langle f, \tilde{\psi}_{\ell,j,\lambda} \rangle|^2 \right)^{1/2} \left(\sum_{\ell=1}^{2N-1} \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda} |\langle f, \psi_{\ell,j,\lambda} \rangle|^2 \right)^{1/2} \\ &\leq (\tilde{B})^{1/2} \|f\|_2 \left(\sum_{\ell=1}^{2N-1} \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda} |\langle f, \psi_{\ell,j,\lambda} \rangle|^2 \right)^{1/2}. \end{aligned}$$

Hence,

$$\frac{1}{\tilde{B}} \|f\|_2^2 \leq \sum_{\ell=1}^{2N-1} \sum_{j \in \mathbb{Z}} \sum_{\lambda \in \Lambda} |\langle f, \psi_{\ell,j,\lambda} \rangle|^2.$$

Same is the case for dual one. This completes the proof. \square

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