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INTERPOLATIONS OF JENSEN'S INEQUALITY

S. S. DRAGOMIR, C. E. M. PEARCE AND J. PEČARIĆ

Abstract. Weighted and unweighted interpolations of general order are given for Jensen's integral inequality. Various upper-bound estimates are made for the differences between the interpolates and some convergence results derived. The results generalise and subsume a body of earlier work and employ streamlined proofs.

1. Introduction

A central tool in the applied literature is Jensen's weighted integral inequality, the basic form of which is as follows.

Theorem 1. *Let $f, g : [a, b] \to \mathbb{R}$ be measurable and denote by I the convex hull of the image of $[a, b]$ under f. Let $\phi : I \to R$ be convex and suppose that g, fg and $(\phi \circ f) \cdot g$ are all integrable on $[a, b]$. If $g(t) \geq 0$ on $[a, b]$ and $\int_a^b g(t)\,dt > 0$, then*

$$\phi\left(\frac{\int_a^b f(t)\,g(t)\,dt}{\int_a^b g(t)\,dt}\right) \leq \frac{\int_a^b (\phi \circ f)(t)\,g(t)\,dt}{\int_a^b g(t)\,dt}. \tag{1.1}$$

A convenient standardisation is suggested by the ubiquitous applications of Jensen's inequality in probability. If we define

$$p(t) := g(t) \Big/ \int_a^b g(t)dt,$$

then p is nonnegative and satisfies $\int_a^b p(t)dt = 1$ and so may be regarded as a probability density function on $[a, b]$. With this notation, (1.1) takes the simple form

$$\phi\left(\int_a^b f(t)p(t)dt\right) \leq \int_a^b (\phi \circ f)(t)p(t)dt. \tag{1.2}$$

Without loss of generality we may work with this simpler canonical form.

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Recently Dragomir and Goh [10] derived an estimate for the difference between the two sides of a multivariate version of (1.1). In our present notation, the univariate case of their estimate is

$$\begin{aligned} 0 &\leq \int_a^b (\phi \circ f)(t)\, p(t)\, dt - \phi\left(\int_a^b f(t)\, p(t)\, dt\right) \\ &\leq \int_a^b (\phi' \circ f)(t) \cdot f(t)\, p(t)\, dt \\ &\qquad - \int_a^b (\phi' \circ f)(t)\, p(t)\, dt \cdot \int_a^b f(t)\, p(t)\, dt, \end{aligned} \tag{1.3}$$

provided that all the integrals exist and ϕ is differentiable convex on $\mathbb{R}$.

In this paper we give some refinements of these results. For notational convenience, we introduce the k–variate linear integral operator

$$I_k\{\cdot\} := \int_a^b \dots \int_a^b (\cdot) p(t_1) \dots p(t_k) dt_1 \dots dt_k.$$

In this notation, (1.2) now becomes

$$\phi(I_1\{f(t)\}) \leq I_1\{(\phi \circ f)(t)\} \tag{1.4}$$

and (1.3) reads

$$\begin{aligned} 0 &\leq I_1\{(\phi \circ f)(t)\} - \phi(I_1\{f(t)\}) \\ &\leq I_1\{(\phi' \circ f)(t) \cdot f(t)\} - I_1\{(\phi' \circ f)(t)\} \cdot I_1\{f(t)\}. \end{aligned} \tag{1.5}$$

In Section 2 we interpolate (1.4), using both weighted and unweighted (that is, uniformly weighted) forms. The k–th order weighted and unweighted interpolates are respectively

$$\varphi_k(u) := I_k\left\{\phi\left(\sum_{i=1}^k u_i f(t_i)\right)\right\}$$

and

$$\varphi_k := I_k\left\{\phi\left(\frac{1}{k}\sum_{i=1}^k f(t_i)\right)\right\}.$$

Here $u = (u_1, \dots, u_k)$ is a set of probability weights, that is, each $u_i \geq 0$ and $\sum_{i=1}^k u_i = 1$, and it is envisaged that k is a fixed positive integer. When we wish to vary the order k the extended notation $u^{(k)} = (u_{1,k}, \dots, u_{k,k})$ will be used.

The basic result is Theorem 2, which generalises a number of known results. We shall see that the k–th order weighted interpolate $\varphi_k(u)$ is minimised by the unweighted interpolate φ_k, that is, when each $u_i = 1/k$. In Section 3 we give upper bounds for the difference between the first and third terms in (2.1) below. By virtue of the noted minimisation result, our estimates include as a special case an upper bound for the difference between the first and second terms in (2.1). A convergence theorem is established for the difference with $k \to \infty$.

In Section 4 we treat the sequence $(\varphi_k(u^{(k)}) - \varphi_k)_{k\geq 1}$. Some results for the sequence $(\varphi_k - \varphi_{k+1})_{k\geq 1}$ are deduced in Section 5. We conclude in Section 6 with some remarks on applications to Hadamard's inequalities.

Our arguments exploit the standardisation of p being a probability density. Suppose $Y_1, \dots, Y_k$ are independent random variables with common density function p and define $X_1, \dots, X_k$ by $X_i = f(Y_i)$ $(i = 1, \dots, k)$. We shall also write X, Y for a generic pair X_i, Y_i. Then I_k is simply the expectation operator with respect to the minimal completed sigma field $\mathbf{F}_k$ generated by $Y_1, \dots, Y_k$. Denoting the mean of X_1 by $E(X_1)$, as is customary, we then have $E(X_1) = I_1\{f(t_1)\}$. Since $\mathbf{F}_1$ is a sub sigma field of $\mathbf{F}_k$, we have also $E(X_1) = I_k\{f(t_1)\}$. We may now express (1.2), (1.5) slightly more succinctly and considerably more evocatively as respectively

$$\phi(E(X)) \leq E(\phi(X))$$

and

$$0 \leq E(\phi(X)) - \phi(E(X)) \leq E(X\phi'(X)) - E(\phi'(X))E(X).$$

We shall lean heavily on this probabilistic formulation both for compact notation within our proofs and for streamlining the algebra involved in them. The assumptions of Theorem 1 are presumed throughout without further comment and with the standardision that g is replaced by a probability density function p. A number of useful bounds arise *via* the Cauchy–Schwarz inequality. In each such connection we shall assume in addition without further comment that f^2 is integrable, and introduce

$$\sigma := \left[I_1\left\{f^2(t)\right\} - \left(I_1\left\{f(t)\right\}\right)^2\right]^{1/2}.$$

Probabilistically this states that

$$\sigma^2 = E(X^2) - [E(X)]^2 =: \ \mathrm{var}(X),$$

the variance of X. The basic probabilistic results we shall invoke are that $E(U^2) = \mathrm{var}(U)$ when $E(U) = 0$ and that for independent random variables $X_1, \dots, X_k$ and constants $u_1, \dots, u_k$, we have

$$\mathrm{var}\left(\sum_{i=1}^{k} u_i X_i\right) = \sum_{i=1}^{k} u_i^2 \mathrm{var}(X_i).$$

For notation convenience, it will be convenient to introduce into our discussion the auxiliary random variables

$$Z_1 = Z_{1,k} := \sum_{i=1}^{k} u_i X_i$$

and

$$W_k := \frac{1}{k}\sum_{i=1}^{k} X_i.$$

It is immediate that $E(Z_1) = E(W_k) = E(X)$ and that $\varphi_k(u) = E(\phi(Z_1))$ and $\varphi_k = E(\phi(W_k))$.

2. Basic Results

Our first result relates expectations involving weighted and unweighted interpolates and refines (1.1).

Theorem 2. *For each $k \geq 1$ and set of probability weights $u^{(k)}$, we have*

$$\phi\left(I_1\left\{f(t)\right\}\right) \leq \varphi_k \leq \varphi_k(u^{(k)}) \leq I_1\left\{(\phi \circ f)(t)\right\}. \tag{2.1}$$

Proof. In probabilistic terms, the result to be proved is that

$$\phi(E(X)) \leq E(\phi(W_k)) \leq E(\phi(Z_1)) \leq E(\phi(X)). \tag{2.2}$$

By Jensen's integral inequality we have

$$E\left\{\phi\left(W_k\right)\right\} \geq \phi\left(E\left\{(W_k)\right\}\right) = \phi(E(X)),$$

the first inequality in the enunciation.

For fixed k put $X_{i+k} := X_i$ and for $1 \leq j \leq k$ define

$$Z_j := \sum_{i=1}^{k} u_i X_{i+j-1},$$

which is consistent with the definition of Z_1. Then $E(Z_j) = E(X)$ and $E(\phi(Z_j)) = E(\phi(Z_1))$.

By Jensen's discrete inequality we have

$$\phi\left(\frac{1}{k}\sum_{i=1}^{k} Z_i\right) \leq \frac{1}{k}\sum_{i=1}^{k} \phi\left(Z_i\right),$$

and since $\sum_{i=1}^{k} Z_i = kW_k$, we derive

$$\phi\left(W_k\right) \leq \frac{1}{k}\sum_{j=1}^{k} \phi\left(Z_j\right).$$

Taking expectations provides

$$E\left\{\phi\left(W_k\right)\right\} \leq \frac{1}{k} E\left\{\sum_{j=1}^{k} \phi\left(Z_j\right)\right\} = E\{\phi(Z_1)\}. \tag{2.3}$$

This gives the second inequality in the enunciation.

Finally, by Jensen's discrete inequality again, we have

$$\phi(Z_1) \leq \sum_{i=1}^{k} u_i \phi(X_i).$$

Taking expectations provides the final desired inequality.

If we choose $u_{i,k+1} = 1/k$ for $1 \leq i \leq k$ and $u_{k+1,k+1} = 0$, then $\varphi_{k+1}(u)$ becomes φ_k. Thus we have $\varphi_{k+1} \leq \varphi_k$ and so $(\varphi_k)_{k\geq 1}$ is a nonincreasing sequence. We have also $\varphi_1 = E(\phi(X))$, of course.

The choices $f(t) := t$ and $p(t) := 1/(b-a)$ provide the following interpolation of the Hadamard integral inequalities, which we exhibit *in extenso.*

Corollary 1. *Suppose ϕ is convex on $[a,b]$ and that u_i $(1 \leq i \leq k)$ is a set of probability weights. Then*

$$\begin{aligned}\phi\left(\frac{a+b}{2}\right) &\leq \frac{1}{(b-a)^k}\int_a^b \cdots \int_a^b \phi\left(\frac{1}{k}\sum_{i=1}^{k} t_i\right) dt_1 \ldots dt_k \\ &\leq \frac{1}{(b-a)^k}\int_a^b \cdots \int_a^b \phi\left(\sum_{i=1}^{k} u_i t_i\right) dt_1 \ldots dt_k \\ &\leq \frac{1}{b-a}\int_a^b \phi(t)\,dt. \qquad (2.4)\end{aligned}$$

This subsumes several known results: the first inequality was proved in [11], the second in [8] and the last in [4]. We pick up these threads again in Section 6.

3. Bounds for the Difference $\varphi_k(u) - \phi(I_1\{f(t)\})$

Theorem 3. *Denote by ϕ'_+ the right derivative of ϕ on the interior $\mathring{I}$ of I. Then*

$$\begin{aligned}&0 \leq \varphi_k(u) - \phi(I_1\{f(t)\}) \\ &\leq I_k\left\{\phi'_+\left(\sum_{i=1}^{k} u_i f(t_i)\right)\sum_{j=1}^{k} u_j f(t_j)\right\} - I_1\left\{f(t)\right\}\cdot I_k\left\{\phi'_+\left(\sum_{i=1}^{k} u_i f(t_i)\right)\right\}. \qquad (3.1)\end{aligned}$$

Proof. We already have the first inequality and wish to prove the second. We may express (3.1) probabilistically as

$$0 \leq E\{\phi Z_1\} - E(X) \leq E\left\{Z_1 \phi'_+\right\} - E(X)\cdot E\left\{\phi'_+\right\}.$$

Since ϕ is convex on I,

$$\phi(x) - \phi(y) \geq \phi'_+(y)(x-y) \text{ for all } x, y \in \mathring{I}$$

and $\phi'_+(\cdot)$ is nonnegative on $\mathring{\mathrm{I}}$. Taking $x = E(X)$ and $y = Z_1$, we deduce that

$$\phi(E(X)) - \phi(Z_1) \geq \phi'_+(Z_1)[E(X) - Z_1].$$

Taking expectations yields

$$E\{\phi(Z_1)\} - \phi(E(Z_1)) \leq E\{Z_1\phi'_+(Z_1)\} - E\{Z_1\} \cdot E\{\phi'_+(Z_1)\}, \qquad (3.2)$$

whence we have the desired result.

When each $u_i = 1/k$, we may exploit symmetry in j of the summand in (3.1) to simplify the conclusion of the last theorem to

$$\begin{aligned} 0 &\leq \varphi_k(u) - \phi\Big(I_1\{f(t)\}\Big) \\ &\leq I_k\Big\{f(t_1)\phi'_+\Big(\frac{1}{k}\sum_{i=1}^{k} f(t_i)\Big)\Big\} - I_1\{f(t)\} \cdot I_k\Big\{\phi'_+\Big(\frac{1}{k}\sum_{i=1}^{k} f(t_i)\Big)\Big\}. \end{aligned}$$

The previous theorem may be extended as follows.

Theorem 4. *For $k \geq 1$ we have*

$$\begin{aligned} 0 &\leq I_k\Big\{\phi'_+\Big(\sum_{i=1}^{k} u_i f(t_i)\Big)\sum_{j=1}^{k} u_j f(t_j)\Big\} - I_1\Big\{f(t)\Big\} \cdot I_k\Big\{\phi'_+\Big(\sum_{i=1}^{k} u_i f(t_i)\Big)\Big\} \\ &\leq \sigma\sqrt{\sum_{j=1}^{k} u_j^2}\Big[I_k\Big\{\Big[\phi'_+\Big(\sum_{i=1}^{k} u_i f(t_i)\Big)\Big]^2\Big\}\Big]^{1/2}. \end{aligned} \qquad (3.3)$$

Proof. Since $E(X) = E(Z_1)$, the middle term in (3.3) can be cast probabilistically as

$$E\{\phi'_+(Z_1) \times [Z_1 - E(Z_1)]\},$$

which by the Cauchy–Schwarz inequality is less than or equal to

$$\Big\{E\Big\{[\phi'_+(Z_1)]^2\Big\}\Big\}^{1/2}\Big\{E\Big\{[Z_1 - E\{Z_1\}]^2\Big\}\Big\}^{1/2}.$$

Further $E[Z_1 - E(Z_1)] = 0$, so we have

$$E\{[Z_1 - E(Z_1)]^2\} = \mathrm{var}(Z_1) = \sum_{i=1}^{k} u_i^2 \mathrm{var}(X_i) = \sigma^2 \sum_{i=1}^{k} u_i^2,$$

and the desired result follows.

As with the previous theorem, (3.3) simplifies when $u_i = 1/k$ for each i, becoming

$$0 \le I_k\left\{\phi'_+\left(\frac{1}{k}\sum_{i=1}^{k} f(t_i)\right) f(t_1)\right\} - I_1\{f(t\} \cdot I_k\left\{\phi'_+\left(\frac{1}{k}\sum_{i=1}^{k} f(t_i)\right)\right\}$$
$$\le \sigma k^{-1/2}\left[I_k\left\{\left[\phi'_+\left(\frac{1}{k}\sum_{i=1}^{k} f(t_i)\right)\right]^2\right\}\right]^{1/2}.$$

for all $k \ge 1$.

Corollary 2. *Suppose that*

$$M := \sup_{x\in I}\left|\phi'_+(x)\right| < \infty \tag{3.4}$$

and

$$\sum_{j=1}^{k} u_{j,k}^2 \to 0 \text{ as } k \to \infty. \tag{3.5}$$

Then

$$\varphi_k\left(u^{(k)}\right) \to \phi\left(I_1\{f(t)\}\right) \text{ as } k \to \infty.$$

We note that the second assumption is automatically satisfied in the particular case $u_{j,k} = 1/k$ for $1 \le j \le k$.

The conclusion of the corollary may be expressed probabilistically as

$$E\{Z_{1,k}\} \to \phi(E(X)) \text{ as } k \to \infty.$$

4. Bounds for $\varphi_k(u) - \varphi_k$

The difference between the outermost terms in (2.1) can be used to provide a crude upper bound for $\varphi_k(u) - \varphi_k$. Here we provide tighter bounds.

Theorem 5. *For $k \ge 1$ we have*

$$0 \le \varphi_k(u) - \varphi_k$$
$$\le I_k\left\{\phi'_+\left(\sum_{i=1}^{k} u_i f(t_i)\right)\sum_{j=1}^{k} u_j f(t_j)\right\} - I_k\left\{\phi'_+\left(\sum_{i=1}^{k} u_i f(t_i)\right)\frac{1}{k}\sum_{j=1}^{k} f(t_j)\right\}.$$

Proof. By the convexity of ϕ

$$\phi(W_k) - \phi(Z_1) \ge \phi'_+(Z_1)(W_k - Z_1).$$

Taking expectations provides

$$E(\phi(Z_1)) - E(\phi(W_k)) \le E\left\{Z_1\phi'_+(Z_1)\right\} - E\left\{W_k\phi'_+(Z_1)\right\},$$

which is the desired result in probabilistic form.

The estimate is continued by the next theorem.

Theorem 6. *For each $k \geq 1$,*

$$I_k\left\{\phi'_+\left(\sum_{i=1}^k u_i f(t_i)\right)\sum_{j=1}^k u_j f\left(t_j\right)\right\}-I_k\left\{\phi'_+\left(\sum_{i=1}^k u_i f\left(t_i\right)\right)\frac{1}{k}\sum_{j=1}^k f\left(t_j\right)\right\}$$
$$\leq \sigma\sqrt{\sum_{i=1}^k \left(u_i - 1/k\right)^2} \times \left[I_k\left\{\left[\phi'_+\left(\sum_{j=1}^k u_j f\left(t_j\right)\right)\right]^2\right\}\right]^{1/2}.$$

Proof. The left–hand side of this inequality is

$$E\left\{\phi'_+\left(Z_1\right) \times \left(Z_1 - W_k\right)\right\},$$

which by the Cauchy–Schwarz inequality is less than or equal to

$$\left(E\left\{\left[\phi'_+\left(Z_1\right)\right]^2\right\}\right)^{1/2} \times \left(E\left\{\left[Z_1 - W_k\right]^2\right\}\right)^{\frac{1}{2}}.$$

Since $E(Z_1 - W_k) = 0$, we may compute the second term in parentheses as

$$\operatorname{var}\left\{\sum_{j=1}^k\left(u_j - \frac{1}{k}\right)X_j\right\} = \sum_{j=1}^k\left(u_j - \frac{1}{k}\right)^2 \operatorname{var}(X_j) = \sigma^2\sum_{j=1}^k\left(u_j - \frac{1}{k}\right)^2,$$

from which we deduce the desired estimate.

Corollary 3. *If* (3.4) *applies, then*

$$0 \leq \varphi_k\left(u\right) - \varphi_k \leq M\sigma\left[\sum_{i=1}^k\left(u_i - \frac{1}{k}\right)^2\right].$$

It follows that subject to (3.4), a sufficient condition for

$$\lim_{k\to\infty}\varphi_k\left(u\right) = \phi\left(I_1\left\{f\left(t\right)\right\}\right)$$

is that

$$\lim_{k\to\infty}\sum_{i=1}^k\left(u_{i,k} - 1/k\right)^2 = 0.$$

By virtue of the relation $\sum_{i=1}^k u_{i,k} = 1$, this is the same condition as (3.5).

5. Upper Bounds for $\varphi_k - \varphi_{k+1}$

From (2.2), we have

$$\phi(E(X)) \leq \varphi_{k+1} \leq \varphi_k \leq ... \leq E(\phi(X)) \tag{5.1}$$

for $k \geq 1$, so that the difference $\varphi_k - \varphi_{k+1}$ is nonnegative and can be ascribed a uniform upper bound $E(\phi(X)) - \phi(E(X))$ which is independent of k. The next theorem refines this to a tighter and k–dependent bound.

Theorem 7. *For each $k \geq 1$,*

$$\begin{aligned} 0 &\leq \varphi_k - \varphi_{k+1} \\ &\leq \frac{1}{k+1}\left[I_k\left\{\phi'_+\left(\frac{1}{k}\sum_{i=1}^{k} f(t_i)\right)f(t_1)\right\} - I_k\left\{\phi'_+\left(\frac{1}{k}\sum_{i=1}^{k} f(t_i)\right)\right\}I_1\{f(t)\}\right]. \end{aligned} \tag{5.2}$$

Proof. As ϕ is convex,

$$\begin{aligned} \phi\left(\frac{1}{k+1}\sum_{i=1}^{k+1} X_i\right) - \phi\left(\frac{1}{k}\sum_{i=1}^{k} X_i\right) &\geq \phi'_+\left(\frac{1}{k}\sum_{i=1}^{k} X_i\right)\left(\frac{1}{k+1}\sum_{j=1}^{k+1} X_j - \frac{1}{k}\sum_{j=1}^{k} X_j\right) \\ &= \phi'_+\left(\frac{1}{k}\sum_{i=1}^{k} X_i\right)\left[\frac{X_{k+1}}{k+1} - \frac{1}{k(k+1)}\sum_{j=1}^{k} X_j\right] \end{aligned}$$

for all $k \geq 1$.

Taking expectations provides

$$\begin{aligned} \varphi_{k+1} - \varphi_k &\geq \frac{1}{k+1}\left[E\left\{\phi'_+\left(\frac{1}{k}\sum_{i=1}^{k} X_i\right)\right\}E(X) - E\left\{\phi'_+\left(\frac{1}{k}\sum_{i=1}^{k} X_i\right)\frac{1}{k}\sum_{j=1}^{k} X_j\right\}\right] \\ &= \frac{1}{k+1}\left[E\left\{\phi'_+\left(\frac{1}{k}\sum_{i=1}^{k} X_i\right)\right\}E(X) - E\left\{X_1\phi'_+\left(\frac{1}{k}\sum_{i=1}^{k} X_i\right)\right\}\right], \end{aligned}$$

where symmetry has been coupled with a change of variables to provide the last step.

The above result is continued by the following one.

Theorem 8. *For each $k \geq 1$ we have*

$$\begin{aligned} &\frac{1}{k+1}\left[I_k\left\{\phi'_+\left(\frac{1}{k}\sum_{i=1}^{k} f(t_i)\right)f(t_1)\right\} - I_k\left\{\phi'_+\left(\frac{1}{k}\sum_{i=1}^{k} f(t_i)\right)\right\}I_1\{f(t)\}\right] \\ &\quad\leq \frac{\sigma}{\sqrt{k(k+1)}}\left[I_k\left\{\left[\phi'_+\left(\frac{1}{k}\sum_{i=1}^{k} f(t_i)\right)\right]^2\right\}\right]^{1/2}. \end{aligned}$$

Proof. The left–hand side can be expressed as

$$\frac{1}{k+1}E\Big\{\phi'_+\Big(\frac{1}{k}\sum_{i=1}^{k}X_i\Big)\Big[X_{k+1}-\frac{1}{k}\sum_{j=1}^{k}X_j\Big]\Big\},$$

which by the Cauchy–Schwarz inequality is less than or equal to

$$\Big(E\Big\{\Big[\phi'_+\Big(\frac{1}{k}\sum_{i=1}^{k}X_i\Big)\Big]^2\Big\}\Big)^{1/2}\times\Big(E\Big\{\Big[X_{k+1}-\frac{1}{k}\sum_{i=1}^{k}X_i\Big]^2\Big\}\Big)^{1/2}.$$

Since $E(X_{k+1}-(1/k)\sum_{i=1}^{k}X_i)=0$, the expression within the second pair of parentheses is

$$\operatorname{var}\Big(X_{k+1}-\frac{1}{k}\sum_{i=1}^{k}X_i\Big)=\operatorname{var}(X_{k+1})+\frac{1}{k^2}\sum_{i=1}^{k}\operatorname{var}(X_i)=\frac{(k+1)\sigma^2}{k},$$

from which we deduce the desired result.

Finally we have the following corollary.

Corollary 4. *If* (3.4) *holds, then for all* $\alpha\in[0,1)$ *we have*

$$\lim_{n\to\infty}(\varphi_n-\varphi_{n+1})\,n^{\alpha}=0.$$

Proof. By the two preceding theorems,

$$0\le\varphi_n-\varphi_{n+1}\le\frac{M\sigma}{\sqrt{n(n+1)}}\tag{5.3}$$

for all $n\ge 1$, whence the result.

6. Applications to Hadamard's Inequalities

We conclude by resuming from Corollary 1 and the observations made there. Hadamard's inequality states that if $\phi: I\to\mathbb{R}$ is convex on the interval $I=[a,b]$ of real numbers, then

$$\phi\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b\phi(x)dx\le\frac{\phi(a)+\phi(b)}{2}.\tag{6.1}$$

Denote by

$$J_k\{\cdot\}:=\frac{1}{(b-a)^k}\int_a^b\cdots\int_a^b(\cdot)\,dx_1\ldots dx_k$$

the special case of I_k when $p(x) := 1/(b-a)$ on $[a,b]$. Dragomir, Pečarić and Sándor [11] have interpolated the first inequality in (6.1) as

$$\phi\left(\frac{a+b}{2}\right) \leq J_{k+1}\left\{\phi\left(\frac{1}{k+1}\sum_{i=1}^{k+1} x_i\right)\right\} \leq J_k\left\{\phi\left(\frac{1}{k}\sum_{i=1}^{k} x_i\right)\right\} \leq ... \leq J_1\{\phi(x)\} \tag{6.2}$$

for all $k \geq 1$. This is a particular case of (5.1).

Dragomir [4] has also established a weighted interpolation, in our notation

$$\phi\left(\frac{a+b}{2}\right) \leq J_k\left\{\phi\left(\sum_{i=1} u_i x_i\right)\right\} \leq J_1\left\{\phi(x)\right\}, \tag{6.3}$$

of Hadamard's first inequality. This was subsequently improved by Dragomir and Buşe [8] who proved *inter alia* that

$$J_k\left\{\phi\left(\frac{1}{k}\sum_{i=1}^{k} x_i\right)\right\} \leq J_k\left\{\phi\left(\sum_{i=1}^{k} u_i x_i\right)\right\}. \tag{6.4}$$

This is Theorem 2 with $f(x) := x$ (and so $X_i = Y_i$).

From Corollary 2 we can obtain the following result which was derived by a different argument in [9].

Suppose $\phi : I \to \mathbb{R}$ is convex, (3.4) holds and that

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} v_i^2}{\left(\sum_{i=1}^{n} v_i\right)^2} = 0.$$

Then if $V_n := \sum_{i=1}^{n} v_i > 0$, we have

$$\lim_{n\to\infty} J_n\left\{\phi\left(\sum_{i=1}^{n} v_i x_i / V_n\right)\right\} = \phi\left(\frac{a+b}{2}\right).$$

Write h_n, $h_n(u)$ respectively for φ_n, $\varphi_n(u)$ in the case $p(x) = 1/(b-a)$ on $[a,b]$. We have the following.

Proposition 1. *Let $\phi : I \to \mathbb{R}$ be convex and suppose* (3.4) *holds. Then for all $a, b \in I$ with $a < b$, we have*

$$0 \leq h_n - h_{n+1} \leq \frac{M(b-a)}{2\sqrt{3}\sqrt{n(n+1)}}$$

for all positive integers n.

Proof. The result is (5.3) with

$$\sigma^2 = \frac{\int_a^b t^2 dt}{b-a} - \left(\frac{\int_a^b t dt}{b-a}\right)^2 = \frac{(b-a)^2}{12}.$$

The consequence

$$\lim_{n\to\infty}\left[n^{\alpha}\left(h_n-h_{n+1}\right)\right]=0 \text{ for } \alpha\in[0,1)$$

is an improvement on the results of [7].

The weighted case is embodied in the following proposition.

Proposition 2. *With the assumptions of Proposition* 1*,*

$$0\le h_n\left(u\right)-h_n\le\frac{M\left(b-a\right)}{2\sqrt{3}}\left[\sum_{i=1}^{n}\left(u_i-1/n\right)^2\right]^{1/2}$$

for all $n\ge 1$*.*

For other results connected with Hadamard's inequality see [1]−[9], where further references are given.

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School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City, MC 8001 Australia.

E-mail: cpearce@maths.adelaide.edu.au

Applied Mathematics Department, Adelaide University, Adelaide, SA 5005, Australia.

E-mail: Sever.Dragomir@vu.edu.au

Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia.

E-mail: pecaric@mahazu.hazu.hr