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ON THE STUDY OF DOUBLE FOURIER SERIES BY DOUBLE MATRIX SUMMABILITY METHOD

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Abstract. In this paper a new theorem on double matrix summability of double Fourier series has been established. This theorem is a generalization of several known and unknown results.

1. Introduction

Harmonic and (N, p_n) summability of single Fourier series have been studied by a number of researchers like lyengar (1943), Siddiqui (1948), Pati (1961), Singh (1963), Hirokawa (1968) and Izumi and Izumi (1968). In 1953, Chow for the first time studied Cesàro summability of double Fourier series. In 1958, Sharma extended the result of Chow for $(H, 1, 1)$ summability which is weaker than $(C, 1, 1)$ summability of double Fourier series. Working in the same direction in 1932, Hille and Tamarkin defined double Nörlund summability (N, p_m, q_n) of double Fourier series. After this double Nörlund summability of double Fourier series was studied by several of researchers like Tripathi and Singh (1981), Tripathi and Ojha (1995), Tripathi and Lal (1984), Mishra (1985), Lal (1992), Singh, Lal and Singh (1995) and Lal and Verma (1998). But nothing seems to have been done so far to study double Fourier series by a double factorable summability method which, as known, includes as special cases, the methods of $(C, 1, 1)$, $(H, 1, 1)$ and (N, p_m, q_n). In this paper a more general result than those of Chow (1953), Lal (1992) and Singh, Lal and Singh (1995) have been established, which include their results as particular cases.

2. Definitions and Notations

Let $f(u, v)$ be a function of (u, v), periodic with respect to u and with respect to v, in each case, with period 2π, and summable in the square $Q(-\pi, -\pi, \pi, \pi)$. The double Fourier series of a function $f(u, v)$ is

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\lambda_{mn}[a_{mn}\cos \mathrm{mu}\cos \mathrm{nv} + b_{mn}\mathrm{mu}\cos \mathrm{nv} + C_{mn}\cos \mathrm{mu}\sin \mathrm{nv} + d_{mn}\sin \mathrm{mu}\sin \mathrm{nv}]$$

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$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \lambda_{mn} A_{mn(u,v)}, \tag{2.1}$$

where

$$\lambda_{mn} = \begin{cases} 1/4 & \text{for } m=0, \quad n = (\text{or } m=n=0) \\ 1/2 & \text{for } m>0, \quad n>0 \text{ and } \ m=0, \quad n>0 \\ 1 & \text{for } m>0, \quad n>0. \end{cases}$$

and $a_{m,n} = \frac{1}{\pi^2}\int\int_Q f(u,v)\cos$ mu con nv $du\ dv$, with similar three similar expressions for b_{mn}, c_{mn} and d_{mn}, where Q denotes the fundamental square $(-\pi,\pi)\times(-\pi,\pi)$. The double series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}$ with the sequence of $(m,n)^{\text{th}}$ parital sums $\{S_{mn}\}$ is said to be summable by a double matrix summability method or summable (T,S) if $t_{m,n}$ tends to a limit s as $(m,n)\to\infty$, where the double matrix mean t_{mn} is given by

$$t_{m,n} = \sum_{i=0}^{m}\sum_{k=0}^{n} a_{m,i} b_{n,k} s_{i,k} = \sum_{i=0}^{m}\sum_{k=0}^{n} a_{m,m-i} b_{n,n-k} s_{m-i,n-k} \tag{2.2}$$

The regularity conditions of double matrix summability means are given by

$$\sum_{i=0}^{m}\sum_{k=0}^{n} a_{m,i} b_{n,k} \to 1 \qquad \text{as}(m,n)\to\infty$$
$$\lim_{m,n}\sum_{k=0}^{n} |a_{mi}\, b_{nk}| = 0, \qquad \text{for each} \quad i = 1,2,\ldots$$
$$\lim_{m,n}\sum_{i=0}^{n} |a_{mi}\, b_{nk}| = 0, \qquad \text{for each} \quad k = 1,2,\ldots$$

Three important particular cases of the double matrix summability method are given by:

(i) $(C,1,1)$ summability mean [Chow (1953)] if $a_{m,i} = \dfrac{1}{m+1}\forall m$ and $b_{n,k} = \dfrac{1}{n+1}\forall n$

(ii) $(H,1,1)$ summability mean [Sharma (1958)] if $a_{m,i} = \dfrac{1}{(m-i+1)\log m}$ and $b_{n,k} = \dfrac{1}{(n-k+1)\log n}$

(iii) (N,p_m,q_n) summability mean [Hille and Tamarkin (1932)] if $a_{m,i} = \dfrac{p_{m-i}}{P_m}$ and $b_{n,k} = \dfrac{q_{n-k}}{Q_n}$ provided $P_m = \sum_{i=0}^{m} p_i$ and $Q_n = \sum_{k=0}^{n} q_k$.

We write

$$\begin{aligned}\phi(u,v) &= \phi(x,y;u,v)\\ &= 1/4[f(x+u,y+v) + f(x+u,y-v) + f(x-u,y+v)\\ &\quad + f(x-u,y-v) - 4f(x,y)]\end{aligned}$$

$$\phi(u,v) = \int_0^u \int_0^v |\phi(s,t)| ds\, dt$$
$$\phi_1(u,t) = \int_0^u |\phi(s,t)| ds$$
$$\phi_2(s,v) = \int_0^v |\phi(s,t)| dt$$
$$\tau = \left(\frac{1}{t}\right) = \text{ integral part of } \frac{1}{t},$$
$$\sigma = \left(\frac{1}{s}\right) = \text{ integral part of } \frac{1}{s}$$

and

$$K_m(s) = \frac{1}{2\pi} \sum_{i=0}^{m} a_{m,m-i} \frac{\sin(m-i+\frac{1}{2})s}{\sin\frac{1}{2}s} \tag{2.3}$$

$$K_n(t) = \frac{1}{2\pi} \sum_{k=0}^{n} b_{n,n-k} \frac{\sin(n-k+\frac{1}{2})t}{\sin\frac{1}{2}t} \tag{2.4}$$

3. Main Theorem

Much is known for the factorable (C, 1, 1), (H,1,1) and (N, p_m, q_n) summabilities of double Fourier series. But till now no work seems to have been done on double matrix summability of double Fourier series. The object of this theorem is to extend the results of Chow (1953), Sharma (1958), Tripathi and Singh (1981) and Tripathi and Ojha (1982), Mishra (1985), Lal (1992) and Singh, Lal and Singh (1995) on (C, 1, 1), (H, 1, 1) and double Nörlund summability method to a more general class of double matrix summability of double Fourier series in two ways. In fact we shall establish the following theorem.

Theorem. *Let $\|T\| = (a_{m,j})$ and $\|S\| = (b_{n,k})$ be two inifinite triangular matrices with $a_{m,i} \geq 0$, $b_{n,k} \geq 0$. $A_{m\tau} = \sum_{i=0}^{\tau} a_{m,m-i}$, $B_{n,\eta} = \sum_{k=0}^{\eta} b_{n,n-k}$, $A_{mm} = 1$ for each $m \geq 0$ and $B_{n,n} = 1 \, \forall n \geq 0$. Let $\{a_{m,i}\}_{i=0}^{m}$ and $\{b_{n,k}\}_{k=0}^{n}$ be two real non-negative and non-decreasing sequences with respect to i and k, respectively. Assume that (T, S) is regular.*
If the conditions

$$\Phi|u,v| = \int_0^u \int_0^v |\Phi(s,t)| ds\, dt = o\left(\frac{\alpha(\frac{1}{u})u}{\log\frac{1}{u}} \frac{\beta(\frac{1}{v})v}{\log\frac{1}{v}} \right) \quad \text{as } (u,v) \to 0 \tag{3.1}$$

$$\int_0^\pi \phi_1(u,t) dt = o\left(\frac{\alpha(\frac{1}{u})u}{\log\frac{1}{u}} \right) \tag{3.2}$$

and

$$\int_0^\pi \phi_2(s,v) ds = o\left(\frac{\beta(\frac{1}{v})v}{\log\frac{1}{v}} \right) \tag{3.3}$$

hold then the double Fourier series (2.1) *is double matrix summable to* $f(x,y)$, *at the point* $(u,v)=(x,y)$, *provided* $\alpha(t)$ *and* $\beta(t)$ *are two positive monotonic non-increasing fucntions such that*

$$\int_1^m \frac{\alpha(x)A_{m,x}}{x\log x}dx = O(1) \tag{3.4}$$

$$\int_1^n \frac{\beta(y)B_{n,y}}{y\log y}dy = O(1), \tag{3.5}$$

and $\frac{\alpha(\frac{1}{s})}{\log(\frac{1}{s})}$, $\frac{\beta(\frac{1}{t})}{\log(\frac{1}{t})}$ *are bounded functions in* $[\frac{1}{m},\delta]$ *and* $[\frac{1}{n},\xi]$ *respectively.*

4. Lemmas

For the proof of our theorem the following lemmas are required.

Lemma 4.1.[Lal and Pratap (1999)] *Let* $K_m(s)$ *and* $K_n(t)$ *be given by* (2.2) *and* (2.3) *respectively then*
(i) $K_m(s)=O(m)$ *for* $0\le s\le (1/m)$
(ii) $K_n(t)=O(n)$ *for* $0\le t\le (1/n)$

Lemma 4.2. [Lal (2000)] (i) *If* $(a_{m,\mu})$ *is non-negative and non-decreasing with* μ *then for* $0\le a\le b\le\infty$ *and* $0\le s\le\pi$ *and any* m.

$$\left|\sum_{\mu=a}^{b} a_{m,m-\mu}e^{i(m-\mu)s}\right| \le O(A_{m,\sigma}) \tag{4.2.1}$$

where σ *=integral part of* $(\frac{1}{s})=[\frac{1}{s}]$

Note. In the proof of this lemma Lal [14] has also shown that

$$\frac{a_{m,m-\sigma}}{s} = O(A_{m,\sigma}) \tag{4.2.1.1}$$

Similarly,

$$\frac{b_{n,n-\tau}}{t} = O(B_{n,\tau}) \tag{4.2.1.2}$$

(ii) If $(b_{n,\nu})$ is non-negative and non-decreasing with ν then for $0\le a\le b\le\infty$, $0\le t\le\pi$ and any n

$$\left|\sum_{\nu=a}^{h} b_{n,n-\nu}e^{i(n-\nu)t}\right| \le O(B_{n,\tau}) \tag{4.2.2}$$

where τ =integral part of $(\frac{1}{t})$

Lemma 4.3. *Let* $K_m(s)$ *be given by* (2.3), *then under the condition of our theorem on* $(a_{m,\mu})$

$$K_m(s) = O\left(\frac{A_{m,\sigma}}{s}\right), \quad \text{for } 0<\frac{1}{m}\le s\le\pi$$

Proof. Since $0 < \frac{1}{m} \leq S \leq \pi$, $\sin\frac{s}{2} > \frac{s}{\pi}$, therefore for $t > 0$ and $s \leq m$. We have

$$\begin{aligned}
|K_m(s)| &= \left|\frac{1}{2\pi}\sum_{\mu=0}^{m} a_{m,m-\mu}\frac{\sin(m-\mu-\frac{1}{2})s}{\sin\frac{s}{2}}\right| \\
&\leq \frac{1}{2\pi}\left|\text{Imaginary part of } \sum_{\mu=0}^{m} a_{m,m-\mu}\frac{e^{i(m-\mu-\frac{1}{2})s}}{\sin\frac{s}{2}}\right| \\
&= O\left(\frac{1}{s}\right)\left|\sum_{\mu=0}^{m} a_{m,m-\mu}e^{i(m-\mu)s}\right|\left|e^{-i\cdot\frac{s}{2}}\right| \\
&= O\left(\frac{1}{s}\right)\left|\sum_{\mu=0}^{m} a_{m,m-\mu}e^{i(m-\mu)s}\right| \\
&= O\left(\frac{A_{m,\sigma}}{s}\right) \text{ by lemma (4.2.1)}
\end{aligned}$$

Lemma 4.4. *Let $K_n(t)$ by given by (2.4) then under the condition of our theorem $(b_{n,\nu})$,*

$$K_n(t) = O\left(\frac{B_{n,\tau}}{t}\right), \quad \text{for } 0 < \frac{1}{n} \leq t \leq \pi$$

Proof. It can be proved similar to lemma (4.3).

5. Proof of the Main Theorem

$(i,k)^{\text{th}}$ partial sum of the series (2.1) at $(u,v) = (x,y)$ is given by

$$s_{i,k} - f(x,y) = \frac{1}{4\pi^2}\left(\int_0^\pi \int_0^\pi \phi(s,t)\frac{\sin(i+\frac{1}{2})s}{\sin\frac{s}{2}}\frac{\sin(k+\frac{1}{2})t}{\sin\frac{t}{2}}\right)ds\,dt$$

Then

$$\sum_{i=0}^{m}\sum_{k=0}^{n} a_{m,i}b_{n,k}\{s_{i,k} - f(x,y)\} = \sum_{i=0}^{m}\sum_{k=0}^{n} a_{m,m-i}b_{n,n-k}\{s_{m-i,n-k} - f(x,y)\}$$

or

$$\begin{aligned}
t_{m,n} - f(x,y) &= \int_0^\pi \int_0^\pi \phi(s,t)K_m(s)K_n(t)ds\,dt \\
&= \left(\int_0^\delta\int_0^\xi + \int_0^\delta\int_\xi^\pi + \int_\delta^\pi\int_0^\xi + \int_\delta^\pi\int_\xi^\pi\right)\phi(s,t)K_m(s)K_n(t)ds\,dt \\
&\equiv I_1 + I_2 + I_3 + I_4, \quad \text{say}
\end{aligned}$$

Now, considering

$$\begin{aligned}
I_1 &= \left(\int_0^{\delta}\int_0^{\xi}\right)\phi(s,t)K_m(s)K_n(t)ds\,dt \\
&= \left(\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} + \int_{\frac{1}{m}}^{\delta}\int_0^{\frac{1}{n}} + \int_0^{\frac{1}{m}}\int_{\frac{1}{n}}^{\xi} + \int_{\frac{1}{m}}^{\delta}\int_{\frac{1}{n}}^{\xi}\right)\phi(s,t)K_m(s)K_n(t)ds\,dt \\
&= I_{1.1} + I_{1.2} + I_{1.3} + I_{1.4}, \quad \text{say}
\end{aligned}$$

where, for $s \leq \delta$, $t \leq \xi$, (3.1) hold
Now, let us consider

$$\begin{aligned}
I_{1.1} &= O(mn)\int_0^{\frac{1}{m}}\int_0^{\frac{1}{n}} |\phi(s,t)|ds\,dt \\
&= O(mn)o\left(\frac{\alpha(m)\beta(n)}{mn\log(m)\log(n)}\right) \text{ by } (3.1) \\
&= o\left(\frac{\alpha(m)\beta(n)}{\log m \log n}\right) \\
&= o(1),
\end{aligned}$$

as $(m,n) \to \infty$ ($\because \frac{\alpha(m)}{\log m} = o(1)$ as $m \to \infty$, as $\alpha(t)$ is a monotonic decreasing function of t, and similarly $\frac{\beta(n)}{\log n} = o(1)$ as $n \to \infty$).

Now

$$\begin{aligned}
I_{1.2} &= \left(\int_{\frac{1}{m}}^{\delta}\int_0^{\frac{1}{n}}\right)\phi(s,t)K_m(s)K_n(t)ds\,dt \\
&\leq \left(\int_0^{\frac{1}{n}} |K_n(t)|dt \int_{\frac{1}{m}}^{\delta} |\phi(s,t)|\frac{A_{m,[1/s]}}{s}ds\right) \text{ by lemma } (4.3) \\
&= \left(\int_0^{\frac{1}{n}} O(n)dt \int_{\frac{1}{m}}^{\delta} |\phi(s,t)|\frac{A_{m,[1/s]}}{s}ds\right) \text{ by lemma } (4.1) \\
&= O(n)\left(\int_0^{\frac{1}{n}} dt \int_{\frac{1}{m}}^{\delta} |\phi(s,t)|\frac{A_{m,[1/s]}}{s}ds\right) \\
&= O(n)\left(\int_0^{\frac{1}{n}} dt\left\{\left(\phi_1(s,t)\frac{A_{m,[1/s]}}{s}\right)_{\frac{1}{m}}^{\delta} - \int_{\frac{1}{m}}^{\delta}\phi_1(s,t)\frac{d}{ds}\left(\frac{A_{m,[1/s]}}{s}\right)ds\right\}\right) \\
&= O(n)\left(\int_0^{\frac{1}{n}}\phi_1(\delta,t)\frac{A_{m,[1/s]}}{\delta}dt\right) + O(n)m\,A_{m,m}\int_0^{\frac{1}{n}}\phi_1\left(\frac{1}{m},t\right)dt \\
&\quad + O(n)\left(\int_0^{\frac{1}{n}} dt \int_{\frac{1}{m}}^{\delta} |\phi_1(s,t)|\frac{d}{ds}\left(\frac{A_{m,[1/s]}}{s}\right)ds\right) \\
&= I_{1.2.1} + I_{1.2.2} + I_{1.2.3}, \quad \text{say}
\end{aligned}$$

Now

$$\begin{aligned}
I_{1.2.1} &= O(n)\int_0^{\frac{1}{n}} \Phi_1(\delta,t)dt \\
&= O(n)\Phi\Big(\delta,\frac{1}{n}\Big) \\
&= O(n)o\bigg(\frac{\alpha(\frac{1}{\delta})\delta}{\log\frac{1}{\delta}}\cdot\frac{\beta(n)\frac{1}{n}}{\log n}\bigg) \\
&= o\bigg(\frac{\beta(n)}{\log n}\bigg) \\
&= o(1),\ \text{as } (m,n)\to\infty, \text{ by hypothesis of the theorem.}
\end{aligned}$$

Next

$$\begin{aligned}
I_{1.2.2} &= O(mn)\int_0^{\frac{1}{n}} \Phi_1(\frac{1}{m},t)dt \\
&= O(mn)\Phi(\frac{1}{m},1/n) \\
&= O(mn)o\bigg(\frac{\alpha(m)}{m\log m}\cdot\frac{\beta(n)}{n\log n}\bigg) \\
&= o\bigg(\frac{\alpha(m)}{\log m}\cdot\frac{\beta(n)}{logn}\bigg) \\
&= o(1),\quad \text{as}\quad (m,n)\to\infty,\quad \text{by hypothesis of the theorem.}
\end{aligned}$$

Lastly

$$\begin{aligned}
I_{1.2.3} &= O(n)\int_0^{\frac{1}{n}} dt\int_{\frac{1}{m}}^{\delta}\Phi_1(s,t)\frac{A_{m,[1/s]}}{s^2}ds + O(n)\int_0^{\frac{1}{n}} dt\int_{\frac{1}{m}}^{\delta}\frac{\Phi_1(s,t)}{s}\frac{d}{ds}(A_{m,[1/s]})ds \\
&= I_{1.2.3.1} + I_{1.2.3.2}
\end{aligned}$$

Now

$$\begin{aligned}
I_{1.2.3.1} &= O(n)\int_0^{\frac{1}{n}} dt\int_{\frac{1}{m}}^{\delta}\Phi_1(s,t)\frac{A_{m,[1/s]}}{s^2}ds \\
&\le O(n)\int_{\frac{1}{m}}^{\delta}\bigg\{\int_0^{\frac{1}{n}}\Phi_1(s,t)dt\bigg\}\frac{A_{m,[1/s]}}{s^2}ds \\
&= O(n)\int_{\frac{1}{m}}^{\delta}\Phi(s,\frac{1}{n})\frac{A_{m,[1/s]}}{s^2}ds \\
&= O(n)\int_{\frac{1}{m}}^{\delta} o\bigg(\frac{\alpha(\frac{1}{s})s}{\log\frac{1}{s}}\cdot\frac{\beta(n)}{n\log n}\bigg)\frac{A_{m,[1/s]}}{s^2}ds \\
&= o\bigg(\frac{\beta(n)}{\log n}\bigg)\int_{\frac{1}{m}}^{\delta}\frac{\alpha(\frac{1}{s})A_{m,[1/s]}}{s\log[1/s]}ds
\end{aligned}$$

$$\begin{aligned}
&= o\left(\frac{\beta(n)}{\log n}\right)\int_{\frac{1}{\delta}}^{m}\frac{\alpha(x)A_{m,x}}{x\,\log x}dx\\
&= o\left(\frac{\beta(n)}{\log n}\right)\int_{1}^{m}\frac{\alpha(x)A_{m,x}}{x\,\log x}dx\\
&= o\left(\frac{\beta(n)}{\log n}\right)O(1) \qquad \text{by (3.4)}\\
&= o(1), \quad \text{as} \quad (m,n)\to\infty.
\end{aligned}$$

Nextly,

$$\begin{aligned}
I_{1.2.3.2} &= O(n)\int_0^{\frac{1}{n}}dt\int_{\frac{1}{m}}^{\delta}\frac{\Phi_1(s,t)}{s}\frac{d}{ds}(A_{m,[1/s]})ds\\
&\le O(n)\int_{\frac{1}{m}}^{\delta}\left\{\int_0^{\frac{1}{n}}\Phi_1(s,t)dt\right\}\frac{1}{s}\frac{d}{ds}(A_{m,[1/s]})ds\\
&= O(n)\int_{\frac{1}{m}}^{\delta}\Phi(s,1/n)\frac{1}{s}\frac{d}{ds}(A_{m,[1/s]})ds\\
&= O(n)\int_{\frac{1}{m}}^{\delta}o\left(\frac{\alpha(\frac{1}{s})s}{\log\frac{1}{s}}\cdot\frac{\beta(n)}{n\,\log n}\right)\frac{d}{ds}(A_{m,[1/s]})ds\\
&= o\left(\frac{\beta(n)}{\log n}\right)\int_{m}^{\frac{1}{\delta}}\frac{\alpha(x)}{\log x}\frac{d}{dx}(A_{m,[1/x]})dx\\
&= o\left(\frac{\beta(n)}{\log n}\right)\int_{m}^{\frac{1}{\delta}}O(1)\frac{d}{dx}(A_{mx})dx\\
&= o\left(\frac{\beta(n)}{\log n}\right)|A_{mx}|_m^{\frac{1}{\delta}}\\
&= o\left(\frac{\beta(n)}{\log n}\right)O(1) = o(1).
\end{aligned}$$

Thus,

$$I_{1.2} = o(1) \text{ as } (m,n)\to\infty.$$

Similarly $I_{1.3} = o(1)$ as $(m,n)\to\infty$.
Now,

$$\begin{aligned}
I_{1.4} &= \left(\int_{\frac{1}{m}}^{\delta}\int_{\frac{1}{n}}^{\xi}|\phi(s,t)|\frac{A_{m,[1/s]}}{s}\cdot\frac{B_{n\cdot[1/t]}}{t}dt\,ds\right)\\
&= O\left(\frac{B_{n,[1/\xi]}A_{m,[1/\delta]}\Phi(\delta,\xi)}{\delta\xi}-\frac{m\,B_{n,[1/\xi]}A_{mm}\Phi(1/m,\,\xi)}{\xi}\right.\\
&\quad -\frac{B_{n,[1/\xi]}}{\xi}\cdot\int_{\frac{1}{m}}^{\delta}\Phi(s,\xi)\frac{d}{ds}\left(\frac{A_{m,[1/s]}}{s}\right)ds
\end{aligned}$$

$$-\frac{n\,B_{nn}A_{m,[1/\delta]}\Phi(\delta,\,1/n)}{\delta}+mn\,B_{nn}A_{mm}\Phi(1/m,\,1/n)$$
$$+n\,B_{nn}\int_{\frac{1}{m}}^{\delta}\Phi(s,\,1/n)\frac{d}{ds}\left(\frac{A_{m,[1/s]}}{s}\right)ds$$
$$+\frac{A_{m,[1/\delta]}}{\delta}\int_{\frac{1}{n}}^{\delta}\Phi_2(s,t)\frac{d}{dt}\left(\frac{B_{n,[1/t]}}{t}\right)dt-m\,A_{mm}\int_{\frac{1}{n}}^{\xi}\Phi_2(1/m,\,t)\frac{d}{dt}\left(\frac{B_{n,[1/t]}}{t}\right)dt$$
$$-\int_{\frac{1}{m}}^{\delta}\int_{\frac{1}{n}}^{\xi}\Phi(s,t)\frac{d}{ds}\left(\frac{A_{m,[1/s]}}{s}\right)\frac{d}{dt}\left(\frac{B_{n,[1/t]}}{t}\right)dt\,ds\Bigg)$$
$$=I_{1.4.1}+I_{1.4.2}+I_{1.4.3}+I_{1.4.4}+I_{1.4.5}+I_{1.4.6}+I_{1.4.7}+I_{1.4.8}+I_{1.4.9},\quad \text{say}$$

Let us consider

$$\begin{aligned}
I_{1.4.1}&=\frac{B_{n,[1/\xi]}A_{m,[1/\delta]}\Phi(\delta,\,\xi)}{\delta\xi}\\
&=\frac{B_{n,[1/\xi]}A_{m,[1/\delta]}}{\delta\xi}o\left(\frac{\alpha(\frac{1}{\delta})\delta}{\log(\frac{1}{\delta})}\cdot\frac{\beta(\frac{1}{\xi})\xi}{\log(\frac{1}{\xi})}\right)\\
&=o\left(\frac{B_{n,[1/\xi]}}{\xi}\cdot\frac{A_{m,[1/\delta]}}{\delta}\cdot\frac{\alpha(\frac{1}{\delta})\delta}{\log(\frac{1}{\delta})}\cdot\frac{\beta(\frac{1}{\xi})\xi}{\log(\frac{1}{\xi})}\right)\\
&=o(1),\quad as(m,n)\to\infty,\quad \text{by the regularity of}(T,S)
\end{aligned}$$

Nextly

$$\begin{aligned}
I_{1.4.2}&=\frac{-mB_{n,[1/\xi]}A_{mm}\Phi(1/m,\,\xi)}{\xi}\\
&=o\left(m,B_{n,[1/\xi]}\left(\frac{\alpha(m)}{m\log m}\cdot\frac{\beta(\frac{1}{\xi})\xi}{\log\xi}\right)\right)\qquad(\because A_{m,m}=1)\\
&=o\left(B_{n,[1/\xi]}\frac{\alpha(m)}{\log\,m}\cdot\frac{\beta(\frac{1}{\xi})\xi}{\log\,\xi}\right)\\
&=o(1)\left(\frac{\alpha(m)}{\log m}\right)\\
&=o(1)\quad \text{as }(m,n)\to\infty,\quad \text{by the hypothesis of the theorem.}
\end{aligned}$$

Now let us consider

$$\begin{aligned}
I_{1.4.3}&=-\frac{B_{n,[1/\xi]}}{\xi}\int_{\frac{1}{m}}^{\delta}\Phi(s,\xi)\frac{d}{ds}\left(\frac{A_{m,[1/s]}}{s}\right)ds\\
&=o\left(\frac{B_{n,[1/\xi]}}{\xi}\right)\int_{\frac{1}{m}}^{\delta}\left(\frac{\alpha(\frac{1}{s})s}{\log\,\frac{1}{s}}\cdot\frac{\beta(\frac{1}{\xi})\xi}{\log\,\frac{1}{\xi}}\right)\left(\frac{A_{m,[1/s]}}{s^2}\right)ds\\
&\quad+o\left(\frac{B_{n\cdot[1/s]}}{\xi}\right)\int_{\frac{1}{m}}^{\delta}o\left(\frac{\alpha(\frac{1}{s})s}{\log\frac{1}{s}}\,\frac{\beta(\frac{1}{\xi})\xi}{\log\,\frac{1}{\xi}}\right)\frac{1}{s}\frac{d}{ds}(A_{m,[1/s]})ds\\
&=I_{1.4.3.1}+I_{1.4.3.2},\quad \text{say}
\end{aligned}$$

Let us consider

$$\begin{aligned}
I_{1.4.3.1} &= o\bigg(\frac{B_{n,[1/\xi]}\beta(\frac{1}{\xi})}{\log\frac{1}{\xi}}\bigg)\int_{\frac{1}{m}}^{\delta}\frac{\alpha(\frac{1}{s})}{s\log\frac{1}{s}}(A_{m,[1/s]})ds\\
&= o\bigg(\frac{B_{n,[1/\xi]}}{\log\frac{1}{\xi}}\,\beta\left(\frac{1}{\xi}\right)\frac{m\,\alpha(m)}{\log m}A_{mm}\bigg)\int_{\frac{1}{m}}^{\delta}ds\\
&= o\bigg(\frac{B_{n,[1/\xi]}}{\log\frac{1}{\xi}}\,\beta\left(\frac{1}{\xi}\right)\frac{m\,\alpha(m)}{\log m}A_{mm}(1/m)\bigg)\\
&= o\bigg(\frac{B_{n,[1/\xi]}\beta(\frac{1}{\xi})}{\log\frac{1}{\xi}}\frac{\alpha(m)}{\log m}\bigg)= o(B_{n,[1/\xi]})o\bigg(\frac{\alpha(m)}{\log m}\bigg)\\
&= o(1), \qquad \text{as } (m,n)\to\infty.
\end{aligned}$$

Further,

$$\begin{aligned}
I_{1.4.3.2} &= O\bigg(\frac{B_{n,[1/\xi]}}{\xi}\bigg)\int_{1/m}^{\delta}o\bigg(\frac{\alpha(\frac{1}{s})s}{\log\frac{1}{s}}\frac{\beta(\frac{1}{\xi})\xi}{\log\frac{1}{\xi}}\bigg)\frac{1}{s}\frac{1}{ds}(A_{m,[1/s]})ds\\
&= o\bigg(\frac{B_{n,[1/\xi]\beta(\frac{1}{\xi})}}{\log\frac{1}{\xi}}\bigg)\int_{1/m}^{\delta}\frac{\alpha(\frac{1}{s})}{\log\frac{1}{s}}\frac{d}{ds}(A_{m,[1/s]})ds\\
&= o(1)\int_{1/m}^{\delta}O(1)\frac{d}{ds}(A_{m,[1/s]})ds, \quad \text{by hypothesis of the theorem}\\
&= o(1)\int_{1/\delta}^{m}\frac{d}{dx}(A_{m,x})dx\\
&= o(1)[A_{m,x}]_{1/\delta}^{m}\\
&= o(1), \quad \text{as } (m,n)\to\infty.
\end{aligned}$$

Thus we get

$$I_{1.4.3} = o(1), \quad \text{as } (m,n)\to\infty.$$

Similar to $I_{1.4.2}$

$$I_{1.4.4} = o(1), \quad \text{as } (m,n)\to\infty.$$

Further

$$\begin{aligned}
I_{1.4.5} &= mn\,B_{nn}A_{mm}\Phi(1/m,\,1/n)\\
&= o\bigg(mn\,A_{mm}B_{nn}\frac{\alpha(m)}{m\log m}\frac{\beta(n)}{n\log n}\bigg)\\
&= o\bigg(\frac{mn\,\alpha(m)\,\beta(n)}{m\log mn\log n}\bigg)\\
&= o(1), \quad \text{as } (m,n)\to\infty.
\end{aligned}$$

Let us consider

$$\begin{aligned}
I_{1.4.6} &= n\,B_{nn}\int_{1/m}^{\delta}\Phi(s,1/n)\frac{d}{ds}\left(\frac{A_{m,[1/s]}}{s}\right)ds\\
&= n\,B_{nn}\int_{1/m}^{\delta}\Phi(s,\,1/n)\frac{A_{m,[1/s]}}{S^2}ds + n\,B_{nn}\int_{1/m}^{\delta}\Phi(s,\,1/n)\frac{1}{s}\frac{d}{ds}A_{m,[1/s]}\\
&= I_{1.4.6.1}+I_{1.4.6.2} \quad \text{say}
\end{aligned}$$

Taking

$$\begin{aligned}
I_{1.4.6.1} &= n\,B_{nn}\int_{1/m}^{\delta}\Phi(s,\,\frac{1}{n})\frac{A_{m,[1/s]}}{s^2}ds\\
&= n\,B_{nn}\int_{1/m}^{\delta}o\left(\frac{\alpha(\frac{1}{s})s}{\log\frac{1}{s}}\frac{\beta(n)}{n\,\log\,n}\right)\cdot\frac{A_{m,[1/s]}}{s^2}ds\\
&= o\left(\frac{n\,B_{nn}\cdot\beta(n)}{n\,\log\,n}\right)\int_{1/m}^{\delta}\frac{\alpha(\frac{1}{s})}{\log\frac{1}{s}}\cdot\frac{A_{m,[1/s]}}{s}ds\\
&= o\left(\frac{\beta(n)}{\log\,n}\right)\int_{1/m}^{\delta}\frac{\alpha(\frac{1}{s})}{\log\frac{1}{s}}\frac{A_{m,[1/s]}}{s}ds\\
&= o(1), \quad \text{as similar to } I_{1.2.3.1}
\end{aligned}$$

Further

$$\begin{aligned}
I_{1.4.6.2} &= n\,B_{nn}\int_{1/m}^{\delta}\Phi\left(s,\frac{1}{n}\right)\frac{1}{s}\frac{d}{ds}A_{m,[1/s]}ds\\
&= n\,B_{nn}\int_{1/m}^{\delta}\left(\frac{\alpha(\frac{1}{s})s}{\log\frac{1}{s}}\frac{\beta(n)}{n\log n}\right)\frac{1}{s}\frac{d}{ds}A_{m,[1/s]}ds\\
&= o\left(\frac{\beta(n)}{\log n}\int_{1/m}^{\delta}\frac{\alpha(\frac{1}{s})}{\log\frac{1}{s}}O(1)ds\right)\\
&= o\left(\frac{\beta(n)}{\log n}\int_{m}^{1/\delta}\frac{\alpha(x)}{\log\,x}\left(\frac{-dx}{x^2}\right)\right)\\
&= o\left(\frac{\beta(n)}{m\,\log\,n}\int_{1/\delta}^{m}\frac{\alpha(x)}{x\,\log\,x}dx\right)\\
&= o\left(\frac{\beta(n)}{m\,\log\,n}O(1)\right)=o(1).
\end{aligned}$$

Thus we get

$$I_{1.4.6}=o(1), \quad \text{as } (m,n)\to\infty$$

As similar to $I_{1.4.3}$,

$$I_{1.4.7}=o(1), \quad \text{as } (m,n)\to\infty$$

As similar to $I_{1.4.6}$

$$I_{1.4.8} = o(1), \quad \text{as } (m,n) \to \infty$$

Lastly

$$I_{1.4.9} = O\left(\int_{1/m}^{\delta}\int_{1/n}^{\xi} \Phi(s,t)\left(\frac{A_{m,[1/s]}}{s^2} + \frac{1}{s}\frac{d}{ds}(A_{m,[1/s]})\right)\times\left(\frac{B_{n,[1/t]}}{t^2} + \frac{1}{t}\frac{d}{dt}(B_{n,[1/t]})\right)dtds\right)$$
$$= I_{1.4.9.1} + I_{1.4.9.2} + I_{1.4.9.3} + I_{1.4.9.4}.$$

$$\begin{aligned}
I_{1.4.9.1} &= O\left(\int_{1/m}^{\delta}\int_{1/n}^{\xi} \Phi(s,t)\left(\frac{A_{m,[1/s]}}{s^2}\frac{B_{m,[1/t]}}{t^2}\right)dtds\right) \\
&= o\left(\int_{1/m}^{\delta}\int_{1/n}^{\xi}\left(\frac{\alpha(\frac{1}{s})s}{\log\frac{1}{s}}\frac{\beta(\frac{1}{t})t}{\log\frac{1}{t}}\right)\frac{A_{m,[1/s]}}{s^2}\frac{B_{n,[1/s]}}{t^2}dtds\right) \\
&= o\left(\int_{1/m}^{\delta}\int_{1/n}^{\xi}\frac{\alpha(\frac{1}{s})}{\log\frac{1}{s}}\frac{A_{m,[1/s]}}{s}\frac{\beta(\frac{1}{t})}{\log\frac{1}{t}}\frac{B_{n,[1/s]}}{t}dtds\right) \\
&= o\left(\int_{1/\delta}^{m}\frac{\alpha(x)A_{m,x}}{x\log x}dx\int_{1/\xi}^{n}\frac{\beta(y)B_{n,y}}{y\log y}dy\right) = o(1)
\end{aligned}$$

$$\begin{aligned}
I_{1.4.9.2} &= O\left(\int_{1/m}^{\delta}\int_{1/n}^{\xi}\Phi(s,t)\frac{A_{m,[1/s]}}{s^2}\frac{1}{t}\frac{d}{dt}(B_{n,[1/t]})dtds\right) \\
&= O\left(\int_{1/m}^{\delta}\int_{1/n}^{\xi}\left(\frac{a(\frac{1}{s})s}{\log\frac{1}{s}}\frac{\beta(\frac{1}{t})t}{\log\frac{1}{t}}\right)\frac{A_{m,[1/s]}}{s^2}\frac{1}{t}\frac{d}{dt}(B_{n,[1/t]})dtds\right) \\
&= o\left(\int_{1/m}^{\delta}\int_{1/n}^{\xi}\frac{\alpha(\frac{1}{s})}{\log\frac{1}{s}}\frac{A_{m,[1/s]}}{s}\frac{\beta(\frac{1}{t})}{\log\frac{1}{t}}\frac{d}{dt}(B_{n,[1/t]})dtds\right) \\
&= o\left(\int_{1/\delta}^{m}\frac{\alpha(x)}{\log x}\frac{A_{m,x}}{x}dx\int_{1/n}^{\xi}\frac{\beta(\frac{1}{t})}{\log\frac{1}{t}}\frac{d}{dt}(B_{n,[1/t]})dtds\right) \\
&= o\left(\int_{1/m}^{\delta}\frac{\alpha(x)A_{m,x}}{x\log x}dx\right)\int_{1/n}^{\xi}\frac{\beta(\frac{1}{t})}{\log\frac{1}{t}}\frac{d}{dt}\Big(B_{n,[1/t]}\Big)dtds \\
&= O(1)\int_{1/n}^{\xi}O(1)\frac{d}{dt}\Big(B_{n,[1/t]}\Big)dt, \quad \text{by the hypothesis of the theorem} \\
&= o(1)\Big(B_{n,[1/t]}\Big)_{1/n}^{\xi} \\
&= o(1), \qquad as, (m,n) \to \infty
\end{aligned}$$

Similarly $I_{1.4.9.3} = o(1)$

$$I_{1.4.9.3} = O\left(\int_{1/m}^{\delta}\int_{1/n}^{\xi}\Phi(s,t)\frac{1}{s}\frac{d}{ds}\Big(A_{m,[1/s]}\Big)\right)\frac{1}{t}\frac{d}{dt}\Big(B_{n,[1/t]}\Big)dtds$$

$$
\begin{aligned}
&= o\bigg(\int_{1/m}^{\delta}\int_{1/n}^{\xi}\bigg(\frac{\alpha(\frac{1}{s})\beta(\frac{1}{t})t}{\log\frac{1}{s}\log\frac{1}{t}}\bigg)\frac{1}{s}\frac{d}{ds}\Big(A_{m,[1/s]}\Big)\bigg)\frac{1}{t}\frac{d}{dt}\Big(B_{n,[1/t]}\Big)dtds\\
&= o\bigg(\int_{1/m}^{\delta}\int_{1/n}^{\xi}\bigg(\frac{\alpha(\frac{1}{s})\beta(\frac{1}{t})t}{\log\frac{1}{s}\log\frac{1}{t}}\bigg)\frac{d}{ds}(A_{m,[1/s]})\bigg)\frac{d}{dt}(B_{n,[1/t]})dtds\\
&= o(1)\int_{1/m}^{\delta}\frac{\alpha(\frac{1}{s})}{\log\frac{1}{s}}\frac{d}{ds}\Big(A_{m,[1/s]}\Big)ds\int_{1/n}^{\xi}\frac{\beta(1/t)}{\log\frac{1}{t}}\frac{d}{dt}\Big(B_{n,[1/t]}\Big)dt\\
&= o(1)\int_{1/m}^{\delta}O(1)\frac{d}{ds}\Big(A_{m,[1/s]}\Big)ds\int_{1/n}^{\xi}O(1)\frac{d}{dt}\Big(B_{n,[1/t]}\Big)dtds,\\
&\qquad \text{by the hypothesis of the theorem}\\
&= o(1)\Big[A_{m,[1/s]}\Big]_{1/m}^{\delta}\Big[B_{n,[1/t]}\Big]_{1/n}^{\xi}\\
&= o(1),\ \text{as } (m,n)\to\infty.
\end{aligned}
$$

Thus,

$$I_{1.4} = o(1),\ \text{as } (m,n)\to\infty.$$

The above estimations, we get

$$I_1 = o(1) \text{ as } (m,n)\to\infty$$

Now, $m^{-1} < \delta < \pi$, $n^{-1} < \xi < \pi$. Then we obtain

$$
\begin{aligned}
I_3 &\le \bigg(\int_{\delta}^{\pi}\int_{0}^{\xi}\bigg)|\phi(s,t)|K_m(s)K_n(t)ds\,dt\\
&= \int_{\delta}^{\pi}|K_m(s)|ds\int_{0}^{1/n}|\phi(s,t)|\,|K_n(t)|dt + \int_{\delta}^{\pi}|K_m(s)|ds\int_{1/n}^{\xi}|\phi(s,t)|\,|K_n(t)|dt\\
&= I_{3.1} + I_{3.2}
\end{aligned}
$$

Taking

$$
\begin{aligned}
I_{3.1} &= \int_{\delta}^{\pi}|K_m(s)|ds\int_{0}^{1/n}|\phi(s,t)|\,|K_n(t)|dt\\
&= O(n)\bigg(\int_{\delta}^{\pi}|K_m(s)|ds\int_{0}^{1/n}|\phi(s,t)|dt\bigg) \quad \text{by Lemma (4.1) (ii)}\\
&= O(n)\int_{\delta}^{\pi}\frac{A_{m,\sigma}}{s}ds\int_{0}^{1/n}|\phi(s,t)|dt \quad \text{by Lemma (4.3)}\\
&= O(n)\int_{0}^{\pi}\Phi_2\bigg(s,\frac{1}{n}\bigg)ds\\
&= O(n)o\bigg(\frac{\beta(n)}{n\log n}\bigg)\\
&= o\bigg(\frac{\beta(n)}{\log n}\bigg)\\
&= o(1),\ \text{as } n\to\infty.
\end{aligned}
$$

Further

$$\begin{aligned}
I_{3.2} &= \int_\delta^\pi \Big|K_m(s)\Big| ds \int_{1/n}^{\xi} |\phi(s,t)| \Big|K_n(t)\Big| dt \\
&= \left[\int_\delta^\pi \frac{A_{m,\sigma}}{s} ds \int_{1/n}^{\xi} \phi(s,t) \frac{B_{n,\tau}}{t} dt\right] \quad \text{by Lemma (4.3) and (4.4)} \\
&= O\left[\int_\delta^\pi ds \left\{ \left[\Phi_2(s,t)\frac{B_{n,\tau}}{t}\right]_{1/n}^{\xi} - \int_{1/n}^{\xi} \Phi_2(s,t) \frac{d}{dt}\left(\frac{B_{n,\tau}}{t}\right) dt\right\}\right] \\
&= O\left[\int_\delta^\pi ds \left\{ \Phi_2(s,\xi)\frac{B_{n,[1/\xi]}}{\xi} - \Phi_2(s,1/n) n\, B_{nn}\right\}\right] + O\left[\int_\delta^\pi ds \int_{1/n}^{\xi} \Phi_2(s,t)\frac{d}{dt}\left(\frac{B_{n,\tau}}{t}\right)dt\right] \\
&= O\left[\int_\delta^\pi \Phi_2(s,\xi)\frac{B_{n,[1/\xi]}}{\xi} ds\right] + O(n)\int_0^\pi \Phi_2(s,1/n)ds + O\left[\int_\delta^\pi ds \int_{1/n}^{\xi} \Phi_2(s,t)\frac{d}{dt}\left(\frac{B_{n,\tau}}{t}\right)dt\right] \\
&= O\left(\frac{B_{n,[1/\xi]}}{\xi}\right)\int_\delta^\pi \Phi_2(s,\xi)ds + O(n) o\left(\frac{\beta_{(n)}}{n\log n}\right) + O\left[\int_\delta^\pi ds \int_{1/n}^{\xi} \Phi_2(s,t)\frac{d}{dt}\left(\frac{\beta_{n,\tau}}{t}\right)dt\right] \\
&= o(1) + o(1) + o(1), \quad \text{similar to } I_{1.4.9} \\
&= o(1), \quad \text{as } (m,n) \to \infty.
\end{aligned}$$

Hence

$$I_3 = o(1), \text{ as } (m,n) \to \infty.$$

Similarly we get

$$I_2 = o(1) \text{ as } (m,n) \to \infty.$$

By the regularity conditions of matrix summability and the Riemann-Lebesgue theorem we have

$$I_4 = o(1), \text{ as } (m,n) \to \infty$$

Therefore by the above estimations, our theorem is completely established.

6. Particular Cases

1. The result of Chow (1953) becomes a particular case of our theorem if $a_{m,i} = \frac{1}{(m+1)}\ \forall i$, $b_{n,k} = \frac{1}{(n+1)}\ \forall k$ and $\alpha(u) = 1$, $\forall u$, $\beta(v) = 1\ \forall v$.
2. If $a_{m,i} = \frac{1}{(m-i+1)\log m}$, $b_{n,k} = \frac{1}{(n-k+1)\log n}$ and $\alpha(u)$, $\beta(v)$ are defined as in (1) then a result of Sharma (1958) becomes a particular case of our theorem.
3. A result of Tripathi and Singh (1981) becomes a particular case of our theorem if $a_{m,i} = \frac{P_{m-i}}{P_m}$, $b_{n,k} = \frac{q_{n-k}}{Q_n}$ provided $P_m = \sum_{i=0}^n p_i$, $Q_n = \sum_{k=0}^n q_k$ and $\alpha(u) = \frac{up(u)\log u}{P_{(u)}}$ and $\beta(v) = \frac{v\,q(v)\log v}{Q_{(v)}}$.
4. If $a_{m,i}$, $b_{n,k}$ are defined as in (3) and $\alpha(u) = \frac{\log(u)}{P_{(u)}}$, $\beta(v) = \frac{\log(v)}{Q_{(v)}}$ then a result of Tripathi and Ojha (1982) becomes a particular case of our theorem.

5. The result of Mishra (1985) becomes a particular case of our theorem if $a_{m,i} = \frac{p^{(1)}_{m-i}}{P^{(1)}_m}$, and $\alpha(u) = \frac{\log(u)}{\psi^{(1)}(u)}$, $\beta(v) = \frac{\log(v)}{\psi^{(2)}(v)}$.
6. If $a_{m,i}$, $b_{n,k}$ are defined as in (3) and $\alpha(u) = \frac{\log\ u\xi(u)}{\alpha P_u}$, $\beta(v) = \frac{\log\ v\psi(v)}{\alpha\ P_v}$ then a result of Lal (1992) becomes the particular case of our theorem.
7. The result of Singh, Lal and Singh (1994) becomes a particular case of our theorem if $a_{m,i}$, $b_{n,k}$ are defined in (3) and $\alpha(u) = \frac{\xi(u)\log\ u}{P_{(u)}}$, $\quad \beta(v) = \frac{\psi(v)\log\ v}{Q_{(v)}}$.

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