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ON THE FEKETE-SZEGÖ PROBLEM FOR STRONGLY α-QUASICONVEX FUNCTIONS

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Abstract. The purpose of the present paper is to introduce the classes $\mathcal{M}_\alpha(\beta)$ and $\mathcal{Q}_\alpha(\beta)$, respectively, of normalized strongly α-convex and α-quasiconvex functions of order β in the open unit disk and to obtain sharp Fekete-Szegö inequalities for functions belonging to the classes $\mathcal{M}_\alpha(\beta)$ and $\mathcal{Q}_\alpha(\beta)$.

1. Introduction

Let $\mathcal{S}$ denote the class of analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are univalent in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and let $\mathcal{C}$ and $\mathcal{K}$ denote the subclasses of $\mathcal{S}$ consisting of convex and close-to-convex functions, respectively.

Fekete and Szegö [5] showed that for $f \in \mathcal{S}$, given in $\mathcal{U}$ by (1.1),

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu & \text{if} \quad \mu \leq 0, \\ 1 + 2e^{-2\mu/(1-\mu)} & \text{if} \quad 0 \leq \mu \leq 1, \\ 4\mu - 3 & \text{if} \quad \mu \geq 1. \end{cases}$$

This inequality is sharp in the sense that for each μ, there exists a function in $\mathcal{S}$ such that equality holds. There are also several results of this type in the literature (see, [1, 8-11]). Recently, Srivastava, Mishra and Das [16] have obtained the Fekete-Szegö inequalities for certain close-to-convex functions.

Denote by $\mathcal{K}(\beta)$ the class of strongly close-to-convex functions of order β. Thus $f \in \mathcal{K}(\beta)$ if and only if there exists $g \in \mathcal{C}$ such that

$$\left|\arg \frac{f'(z)}{g'(z)}\right| \leq \frac{\pi}{2}\beta \quad (\beta \geq 0;\ z \in \mathcal{U}).$$

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Clearly, $\mathcal{K}(0) = \mathcal{C}$, $\mathcal{K}(1) = \mathcal{K}$ and for $0 \leq \beta \leq 1$, $\mathcal{K}(\beta)$ is a subclass of $\mathcal{K}$ and hence contains only univalent functions. However, Goodman [6] showed that $\mathcal{K}(\beta)$ can contain functions with infinite valence for $\beta > 1$. For the class $\mathcal{K}(\beta)$, the Fekete-Szegö problem has been also solved by London [11] (also, see [1]). We now introduce new classes which cover some well-known classes of univalent functions as follows:

Definition 1.1. A function $f \in \mathcal{S}$ given by (1.1) is said to be strongly α-convex of order β if

$$\left|\arg\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)}\right\}\right| \leq \frac{\pi}{2}\beta \quad (\alpha \geq 0;\ 0 < \beta \leq 1;\ z \in \mathcal{U}). \tag{1.2}$$

Denote by $\mathcal{M}_\alpha(\beta)$ the class of strongly α-convex functions of order β. We note that the class $\mathcal{M}_\alpha(1)$ was introduced by Mocanu [11], which is a generalization of both classes of starlike and convex functions. In particular, $\mathcal{M}_0(\beta)$ is the class of strongly starlike functions of order β studied by Brannan and Kirwan [2].

Definition 1.2. A function $f \in \mathcal{S}$, given by (1.1), is said to be strongly α-quasiconvex of order β if there exists a function $g \in \mathcal{C}$ such that

$$\left|\arg\left\{(1-\alpha)\frac{f'(z)}{g'(z)} + \alpha\frac{(zf'(z))'}{g'(z)}\right\}\right| \leq \frac{\pi}{2}\beta \quad (\alpha,\ \beta \geq 0;\ z \in \mathcal{U}).$$

We denote by $\mathcal{Q}_\alpha(\beta)$ the class of strongly α-quasiconvex functions of order β. Also we note that $\mathcal{Q}_0(1) = \mathcal{K}$ and $\mathcal{Q}_1(1)$ is the class of quasiconvex functions introduced by Noor [13]. Furthermore, the class $\mathcal{Q}_\alpha(1)$, the class of α-quasiconvex functions, have extensively studied by Noor and Alkhorasani [14].

The purpose of the present paper is to prove sharp Fekete-Szegö inequalities for functions belonging to the classes $\mathcal{M}_\alpha(\beta)$ and $\mathcal{Q}_\alpha(\beta)$, which imply the results obtained earlier by Abdel-Gawad and Thomas [1], Keogh and Merkes [8] and London [11].

2. Main Results

To prove our main results, we need the following

Lemma 2.1. *Let p be analytic in $\mathcal{U}$ and satisfy* $\mathrm{Re}\{p(z)\} > 0$ *for $z \in \mathcal{U}$, with $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$. Then*

$$|p_n| \leq 2 \quad (n \geq 1) \tag{2.1}$$

and

$$\left|p_2 - \frac{p_1^2}{2}\right| \leq 2 - \frac{|p_1|^2}{2}. \tag{2.2}$$

The inequality (2.1) was first proved by Carathéodory [3] (also, see Duren [4, p.41]) and the inequality (2.2) can be found in [15, p.166].

With the help of Lemma 2.1, we now derive

Theorem 2.1. *Let $f \in \mathcal{M}_\alpha(\beta)$ and be given by* (1.1). *The for complex number μ,*

$$|a_3 - \mu a_2^2| \leq \frac{\beta}{1+2\alpha} \max\left\{1, \frac{|\alpha^2 + 8\alpha + 3 - 4\mu(1+2\alpha)|\beta}{(1+\alpha)^2}\right\}.$$

For each μ, there is a function in $\mathcal{M}_\alpha(\beta)$ such that equality holds.

Proof. From (1.2), we can write

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} = p^\beta(z), \tag{2.3}$$

where p is given by Lemma 2.1. Equating coefficents, we obtain

$$a_3 - \mu a_2^2 = \frac{\beta}{2(1+2\alpha)}\left(p_2 - \frac{p_1^2}{2}\right) + \frac{(\alpha^2 + 8\alpha + 3 - 4\mu(1+2\alpha))\beta^2 p_1^2}{4(1+2\alpha)(1+\alpha)^2}. \tag{2.4}$$

Therefore, using (2.4) and applying Lemma 2.1, we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{\beta}{1+2\alpha}, & \text{if} \quad k(\alpha) \leq \dfrac{(1+\alpha)^2}{\beta}, \\ \dfrac{|\alpha^2 + 8\alpha + 3 - 4\mu(1+2\alpha)|\beta^2}{(1+2\alpha)(1+\alpha)^2}, & \text{if} \quad k(\alpha) \geq \dfrac{(1+\alpha)^2}{\beta}, \end{cases}$$

where

$$k(\alpha) = |\alpha^2 + 8\alpha + 3 - 4\mu(1+2\alpha)|.$$

Equality is attained for functions in $\mathcal{M}_\alpha(\beta)$, respectively, given by

$$(1+\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} = \left(\frac{1+z^2}{1-z^2}\right)^\beta \tag{2.5}$$

and

$$(1+\alpha)\frac{zf'(z)}{f(z)} + \alpha\frac{(zf'(z))'}{f'(z)} = \left(\frac{1+z}{1-z}\right)^\beta. \tag{2.6}$$

Remark 2.1. It follows at once from (2.3) that $|a_2| \leq 2\beta/(1+\alpha)$ and Theorem 2.1 gives

$$|a_3| \leq \begin{cases} \dfrac{\beta}{1+2\alpha}, & \text{if} \quad (1+\alpha)^2 \geq (\alpha^2 + 8\alpha + 3)\beta, \\ \dfrac{(\alpha^2 + 8\alpha + 3)\beta^2}{(1+2\alpha)(1+\alpha)^2} & \text{if} \quad (1+\alpha)^2 \leq (\alpha^2 + 8\alpha + 3)\beta. \end{cases}$$

The inequality for $|a_2|$ is sharp when f is defined by (2.6) and the inequalities for $|a_3|$ are sharp when f is defined by (2.5) and (2.6), respectively.

Next, we consider the real number μ as follows.

Theorem 2.2. *Let $f \in \mathcal{M}_\alpha(\beta)$ and be given by* (1.1). *Then for real number μ,*

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{(\alpha^2+8\alpha+3-4(1+2\alpha)\mu)\beta^2}{(1+2\alpha)(1+\alpha)^2}, & if \quad \mu\le\dfrac{(\alpha^2+8\alpha+3)\beta-(1+\alpha)^2}{4(1+2\alpha)\beta},\\ \dfrac{\beta}{1+2\alpha}, & if \quad \dfrac{(\alpha^2+8\alpha+3)\beta-(1+\alpha)^2}{4(1+2\alpha)\beta}\le\mu\le\dfrac{(\alpha^2+8\alpha+3)\beta+(1+\alpha)^2}{4(1+2\alpha)\beta},\\ \dfrac{(4(1+2\alpha)\mu-(\alpha^2+8\alpha+3))\beta^2}{(1+2\alpha)(1+\alpha)^2}, & if \quad \mu\ge\dfrac{(\alpha^2+8\alpha+3)\beta+(1+\alpha)^2}{4(1+2\alpha)\beta}.\end{cases}$$

For each μ, there is a function in $\mathcal{M}_\alpha(\beta)$ such that equality holds in all cases.

Proof. We consider two cases. At first, we suppose that $\mu\le(\alpha^2+8\alpha+3)/(4(1+2\alpha))$. Then (2.4) and Lemma 2.1 give

$$|a_3-\mu a_2^2|\le\frac{\beta}{1+2\alpha}+\frac{((\alpha^2+8\alpha+3-4\mu(1+2\alpha))\beta^2-(1+\alpha)^2\beta)|p_1|^2}{4(1+2\alpha)(1+\alpha)^2}.$$

So, by using the fact that $|p_1|\le 2$, we obtain

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{(\alpha^2+8\alpha+3-4(1+2\alpha)\mu)\beta^2}{(1+2\alpha)(1+\alpha^2)}, & \text{if} \quad \mu\le\dfrac{(\alpha^2+8\alpha+3)\beta-(1+\alpha)^2}{4(1+2\alpha)\beta},\\ \dfrac{\beta}{1+2\alpha}, & \text{if} \quad \dfrac{(\alpha^2+8\alpha+3)\beta-(1+\alpha)^2}{4(1+2\alpha)\beta}\le\mu\le\dfrac{\alpha^2+8\alpha+3}{4(1+2\alpha)}.\end{cases}$$

Equality is attained by choosing $p_1=p_2=2$ and $p_1=0$, $p_2=2$, respectively, in (2.4).

Next, we suppose that $\mu\ge(\alpha^2+8\alpha+3)/(4(1+2\alpha))$. In this case, it follows, as in the first case, from (2.4) and Lemma 2.1 that

$$|a_3-\mu a_2^2|\le\begin{cases}\dfrac{\beta}{1+2\alpha}, & \text{if} \quad \dfrac{\alpha^2+8\alpha+3}{4(1+2\alpha)}\le\mu\le\dfrac{(\alpha^2+8\alpha+3)\beta+(1+\alpha)^2}{4(1+2\alpha)\beta},\\ \dfrac{(4(1+2\alpha)\mu-(\alpha^2+8\alpha+3))\beta^2}{(1+2\alpha)(1+\alpha)^2}, & \text{if} \quad \mu\ge\dfrac{(\alpha^2+8\alpha+3)\beta+(1+\alpha)^2}{4(1+2\alpha)\beta}.\end{cases}$$

The results are sharp choosing $p_1=0$, $p_2=2$ and $p_1=2i$, $p_2=-2$, respectively, in (2.4).

Finally, we prove

Theorem 2.3. *Let $f\in\mathcal{Q}_\alpha(\beta)$ and be given by* (1.1). *Then for $\alpha\ge0$ and $\beta\ge0$, we have*

$$3(2\alpha+1)|a_3-\mu a_2^2|$$

$$\le \begin{cases} 1+\dfrac{(1+\beta)^2(2(\alpha+1)^2-3(2\alpha+1)\mu)}{(\alpha+1)^2}, & if \quad \mu\le \dfrac{2\beta(1+\alpha)^2}{3(2\alpha+1)(1+\beta)}, \\ 1+2\beta+\dfrac{2(2(\alpha+1)^2-3(2\alpha+1)\mu)}{2(\alpha+1)^2-\beta(2(\alpha+1)^2-3(2\alpha+1)\mu)}, & if \quad \dfrac{2\beta(1+\alpha)^2}{3(2\alpha+1)(1+\beta)}\le\mu\le\dfrac{2(1+\alpha)^2}{3(2\alpha+1)}, \\ 1+2\beta, & if \quad \dfrac{2(1+\alpha)^2}{3(2\alpha+1)}\le\mu\le\dfrac{2(1+\alpha)^2(2+\beta)}{3(2\alpha+1)(1+\beta)}, \\ -1+\dfrac{(1+\beta)^2(3(2\alpha+1)\mu-2(\alpha+1)^2)}{(\alpha+1)^2}, & if \quad \mu\ge\dfrac{2(1+\alpha)^2(2+\beta)}{3(2\alpha+1)(1+\beta)}. \end{cases}$$

For each μ, there is a function in $\mathcal{Q}_\alpha(\beta)$ such that equality holds in all cases.

Proof. Let $f \in \mathcal{Q}_\alpha(\beta)$. Then it follows from the definition that we may write

$$(1-\alpha)\frac{f'(z)}{g'(z)}+\alpha\frac{(zf'(z))'}{g'(z)}=p^\beta(z), \tag{2.7}$$

where g is convex and p has positive real part. Let $g(t) = z + b_2z^2 + b_3z^3 + \cdots$ and let p be given as in Lemma 2.1. Then by comparing the coefficients of both sides of (2.7), we obtain

$$3(2\alpha+1)(a_3-\mu a_2^2)=3\left(b_3+\frac{1}{3}(x-2)b_2^2\right)+\beta\left(p_2+\frac{1}{4}(\beta x-2)p_1^2\right)+\beta x p_1 b_2, \tag{2.8}$$

where

$$x=\frac{2(\alpha+1)^2-3(2\alpha+1)\mu}{(\alpha+1)^2}.$$

Since rotations of f also belong to $\mathcal{Q}_\alpha(\beta)$, without loss of generality, we may assume that $a_3 - \mu a_2^2$ is positive. Thus we now estimate $\mathrm{Re}(a_3 - \mu a_2^2)$.

Since $g \in \mathcal{C}$, there exists $h(z) = 1 + k_1z + k_2z^2 + \cdots (z \in \mathcal{U})$ with positive real part, such that $g'(z) + zg''(z) = g'(z)h(z)$. Hence, by equating coefficients, we get that $b_2 = k_1/2$ and $b_3 = (k_2 + k_1^2)/6$. So, by using Lemma 2.1 and letting $k_1 = 2\rho e^{i\phi}$ $(0 \le \rho \le 1,\ 0 \le \phi \le 2\pi)$ and $p_1 = 2re^{i\theta}$ $(0 \le r \le 1,\ 0 \le \theta \le 2\pi)$ in (2.8), we obtain

$$\begin{aligned} &3(2\alpha+1)\mathrm{Re}(a_3-\mu a_2^2)\\ &\le(1-\rho^2)+(x+1)\rho^2\cos 2\phi+2\beta(1-r^2)+\beta^2 x r^2\cos 2\theta+2\beta x r\rho\cos(\theta+\phi)\\ &=\psi(x),\quad \text{say.} \end{aligned} \tag{2.9}$$

We consider first the case

$$\frac{2\beta(1+\alpha)^2}{3(2\alpha+1)(1+\beta)}\le\mu\le\frac{2(1+\alpha)^2}{3(2\alpha+1)}.$$

Then we have $0 \le x \le 2/(1+\beta)$. Since the expression $-2t^2 + \beta xt^2 \cos 2\theta + 2xt$ is the largest when $t = x/(2 - \beta x \cos 2\theta)$, we have

$$\psi(x)\le 1+2\beta+\frac{2(2(\alpha+1)^2-3(2\alpha+1)\mu)}{2(\alpha+1)^2-\beta(2(\alpha+1)^2-3(2\alpha+1)\mu)}.$$

and with (2.9), we obtain the second inequality of the theorem. Equality occurs only if the function f is defined by

$$(1-\alpha)f'(z)+\alpha(zf'(z))' = \frac{1}{(1-z)^2}\left(\lambda\frac{1+z}{1-z}+(1-\lambda)\frac{1-z}{1+z}\right)^{\beta},$$

where

$$\lambda = \frac{2(\alpha+1)^2+(1-\beta)(2(\alpha+1)^2-3(2\alpha+1)\mu)}{4(\alpha+1)^2-2\beta(2(\alpha+1)^2-3(2\alpha+1)\mu)}.$$

We now prove the first inequality. Let

$$\mu \le \frac{2\beta(1+\alpha)^2}{3(2\alpha+1)(1+\beta)}.$$

Then we have $x \ge 2/(1+\beta)$, and

$$\psi(x) \le 1+\frac{(1+\beta)^2(2(\alpha+1)^2-3(2\alpha+1)\mu)}{(\alpha+1)^2},$$

as required. Equality occurs only if the function f is defined by

$$(1-\alpha)f'(z)+\alpha(zf'(z))' = \frac{1}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\beta}.$$

Let $x_1 = -2/(1+\beta)$. We note that $\psi(x_1) \le 1+2\beta$. Now we consider two possibilities. Firstly, we suppose that $x_1 \le x \le 0$, that is,

$$\frac{2(1+\alpha)^2}{3(2\alpha+1)} \le \mu \le \frac{2(1+\alpha)^2(2+\beta)}{3(2\alpha+1)(1+\beta)}.$$

Since

$$\psi(\lambda x_1) = \lambda\psi(x_1)+(1-\lambda)\psi(0) \le 1+2\beta,$$

for $0 \le \lambda \le 1$, we obtain $\psi(x) \le 1+2\beta$ and this prove the third inequality of the theorem. Equality occurs only if the function f is defined by

$$(1-\alpha)f'(z)+\alpha(zf'(z))' = \frac{(1+z^2)^{\beta}}{(1-z^2)^{1+\beta}}.$$

Secondly, we suppose that $x \le x_1$, that is,

$$\mu \ge \frac{2(1+\alpha)^2(2+\beta)}{3(2\alpha+1)(1+\beta)}.$$

Then we have

$$\psi(x) \le -1+\frac{(1+\beta)^2(3(2\alpha+1)\mu-2(\alpha+1)^2)}{(\alpha+1)^2},$$

and this is the last inequality of the theorem. Equality occurs only if the function f is defined by

$$(1-\alpha)f'(z)+\alpha(zf'(z))' = \frac{1}{(1-iz)^2}\left(\frac{1+iz}{1-iz}\right)^{\beta}.$$

Therefore we complete the proof of the theorem.

From Theorem 2.3, we have immediately the following

Corollary 2.1. *Let $f \in \mathcal{Q}_1(\beta)$ and be given by* (1.1). *Then for $\beta \geq 0$, we have*

$$9|a_3-\mu a_2^2| \leq \begin{cases} 1+\dfrac{(1+\beta)^2(8-9\mu)}{4} & if \quad \mu \leq \dfrac{8\beta}{9(1+\beta)}, \\ 1+2\beta+\dfrac{2(8-9\mu)}{8-\beta(8-9\mu)} & if \quad \dfrac{8\beta}{9(1+\beta)} \leq \mu \leq \dfrac{8}{9}, \\ 1+2\beta & if \quad \dfrac{8}{9} \leq \mu \leq \dfrac{8(2+\beta)}{9(1+\beta)}, \\ -1+\dfrac{(1+\beta)^2(9\mu-8)}{4} & if \quad \mu \geq \dfrac{8(2+\beta)}{9(1+\beta)}. \end{cases}$$

For each μ, there is function in $\mathcal{Q}_1(\beta)$ such that equality holds in all cases.

Remark 2.2. (i). By letting $\beta = 1$ in Corollary 2.1, we have the result by Keogh and Merkes [8].

(ii). For $\alpha = 0$ in Theorem 2.3, we have the corresponding results obtained by Abdel-Gawad and Thomas [1] and London [11].

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References

[1] H. R. Abdel-Gawad and D. K. Thomas, *The Fekete-Szegö problem for strongly close-to-convex functions,* Proc. Amer. Math. Soc. **114**(1992), 345-349.

[2] D. A. Brannan and W. E. Kirwan, *On some classes of bounded univalent functions,* J. London Math. Soc. **1**(1969), 431-443.

[3] C. Carathéodory, *Uber den variabilitätsbereich der fourier'schen konstanten von positiven harmonischen funktionen,* Rend. Circ. Mat. Palermo **32**(1911), 193-127.

[4] P. L. Duren, Univalent functions, Springer-Verlag, New York, 1983.

[5] M. Fekete and G. Szegö, *Eine Bermerkung uber ungerade schlichte function,* J. London Math. Soc. **8**(1933), 85-89.

[6] A. W. Goodman, *On close-to-convex functions of higher oreder,* Ann. Univ. Sci. Badapest Eötvös Sect. Math. **15**(1972), 17-30.

[7] W. Kalplan, *Close-to-convex schlicht functions,* Michigan Math. J. **1**(1952), 169-185.
[8] F. R. Keogh and E. P. Merkes, *A coefficient inequality for certain classes of analytic functions,* Proc. Amer. Math. Soc. **20**(1969), 8-12.
[9] W. Koepf, *On the Fekete-Szegö problem for close-to-convex functions,* Proc. Amer. Math. Soc. **101**(1987), 89-95.
[10] W. Koepf, *On the Fekete-Szegö problem for close-to-convex functions,* Arch. Math. **49** (1987), 420-433.
[11] R. R. London, *Fekete-Szegö inequalities for close-to-convex functions,* Proc. Amer. Math. Soc. **117**(1993), 947-950.
[12] P. T. Mocanu, *Une propriété de convexité généralisée dans la théorie de la représentation conforme,* Mathematica (Cluj) **11**(1969), 127-133.
[13] K. I. Noor, *On quasi-convex functions and related topics,* Internat. J. Math. and Math. Sci. **10**(1987), 234-258.
[14] K. I. Noor and H. A. Alkhorasani, *Properties of close-to-convexing preserved by some integral operators,* J. Math. Anal. Appl. **112**(1985), 509-516.
[15] Ch. Pommerenke, Univalent functions, Vandenhoeck and Ruprecht, Göttingen, 1975.
[16] H. M. Srivastava, A. K. Mishra and M. K. Das, *The Fekete-Szegö problem for a subclass of close-to-convex functions,* Complex Variables Theory Appl. **44**(2001), 145-163.

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