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INEQUALITIES OF JENSEN'S TYPE FOR GENERALIZED k-g-FRACTIONAL INTEGRALS

SILVESTRU SEVER DRAGOMIR

Abstract. In this paper we establish some inequalities of Jensen and Hermite-Hadamard type for the k-g-fractional integrals of convex functions defined an interval $[a,b]$. Some examples for the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ and for classical Riemann-Liouville fractional integrals are also given.

1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f:[a,b]\to\mathbb{R}$, is well known in the literature as the *Hermite-Hadamard inequality*.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the monograph [22], the recent survey paper [15] and the references therein.

Let $f:[a,b]\to\mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$. The *Riemann-Liouville fractional integrals* are defined for $\alpha>0$ by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt$$

for $a<x\leq b$ and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt$$

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for $a \le x < b$, where Γ is the *Gamma function*. For $\alpha = 0$, they are defined as

$$J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x) \text{ for } x \in (a,b).$$

In the recent paper [16] we obtained the following Hermite-Hadamard type inequalities for convex functions and the Riemann-Liouville fractional integrals

$$\begin{aligned} \frac{1}{\alpha+1}\left[\frac{1}{\alpha} f(x) + \frac{f(a)+f(b)}{2}\right] &\ge \frac{1}{2}\Gamma(\alpha)\left[\frac{J_{a+}^{\alpha} f(x)}{(x-a)^{\alpha}} + \frac{J_{b-}^{\alpha} f(x)}{(b-x)^{\alpha}}\right] \\ &\ge \int_0^1 (1-s)^{\alpha-1} f\left(sx + (1-s)\frac{a+b}{2}\right) ds \\ &\ge \frac{1}{\alpha} f\left(\frac{\alpha}{\alpha+1}\left(\frac{x}{\alpha} + \frac{a+b}{2}\right)\right) \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} \frac{1}{\alpha+1}\left[f(x) + \frac{1}{\alpha}\frac{f(a)+f(b)}{2}\right] &\ge \frac{1}{2}\Gamma(\alpha)\left[\frac{J_{x-}^{\alpha} f(a)}{(x-a)^{\alpha}} + \frac{J_{x+}^{\alpha} f(b)}{(b-x)^{\alpha}}\right] \\ &\ge \int_0^1 s^{\alpha-1} f\left(sx + (1-s)\frac{a+b}{2}\right) ds \\ &\ge \frac{1}{\alpha} f\left(\frac{\alpha}{\alpha+1}\left(x + \frac{1}{\alpha}\frac{a+b}{2}\right)\right) \end{aligned} \tag{1.3}$$

for any $x \in (a,b)$ and $\alpha > 0$.

In order to extend these type of inequalities for more general fractional integrals we need the following preparations.

Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. We define the function $K : [0,\infty) \to \mathbb{C}$ by

$$K(t) := \begin{cases} \int_0^t k(s)\, ds & \text{if } 0 < t, \\ 0 & \text{if } t = 0. \end{cases}$$

As a simple example, if $k(t) = t^{\alpha-1}$ then for $\alpha \in (0,1)$ the function k is defined on $(0,\infty)$ and $K(t) := \frac{1}{\alpha} t^{\alpha}$ for $t \in [0,\infty)$. If $\alpha \ge 1$, then k is defined on $[0,\infty)$ and $K(t) := \frac{1}{\alpha} t^{\alpha}$ for $t \in [0,\infty)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). For the Lebesgue integrable function $f : (a,b) \to \mathbb{C}$, we define the k-g-*left-sided fractional integral* of f by

$$S_{k,g,a+} f(x) = \int_a^x k\big(g(x) - g(t)\big) g'(t) f(t)\, dt,\ x \in (a,b] \tag{1.4}$$

and the k-g-*right-sided fractional integral* of f by

$$S_{k,g,b-} f(x) = \int_x^b k\big(g(t) - g(x)\big) g'(t) f(t)\, dt,\ x \in [a,b). \tag{1.5}$$

If we take $k(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$, where Γ is the *Gamma function*, then

$$\begin{aligned} S_{k,g,a+} f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x [g(x) - g(t)]^{\alpha-1} g'(t) f(t)\, dt \\ &=: I^{\alpha}_{a+,g} f(x), \ a < x \le b \end{aligned} \tag{1.6}$$

and

$$\begin{aligned} S_{k,g,b-} f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b [g(t) - g(x)]^{\alpha-1} g'(t) f(t)\, dt \\ &=: I^{\alpha}_{b-,g} f(x), \ a \le x < b, \end{aligned} \tag{1.7}$$

which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a, b]$ as defined in [25, p.100].

For $g(t) = t$ in (1.7) we have the classical *Riemann-Liouville fractional integrals* while for the logarithmic function $g(t) = \ln t$ we have the *Hadamard fractional integrals* [25, p. 111]

$$H^{\alpha}_{a+} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1} \frac{f(t)\, dt}{t}, \ 0 \le a < x \le b \tag{1.8}$$

and

$$H^{\alpha}_{b-} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1} \frac{f(t)\, dt}{t}, \ 0 \le a < x < b. \tag{1.9}$$

One can consider the function $g(t) = -t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R^{\alpha}_{a+} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{f(t)\, dt}{(x-t)^{1-\alpha} t^{\alpha+1}}, \ 0 \le a < x \le b \tag{1.10}$$

and

$$R^{\alpha}_{b-} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(t)\, dt}{(t-x)^{1-\alpha} t^{\alpha+1}}, \ 0 \le a < x < b. \tag{1.11}$$

Also, for $g(t) = \exp(\beta t)$, $\beta > 0$, we can consider the "*β-Exponential fractional integrals*"

$$E^{\alpha}_{a+,\beta} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_a^x [\exp(\beta x) - \exp(\beta t)]^{\alpha-1} \exp(\beta t) f(t)\, dt, \tag{1.12}$$

for $a < x \le b$ and

$$E^{\alpha}_{b-,\beta} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_x^b [\exp(\beta t) - \exp(\beta x)]^{\alpha-1} \exp(\beta t) f(t)\, dt, \tag{1.13}$$

for $a \le x < b$.

If we take $g(t) = t$ in (1.4) and (1.5), then we can consider the following *k-fractional integrals*

$$S_{k,a+} f(x) = \int_a^x k(x-t) f(t)\, dt, \ x \in (a, b] \tag{1.14}$$

for $a < x \le b$ and

$$S_{k,b-} f(x) = \int_x^b k(t-x) f(t)\, dt, \ x \in [a, b). \tag{1.15}$$

In [28], Raina studied a class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) := \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k+\lambda)} x^k, \ |x| < R, \text{ with } R > 0 \tag{1.16}$$

for $\rho, \lambda > 0$ where the coefficients $\sigma(k)$ generate a bounded sequence of positive real numbers. With the help of (1.16), Raina defined the following left-sided fractional integral operator

$$\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w} f(x) := \int_a^x (x-t)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}\left(w(x-t)^{\rho}\right) f(t)\,dt, \ x > a \tag{1.17}$$

where $\rho, \lambda > 0$, $w \in \mathbb{R}$ and f is such that the integral on the right side exists.

In [1], the right-sided fractional operator was also introduced as

$$\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w} f(x) := \int_x^b (t-x)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}\left(w(t-x)^{\rho}\right) f(t)\,dt, \ x < b \tag{1.18}$$

where $\rho, \lambda > 0$, $w \in \mathbb{R}$ and f is such that the integral on the right side exists. Several Ostrowski type inequalities were also established.

We observe that for $k(t) = t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}(wt^{\rho})$ we re-obtain the definitions of (1.17) and (1.18) from (1.14) and (1.15).

In [26], Kirane and Torebek introduced the following *exponential fractional integrals*

$$\mathcal{T}^{\alpha}_{a+} f(x) := \frac{1}{\alpha} \int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(x-t)\right\} f(t)\,dt, \ x > a \tag{1.19}$$

and

$$\mathcal{T}^{\alpha}_{b-} f(x) := \frac{1}{\alpha} \int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(t-x)\right\} f(t)\,dt, \ x < b \tag{1.20}$$

where $\alpha \in (0,1)$.

We observe that for $k(t) = \frac{1}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$, $t \in \mathbb{R}$ we re-obtain the definitions of (1.19) and (1.20) from (1.14) and (1.15).

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). We can define the more general exponential fractional integrals

$$\mathcal{T}^{\alpha}_{g,a+} f(x) := \frac{1}{\alpha} \int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}\left(g(x)-g(t)\right)\right\} g'(t) f(t)\,dt, \ x > a \tag{1.21}$$

and

$$\mathcal{T}^{\alpha}_{g,b-} f(x) := \frac{1}{\alpha} \int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}\left(g(t)-g(x)\right)\right\} g'(t) f(t)\,dt, \ x < b \tag{1.22}$$

where $\alpha \in (0,1)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Assume that $\alpha > 0$. We can also define the *logarithmic fractional integrals*

$$\mathcal{L}^{\alpha}_{g,a+} f(x) := \int_a^x \left(g(x)-g(t)\right)^{\alpha-1} \ln\left(g(x)-g(t)\right) g'(t) f(t)\,dt, \tag{1.23}$$

for $0 < a < x \le b$ and

$$\mathcal{L}^{\alpha}_{g,b-} f(x) := \int_x^b (g(t) - g(x))^{\alpha-1} \ln(g(t) - g(x))\, g'(t) f(t)\, dt, \tag{1.24}$$

for $0 < a \le x < b$, where $\alpha > 0$. These are obtained from (1.14) and (1.15) for the kernel $k(t) = t^{\alpha-1} \ln t$, $t > 0$.

For $\alpha = 1$ we get

$$\mathcal{L}_{g,a+} f(x) := \int_a^x \ln(g(x) - g(t))\, g'(t) f(t)\, dt, \; 0 < a < x \le b \tag{1.25}$$

and

$$\mathcal{L}_{g,b-} f(x) := \int_x^b \ln(g(t) - g(x))\, g'(t) f(t)\, dt, \; 0 < a \le x < b. \tag{1.26}$$

For $g(t) = t$, we have the simple forms

$$\mathcal{L}^{\alpha}_{a+} f(x) := \int_a^x (x-t)^{\alpha-1} \ln(x-t) f(t)\, dt, \; 0 < a < x \le b, \tag{1.27}$$

$$\mathcal{L}^{\alpha}_{b-} f(x) := \int_x^b (t-x)^{\alpha-1} \ln(t-x) f(t)\, dt, \; 0 < a \le x < b, \tag{1.28}$$

$$\mathcal{L}_{a+} f(x) := \int_a^x \ln(x-t) f(t)\, dt, \; 0 < a < x \le b \tag{1.29}$$

and

$$\mathcal{L}_{b-} f(x) := \int_x^b \ln(t-x) f(t)\, dt, \; 0 < a \le x < b. \tag{1.30}$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [2]−[18], [23]−[36] and the references therein.

For k and g as at the beginning of Introduction, we consider the mixed operator

$$\begin{aligned} S_{k,g,a+,b-} f(x) &:= \frac{1}{2}\left[S_{k,g,a+} f(x) + S_{k,g,b-} f(x)\right] \\ &= \frac{1}{2}\left[\int_a^x k(g(x) - g(t))\, g'(t) f(t)\, dt + \int_x^b k(g(t) - g(x))\, g'(t) f(t)\, dt\right] \end{aligned} \tag{1.31}$$

for the Lebesgue integrable function $f : (a,b) \to \mathbb{C}$ and $x \in (a,b)$.

Observe that

$$S_{k,g,x+} f(b) = \int_x^b k(g(b) - g(t))\, g'(t) f(t)\, dt, \; x \in [a,b)$$

and

$$S_{k,g,x-} f(a) = \int_a^x k(g(t) - g(a))\, g'(t) f(t)\, dt, \; x \in (a,b].$$

We can define also the dual mixed operator

$$\check{S}_{k,g,a+,b-} f(x) := \frac{1}{2}\left[S_{k,g,x+} f(b) + S_{k,g,x-} f(a)\right]$$

$$= \frac{1}{2}\left[\int_x^b k\left(g(b)-g(t)\right)g'(t)f(t)\,dt + \int_a^x k\left(g(t)-g(a)\right)g'(t)f(t)\,dt\right]$$

for any $x \in (a,b)$.

In this paper we establish some inequalities of Jensen and Hermite-Hadamard type for the k-g-fractional integrals of convex functions defined an interval $[a,b]$. Some examples for the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ and for classical Riemann-Liouville fractional integrals are also given.

2. The main results

We have the following bounds for the operator $S_{k,g,a+,b-}f$:

Theorem 1. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with nonnegative values and integrable on any finite subinterval. Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). If $f:[a,b]\to\mathbb{R}$ is a convex function, then*

$$\begin{aligned}
&\frac{1}{2}\left[K\left(g(x)-g(a)\right)+K\left(g(b)-g(x)\right)\right]\\
&\quad\times f\left(\frac{K\left(g(x)-g(a)\right)a+K\left(g(b)-g(x)\right)b}{K\left(g(x)-g(a)\right)+K\left(g(b)-g(x)\right)}+\frac{\int_a^x K\left(g(x)-g(t)\right)dt-\int_x^b K\left(g(t)-g(x)\right)dt}{K\left(g(x)-g(a)\right)+K\left(g(b)-g(x)\right)}\right)\\
&\le \frac{1}{2}\left[f\left(a+\frac{1}{K\left(g(x)-g(a)\right)}\int_a^x K\left(g(x)-g(t)\right)dt\right)K\left(g(x)-g(a)\right)\right.\\
&\quad\left.+f\left(b-\frac{1}{K\left(g(b)-g(x)\right)}\int_x^b K\left(g(t)-g(x)\right)dt\right)K\left(g(b)-g(x)\right)\right]\\
&\le S_{k,g,a+,b-}f(x)\\
&\le \frac{1}{2}\left[K\left(g(x)-g(a)\right)f(a)+K\left(g(b)-g(x)\right)f(b)\right]\\
&\quad+\frac{1}{2}\left[\frac{f(x)-f(a)}{x-a}\int_a^x K\left(g(x)-g(t)\right)dt-\frac{f(b)-f(x)}{b-x}\int_x^b K\left(g(t)-g(x)\right)dt\right] \qquad (2.1)
\end{aligned}$$

for $x\in(a,b)$.

Proof. Since $f:[a,b]\to\mathbb{R}$ is convex, then for $x\in(a,b)$

$$f(t)\le\frac{t-a}{x-a}f(x)+\frac{x-t}{x-a}f(a),\ t\in[a,x] \qquad (2.2)$$

and

$$f(t)\le\frac{t-x}{b-x}f(b)+\frac{b-t}{b-x}f(x),\ t\in[x,b]. \qquad (2.3)$$

By (2.2) and (2.3) we have

$$S_{k,g,a+,b-}f(x)=\frac{1}{2}\left[\int_a^x k\left(g(x)-g(t)\right)g'(t)f(t)\,dt+\int_x^b k\left(g(t)-g(x)\right)g'(t)f(t)\,dt\right]$$

$$\leq \frac{1}{2}\int_a^x k\left(g(x)-g(t)\right)g'(t)\left[\frac{t-a}{x-a}f(x)+\frac{x-t}{x-a}f(a)\right]dt$$
$$+\frac{1}{2}\int_x^b k\left(g(t)-g(x)\right)g'(t)\left[\frac{t-x}{b-x}f(b)+\frac{b-t}{b-x}f(x)\right]dt \tag{2.4}$$

for $x \in (a,b)$.

Using the chain rule we have

$$\left(K\left(g(x)-g(t)\right)\right)' = -K'\left(g(x)-g(t)\right)g'(t) = -k\left(g(x)-g(t)\right)g'(t)$$

for $t \in (a,x)$ and

$$\left(K\left(g(t)-g(x)\right)\right)' = K'\left(g(t)-g(x)\right)g'(t) = k\left(g(t)-g(x)\right)g'(t)$$

for $t \in (x,b)$.

Then, integrating by parts, we have

$$\int_a^x k\left(g(x)-g(t)\right)g'(t)\left[\frac{t-a}{x-a}f(x)+\frac{x-t}{x-a}f(a)\right]dt$$
$$= -\int_a^x \left(K\left(g(x)-g(t)\right)\right)'\left[\frac{t-a}{x-a}f(x)+\frac{x-t}{x-a}f(a)\right]dt$$
$$= -\left[K\left(g(x)-g(t)\right)\left[\frac{t-a}{x-a}f(x)+\frac{x-t}{x-a}f(a)\right]\Big|_a^x - \frac{f(x)-f(a)}{x-a}\int_a^x K\left(g(x)-g(t)\right)dt\right]$$
$$= K\left(g(x)-g(a)\right)f(a)+\frac{f(x)-f(a)}{x-a}\int_a^x K\left(g(x)-g(t)\right)dt$$

and

$$\int_x^b k\left(g(t)-g(x)\right)g'(t)\left[\frac{t-x}{b-x}f(b)+\frac{b-t}{b-x}f(x)\right]dt$$
$$= \int_x^b \left(K\left(g(t)-g(x)\right)\right)'\left[\frac{t-x}{b-x}f(b)+\frac{b-t}{b-x}f(x)\right]dt$$
$$= K\left(g(t)-g(x)\right)\left[\frac{t-x}{b-x}f(b)+\frac{b-t}{b-x}f(x)\right]\Big|_x^b - \frac{f(b)-f(x)}{b-x}\int_x^b K\left(g(t)-g(x)\right)dt$$
$$= K\left(g(b)-g(x)\right)f(b)-\frac{f(b)-f(x)}{b-x}\int_x^b K\left(g(t)-g(x)\right)dt$$

for $x \in (a,b)$.

Therefore by (2.4) we have

$$S_{k,g,a+,b-}f(x) \leq \frac{1}{2}\left[\frac{f(x)-f(a)}{x-a}\int_a^x K\left(g(x)-g(t)\right)dt + K\left(g(x)-g(a)\right)f(a)\right]$$
$$+\frac{1}{2}\left[K\left(g(b)-g(x)\right)f(b)-\frac{f(b)-f(x)}{b-x}\int_x^b K\left(g(t)-g(x)\right)dt\right]$$

$$= \frac{1}{2}\left[K\left(g(x)-g(a)\right)f(a)+K\left(g(b)-g(x)\right)f(b)\right]$$
$$+\frac{1}{2}\left[\frac{f(x)-f(a)}{x-a}\int_a^x K\left(g(x)-g(t)\right)dt-\frac{f(b)-f(x)}{b-x}\int_x^b K\left(g(t)-g(x)\right)dt\right]$$

for $x\in(a,b)$, which proves the third inequality in (2.1).

We use the Jensen inequality in the form

$$\frac{\int_c^d w(t)f(t)\,dt}{\int_c^d w(t)\,dt}\ge f\left(\frac{\int_c^d w(t)\,t\,dt}{\int_c^d w(t)\,dt}\right), \tag{2.5}$$

where $f:[c,d]\to\mathbb{R}$ is convex and $w(t)\ge 0$, $t\in[c,d]$ is integrable with $\int_c^d w(t)\,dt>0$.

Therefore

$$\int_a^x k\left(g(x)-g(t)\right)g'(t)f(t)\,dt$$
$$\ge f\left(\frac{\int_a^x k\left(g(x)-g(t)\right)g'(t)\,t\,dt}{\int_a^x k\left(g(x)-g(t)\right)g'(t)\,dt}\right)\int_a^x k\left(g(x)-g(t)\right)g'(t)\,dt \tag{2.6}$$

and

$$\int_x^b k\left(g(t)-g(x)\right)g'(t)f(t)\,dt$$
$$\ge f\left(\frac{\int_x^b k\left(g(t)-g(x)\right)g'(t)\,t\,dt}{\int_x^b k\left(g(t)-g(x)\right)g'(t)\,dt}\right)\int_x^b k\left(g(t)-g(x)\right)g'(t)\,dt \tag{2.7}$$

for $x\in(a,b)$.

We have

$$\int_a^x k\left(g(x)-g(t)\right)g'(t)\,dt=-\int_a^x\left(K\left(g(x)-g(t)\right)\right)'dt=K\left(g(x)-g(a)\right)$$

and

$$\int_a^x k\left(g(x)-g(t)\right)g'(t)\,t\,dt=-\int_a^x\left(K\left(g(x)-g(t)\right)\right)'t\,dt$$
$$=-\left[K\left(g(x)-g(t)\right)t\Big|_a^x-\int_a^x K\left(g(x)-g(t)\right)dt\right]$$
$$=K\left(g(x)-g(a)\right)a+\int_a^x K\left(g(x)-g(t)\right)dt$$

for $x\in(a,b)$.

Also

$$\int_x^b k\left(g(t)-g(x)\right)g'(t)\,dt=\int_x^b\left(K\left(g(t)-g(x)\right)\right)'dt=K\left(g(b)-g(x)\right)$$

and

$$\int_x^b k\left(g(t)-g(x)\right)g'(t)\,t\,dt=\int_x^b\left(K\left(g(t)-g(x)\right)\right)'t\,dt$$

$$= K\left(g(t)-g(x)\right) t\Big|_x^b - \int_x^b K\left(g(t)-g(x)\right) dt$$
$$= K\left(g(b)-g(x)\right) b - \int_x^b K\left(g(t)-g(x)\right) dt$$

for $x \in (a,b)$.

Then by (2.6) and (2.7) we have

$$\int_a^x k\left(g(x)-g(t)\right) g'(t) f(t)\, dt \geq f\left(a + \frac{1}{K\left(g(x)-g(a)\right)} \int_a^x K\left(g(x)-g(t)\right) dt\right) K\left(g(x)-g(a)\right) \tag{2.8}$$

and

$$\int_x^b k\left(g(t)-g(x)\right) g'(t) f(t)\, dt \geq f\left(b - \frac{1}{K\left(g(b)-g(x)\right)} \int_x^b K\left(g(t)-g(x)\right) dt\right) K\left(g(b)-g(x)\right) \tag{2.9}$$

for $x \in (a,b)$.

Using the inequalities (2.8) and (2.9) we have

$$\begin{aligned} S_{k,g,a+,b-} f(x) &= \frac{1}{2}\left[\int_a^x k\left(g(x)-g(t)\right) g'(t) f(t)\, dt + \int_x^b k\left(g(t)-g(x)\right) g'(t) f(t)\, dt\right] \\ &\geq \frac{1}{2} f\left(a + \frac{1}{K\left(g(x)-g(a)\right)} \int_a^x K\left(g(x)-g(t)\right) dt\right) K\left(g(x)-g(a)\right) \\ &\quad + \frac{1}{2} f\left(b - \frac{1}{K\left(g(b)-g(x)\right)} \int_x^b K\left(g(t)-g(x)\right) dt\right) K\left(g(b)-g(x)\right), \end{aligned}$$

which proves the second inequality in (2.1).

By the convexity of f we have for $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$, that

$$\frac{\alpha f(c) + \beta f(d)}{\alpha + \beta} \geq f\left(\frac{\alpha c + \beta d}{\alpha + \beta}\right). \tag{2.10}$$

Then for

$$\alpha = \frac{K\left(g(x)-g(a)\right)}{2}, \qquad \beta = \frac{K\left(g(b)-g(x)\right)}{2}$$

and

$$c = a + \frac{1}{K\left(g(x)-g(a)\right)} \int_a^x K\left(g(x)-g(t)\right) dt,$$
$$d = b - \frac{1}{K\left(g(b)-g(x)\right)} \int_x^b K\left(g(t)-g(x)\right) dt$$

we have

$$f\left(a + \frac{1}{K\left(g(x)-g(a)\right)} \int_a^x K\left(g(x)-g(t)\right) dt\right) \frac{K\left(g(x)-g(a)\right)}{2}$$

$$
\begin{aligned}
&+f\left(b-\frac{1}{K\left(g(b)-g(x)\right)}\int_x^b K\left(g(t)-g(x)\right)dt\right)\frac{K\left(g(b)-g(x)\right)}{2}\\
\geq{}&\frac{1}{2}\left[K\left(g(x)-g(a)\right)+K\left(g(b)-g(x)\right)\right]\\
&\times f\left(\frac{K\left(g(x)-g(a)\right)a+K\left(g(b)-g(x)\right)b}{K\left(g(x)-g(a)\right)+K\left(g(b)-g(x)\right)}+\frac{\int_a^x K\left(g(x)-g(t)\right)dt-\int_x^b K\left(g(t)-g(x)\right)dt}{K\left(g(x)-g(a)\right)+K\left(g(b)-g(x)\right)}\right),
\end{aligned}
$$

which proves the first inequality in (2.1). □

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the g*-mean of two numbers* $a,\ b\in I$ as

$$M_g(a,b):=g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean*. If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq 0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent p*. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g(a,b)=LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right),$$

the *LogMeanExp function.*

Corollary 1. *With the assumptions of Theorem 1 we have*

$$
\begin{aligned}
&K\left(\frac{g(b)-g(a)}{2}\right)f\left(\frac{a+b}{2}+\frac{\int_a^{M_g(a,b)}K\left(\frac{g(a)+g(b)}{2}-g(t)\right)dt-\int_{M_g(a,b)}^b K\left(g(t)-\frac{g(a)+g(b)}{2}\right)dt}{2K\left(\frac{g(b)-g(a)}{2}\right)}\right)\\
\leq{}&\frac{1}{2}\left[f\left(a+\frac{1}{K\left(\frac{g(b)-g(a)}{2}\right)}\int_a^{M_g(a,b)}K\left(\frac{g(a)+g(b)}{2}-g(t)\right)dt\right)\right.\\
&\left.+f\left(b-\frac{1}{K\left(\frac{g(b)-g(a)}{2}\right)}\int_{M_g(a,b)}^b K\left(g(t)-\frac{g(a)+g(b)}{2}\right)dt\right)\right]K\left(\frac{g(b)-g(a)}{2}\right)\\
\leq{}&S_{k,g,a+,b-}f\left(M_g(a,b)\right)\\
\leq{}&\frac{1}{2}\left[f(a)+f(b)\right]K\left(\frac{g(b)-g(a)}{2}\right)\\
&+\frac{1}{2}\frac{f\left(M_g(a,b)\right)-f(a)}{M_g(a,b)-a}\int_a^{M_g(a,b)}K\left(\frac{g(a)+g(b)}{2}-g(t)\right)dt\\
&-\frac{1}{2}\frac{f(b)-f\left(M_g(a,b)\right)}{b-M_g(a,b)}\int_{M_g(a,b)}^b K\left(g(t)-\frac{g(a)+g(b)}{2}\right)dt. \qquad (2.11)
\end{aligned}
$$

For the dual operator $\check{S}_{k,g,a+,b-}f$ we also have the following bounds:

Theorem 2. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with nonnegative values and integrable on any finite subinterval. Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). If $f:[a,b]\to\mathbb{R}$ is a convex function, then*

$$\begin{aligned}
&\frac{1}{2}\left[K\left(g(x)-g(a)\right)+K\left(g(b)-g(x)\right)\right]\\
&\quad\times f\left(x+\frac{\int_x^b K\left(g(b)-g(t)\right)dt-\int_a^x K\left(g(t)-g(a)\right)dt}{K\left(g(x)-g(a)\right)+K\left(g(b)-g(x)\right)}\right)\\
&\le\frac{1}{2}\left[f\left(x-\frac{1}{K\left(g(x)-g(a)\right)}\int_a^x K\left(g(t)-g(a)\right)dt\right)K\left(g(x)-g(a)\right)\right.\\
&\quad\left.+f\left(x+\frac{1}{K\left(g(b)-g(x)\right)}\int_x^b K\left(g(b)-g(t)\right)dt\right)K\left(g(b)-g(x)\right)\right]\\
&\le\check{S}_{k,g,a+,b-}f(x)\\
&\le\frac{1}{2}\left[K\left(g(x)-g(a)\right)+K\left(g(b)-g(x)\right)\right]f(x)\\
&\quad+\frac{1}{2}\left[\frac{f(b)-f(x)}{b-x}\int_x^b K\left(g(b)-g(t)\right)dt-\frac{f(x)-f(a)}{x-a}\int_a^x K\left(g(t)-g(a)\right)dt\right]
\end{aligned}\tag{2.12}$$

for $x\in(a,b)$.

Proof. Using (2.2) and (2.3) we have

$$\begin{aligned}
\check{S}_{k,g,a+,b-}f(x)&=\frac{1}{2}\int_a^x k\left(g(t)-g(a)\right)g'(t)f(t)\,dt+\frac{1}{2}\int_x^b k\left(g(b)-g(t)\right)g'(t)f(t)\,dt\\
&\le\frac{1}{2}\int_a^x k\left(g(t)-g(a)\right)g'(t)\left[\frac{t-a}{x-a}f(x)+\frac{x-t}{x-a}f(a)\right]dt\\
&\quad+\frac{1}{2}\int_x^b k\left(g(b)-g(t)\right)g'(t)\left[\frac{t-x}{b-x}f(b)+\frac{b-t}{b-x}f(x)\right]dt
\end{aligned}\tag{2.13}$$

for $x\in(a,b)$.

Using the chain rule we have

$$\left(K\left(g(b)-g(t)\right)\right)'=-K'\left(g(b)-g(t)\right)g'(t)=-k\left(g(b)-g(t)\right)g'(t)$$

for $t\in(x,b)$ and

$$\left(K\left(g(t)-g(a)\right)\right)'=K'\left(g(t)-g(a)\right)g'(t)=k\left(g(t)-g(a)\right)g'(t)$$

for $t\in(a,x)$.

Then we have

$$\int_a^x k\left(g(t)-g(a)\right)g'(t)\left[\frac{t-a}{x-a}f(x)+\frac{x-t}{x-a}f(a)\right]dt$$

$$= \int_a^x \left(K\left(g(t)-g(a)\right)\right)' \left[\frac{t-a}{x-a} f(x) + \frac{x-t}{x-a} f(a)\right] dt$$
$$= K\left(g(t)-g(a)\right) \left[\frac{t-a}{x-a} f(x) + \frac{x-t}{x-a} f(a)\right]\Bigg|_a^x - \frac{f(x)-f(a)}{x-a} \int_a^x K\left(g(t)-g(a)\right) dt$$
$$= K\left(g(x)-g(a)\right) f(x) - \frac{f(x)-f(a)}{x-a} \int_a^x K\left(g(t)-g(a)\right) dt$$

and

$$\int_x^b k\left(g(b)-g(t)\right) g'(t) \left[\frac{t-x}{b-x} f(b) + \frac{b-t}{b-x} f(x)\right] dt$$
$$= -\int_x^b \left(K\left(g(b)-g(t)\right)\right)' \left[\frac{t-x}{b-x} f(b) + \frac{b-t}{b-x} f(x)\right] dt$$
$$= -\left[K\left(g(b)-g(t)\right) \left[\frac{t-x}{b-x} f(b) + \frac{b-t}{b-x} f(x)\right]\Bigg|_x^b - \frac{f(b)-f(x)}{b-x} \int_x^b K\left(g(b)-g(t)\right) dt\right]$$
$$= K\left(g(b)-g(x)\right) f(x) + \frac{f(b)-f(x)}{b-x} \int_x^b K\left(g(b)-g(t)\right) dt$$

for $x \in (a,b)$.

From (2.13) we get

$$\begin{aligned}
\check{S}_{k,g,a+,b-} f(x) \le{} & \frac{1}{2}\left[K\left(g(x)-g(a)\right) f(x) - \frac{f(x)-f(a)}{x-a} \int_a^x K\left(g(t)-g(a)\right) dt\right] \\
& + \frac{1}{2}\left[K\left(g(b)-g(x)\right) f(x) + \frac{f(b)-f(x)}{b-x} \int_x^b K\left(g(b)-g(t)\right) dt\right] \\
={} & \frac{1}{2}\left[K\left(g(x)-g(a)\right) + K\left(g(b)-g(x)\right)\right] f(x) \\
& + \frac{1}{2}\left[\frac{f(b)-f(x)}{b-x} \int_x^b K\left(g(b)-g(t)\right) dt - \frac{f(x)-f(a)}{x-a} \int_a^x K\left(g(t)-g(a)\right) dt\right],
\end{aligned}$$

which proves the third inequality in (2.12).

By Jensen's inequality (2.5) we also have

$$\begin{aligned}
& \int_a^x k\left(g(t)-g(a)\right) g'(t) f(t)\, dt \\
& \quad \ge f\left(\frac{\int_a^x k\left(g(t)-g(a)\right) g'(t)\, t\, dt}{\int_a^x k\left(g(t)-g(a)\right) g'(t)\, dt}\right) \int_a^x k\left(g(t)-g(a)\right) g'(t)\, dt
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
& \int_x^b k\left(g(b)-g(t)\right) g'(t) f(t)\, dt \\
& \quad \ge f\left(\frac{\int_x^b k\left(g(b)-g(t)\right) g'(t)\, t\, dt}{\int_x^b k\left(g(b)-g(t)\right) g'(t)\, dt}\right) \int_x^b k\left(g(b)-g(t)\right) g'(t)\, dt
\end{aligned} \tag{2.15}$$

for $x \in (a,b)$.

Observe that

$$\int_a^x k\left(g(t)-g(a)\right)g'(t)\,dt=\int_a^x \left(K\left(g(t)-g(a)\right)\right)'\,dt=K\left(g(x)-g(a)\right)$$

and

$$\begin{aligned}\int_a^x k\left(g(t)-g(a)\right)g'(t)\,t\,dt&=\int_a^x \left(K\left(g(t)-g(a)\right)\right)'\,t\,dt\\&=K\left(g(t)-g(a)\right)t\Big|_a^x-\int_a^x K\left(g(t)-g(a)\right)dt\\&=K\left(g(x)-g(a)\right)x-\int_a^x K\left(g(t)-g(a)\right)dt\end{aligned}$$

for $x\in(a,b)$.

Also

$$\int_x^b k\left(g(b)-g(t)\right)g'(t)\,dt=-\int_x^b \left(K\left(g(b)-g(t)\right)\right)'\,dt=K\left(g(b)-g(x)\right)$$

and

$$\begin{aligned}\int_x^b k\left(g(b)-g(t)\right)g'(t)\,t\,dt&=-\int_x^b \left(K\left(g(b)-g(t)\right)\right)'\,t\,dt\\&=-\left[K\left(g(b)-g(t)\right)t\Big|_x^b-\int_x^b K\left(g(b)-g(t)\right)dt\right]\\&=K\left(g(b)-g(x)\right)x+\int_x^b K\left(g(b)-g(t)\right)dt\end{aligned}$$

for $x\in(a,b)$.

Therefore, by (2.14) and (2.15) we have

$$\begin{aligned}\check{S}_{k,g,a+,b-}f(x)&=\frac{1}{2}\int_a^x k\left(g(t)-g(a)\right)g'(t)f(t)\,dt+\frac{1}{2}\int_x^b k\left(g(b)-g(t)\right)g'(t)f(t)\,dt\\&\geq\frac{1}{2}f\left(x-\frac{1}{K\left(g(x)-g(a)\right)}\int_a^x K\left(g(t)-g(a)\right)dt\right)K\left(g(x)-g(a)\right)\\&\quad+\frac{1}{2}f\left(x+\frac{1}{K\left(g(b)-g(x)\right)}\int_x^b K\left(g(b)-g(t)\right)dt\right)K\left(g(b)-g(x)\right),\end{aligned}$$

which prove the second inequality in (2.12).

Using the inequality (2.10) for

$$\alpha=\frac{K\left(g(x)-g(a)\right)}{2},\qquad\beta=\frac{K\left(g(b)-g(x)\right)}{2}$$

and

$$c=x-\frac{1}{K\left(g(x)-g(a)\right)}\int_a^x K\left(g(t)-g(a)\right)dt,$$

$$d=x+\frac{1}{K\left(g\left(b\right)-g\left(x\right)\right)}\int_{x}^{b}K\left(g\left(b\right)-g\left(t\right)\right)dt$$

we have

$$\begin{aligned}&f\left(x-\frac{1}{K\left(g\left(x\right)-g\left(a\right)\right)}\int_{a}^{x}K\left(g\left(t\right)-g\left(a\right)\right)dt\right)\frac{K\left(g\left(x\right)-g\left(a\right)\right)}{2}\\&\quad+f\left(x+\frac{1}{K\left(g\left(b\right)-g\left(x\right)\right)}\int_{x}^{b}K\left(g\left(b\right)-g\left(t\right)\right)dt\right)\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{2}\\&\geq\frac{1}{2}\left[K\left(g\left(x\right)-g\left(a\right)\right)+K\left(g\left(b\right)-g\left(x\right)\right)\right]f\left(x+\frac{\int_{x}^{b}K\left(g\left(b\right)-g\left(t\right)\right)dt-\int_{a}^{x}K\left(g\left(t\right)-g\left(a\right)\right)dt}{K\left(g\left(x\right)-g\left(a\right)\right)+K\left(g\left(b\right)-g\left(x\right)\right)}\right),\end{aligned}$$

which proves the first inequality in (2.12). □

Corollary 2. *With the assumptions of Theorem 2, we have*

$$\begin{aligned}&K\left(\frac{g\left(b\right)-g\left(a\right)}{2}\right)f\left(M_{g}\left(a,b\right)+\frac{\int_{M_{g}\left(a,b\right)}^{b}K\left(g\left(b\right)-g\left(t\right)\right)dt-\int_{a}^{M_{g}\left(a,b\right)}K\left(g\left(t\right)-g\left(a\right)\right)dt}{2K\left(\frac{g\left(b\right)-g\left(a\right)}{2}\right)}\right)\\&\leq\frac{1}{2}\left[f\left(M_{g}\left(a,b\right)-\frac{1}{K\left(\frac{g\left(b\right)-g\left(a\right)}{2}\right)}\int_{a}^{M_{g}\left(a,b\right)}K\left(g\left(t\right)-g\left(a\right)\right)dt\right)\right.\\&\quad\left.+f\left(M_{g}\left(a,b\right)+\frac{1}{K\left(\frac{g\left(b\right)-g\left(a\right)}{2}\right)}\int_{M_{g}\left(a,b\right)}^{b}K\left(g\left(b\right)-g\left(t\right)\right)dt\right)\right]K\left(\frac{g\left(b\right)-g\left(a\right)}{2}\right)\\&\leq\check{S}_{k,g,a+,b-}f\left(M_{g}\left(a,b\right)\right)\\&\leq K\left(\frac{g\left(b\right)-g\left(a\right)}{2}\right)f\left(M_{g}\left(a,b\right)\right)+\frac{1}{2}\frac{f\left(b\right)-f\left(M_{g}\left(a,b\right)\right)}{b-M_{g}\left(a,b\right)}\int_{M_{g}\left(a,b\right)}^{b}K\left(g\left(b\right)-g\left(t\right)\right)dt\\&\quad-\frac{1}{2}\frac{f\left(M_{g}\left(a,b\right)\right)-f\left(a\right)}{M_{g}\left(a,b\right)-a}\int_{a}^{M_{g}\left(a,b\right)}K\left(g\left(t\right)-g\left(a\right)\right)dt.\end{aligned}\tag{2.16}$$

3. Applications for generalized Riemann-Liouville fractional integrals

If we take $k\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}t^{\alpha-1}$, where Γ is the *Gamma function*, then

$$S_{k,g,a+}f\left(x\right)=I_{a+,g}^{\alpha}f\left(x\right):=\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{x}\left[g\left(x\right)-g\left(t\right)\right]^{\alpha-1}g'\left(t\right)f\left(t\right)dt$$

for $a<x\leq b$ and

$$S_{k,g,b-}f\left(x\right)=I_{b-,g}^{\alpha}f\left(x\right):=\frac{1}{\Gamma\left(\alpha\right)}\int_{x}^{b}\left[g\left(t\right)-g\left(x\right)\right]^{\alpha-1}g'\left(t\right)f\left(t\right)dt$$

for $a\leq x<b$, which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ as defined in [25, p.100].

We consider the mixed operators

$$I^{\alpha}_{g,a+,b-}f(x) := \frac{1}{2}\left[I^{\alpha}_{a+,g}f(x) + I^{\alpha}_{b-,g}f(x)\right] \tag{3.1}$$

and

$$\breve{I}^{\alpha}_{g,a+,b-}f(x) := \frac{1}{2}\left[I^{\alpha}_{x+,g}f(b) + I^{\alpha}_{x-,g}f(a)\right] \tag{3.2}$$

for $x \in (a,b)$.

We observe that for $\alpha > 0$ we have

$$K(t) = \frac{1}{\Gamma(\alpha)}\int_0^t s^{\alpha-1}ds = \frac{t^{\alpha}}{\alpha\Gamma(\alpha)} = \frac{t^{\alpha}}{\Gamma(\alpha+1)},\ t \geq 0.$$

In what follows we assume that $f:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$. Using the inequality (2.1) we get

$$\begin{aligned}
&\frac{1}{2\Gamma(\alpha+1)}\left[\left(g(x)-g(a)\right)^{\alpha}+\left(g(b)-g(x)\right)^{\alpha}\right] \\
&\quad\times f\left(\frac{\left(g(x)-g(a)\right)^{\alpha}a+\left(g(b)-g(x)\right)^{\alpha}b+\int_a^x\left(g(x)-g(t)\right)^{\alpha}dt-\int_x^b\left(g(t)-g(x)\right)^{\alpha}dt}{\left(g(x)-g(a)\right)^{\alpha}+\left(g(b)-g(x)\right)^{\alpha}}\right) \\
&\leq \frac{1}{2\Gamma(\alpha+1)}\left[f\left(a+\frac{1}{\left(g(x)-g(a)\right)^{\alpha}}\int_a^x\left(g(x)-g(t)\right)^{\alpha}dt\right)\left(g(x)-g(a)\right)^{\alpha}\right. \\
&\quad\left.+f\left(b-\frac{1}{\left(g(b)-g(x)\right)^{\alpha}}\int_x^b\left(g(t)-g(x)\right)^{\alpha}dt\right)\left(g(b)-g(x)\right)^{\alpha}\right] \\
&\leq I^{\alpha}_{g,a+,b-}f(x) \\
&\leq \frac{1}{2\Gamma(\alpha+1)}\left[\left(g(x)-g(a)\right)^{\alpha}f(a)+\left(g(b)-g(x)\right)^{\alpha}f(b)\right] \\
&\quad+\frac{1}{2\Gamma(\alpha+1)}\left[\frac{f(x)-f(a)}{x-a}\int_a^x\left(g(x)-g(t)\right)^{\alpha}dt-\frac{f(b)-f(x)}{b-x}\int_x^b\left(g(t)-g(x)\right)^{\alpha}dt\right]
\end{aligned} \tag{3.3}$$

while from (2.12) we get

$$\begin{aligned}
&\frac{1}{2\Gamma(\alpha+1)}\left[\left(g(x)-g(a)\right)^{\alpha}+\left(g(b)-g(x)\right)^{\alpha}\right] \\
&\quad\times f\left(x+\frac{\int_x^b\left(g(b)-g(t)\right)^{\alpha}dt-\int_a^x\left(g(t)-g(a)\right)^{\alpha}dt}{\left(g(x)-g(a)\right)^{\alpha}+\left(g(b)-g(x)\right)^{\alpha}}\right) \\
&\leq \frac{1}{2\Gamma(\alpha+1)}\left[f\left(x-\frac{1}{\left(g(x)-g(a)\right)^{\alpha}}\int_a^x\left(g(t)-g(a)\right)^{\alpha}dt\right)\left(g(x)-g(a)\right)^{\alpha}\right. \\
&\quad\left.+f\left(x+\frac{1}{\left(g(b)-g(x)\right)^{\alpha}}\int_x^b\left(g(b)-g(t)\right)^{\alpha}dt\right)\left(g(b)-g(x)\right)^{\alpha}\right] \\
&\leq \breve{I}^{\alpha}_{g,a+,b-}f(x)
\end{aligned}$$

$$\leq \frac{1}{2\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}\right]f(x)$$
$$+\frac{1}{2\Gamma(\alpha+1)}\left[\frac{f(b)-f(x)}{b-x}\int_x^b (g(b)-g(t))^{\alpha}dt-\frac{f(x)-f(a)}{x-a}\int_a^x (g(t)-g(a))^{\alpha}dt\right] \tag{3.4}$$

for $x\in(a,b)$.

Also, by (2.11) and (2.16) we have

$$\frac{1}{2^{\alpha}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}$$
$$\times f\left(\frac{a+b}{2}+\frac{\int_a^{M_g(a,b)}\left(\frac{g(a)+g(b)}{2}-g(t)\right)^{\alpha}dt-\int_{M_g(a,b)}^b\left(g(t)-\frac{g(a)+g(b)}{2}\right)^{\alpha}dt}{2^{1-\alpha}(g(b)-g(a))^{\alpha}}\right)$$
$$\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\left[f\left(a+\frac{1}{\left(\frac{g(b)-g(a)}{2}\right)^{\alpha}}\int_a^{M_g(a,b)}\left(\frac{g(a)+g(b)}{2}-g(t)\right)^{\alpha}dt\right)\right.$$
$$\left.+f\left(b-\frac{1}{\left(\frac{g(b)-g(a)}{2}\right)^{\alpha}}\int_{M_g(a,b)}^b\left(g(t)-\frac{g(a)+g(b)}{2}\right)^{\alpha}dt\right)\right](g(b)-g(a))^{\alpha}$$
$$\leq I^{\alpha}_{g,a+,b-}f(M_g(a,b))$$
$$\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}[f(a)+f(b)](g(b)-g(a))^{\alpha}$$
$$+\frac{1}{2\Gamma(\alpha+1)}\frac{f(M_g(a,b))-f(a)}{M_g(a,b)-a}\int_a^{M_g(a,b)}\left(\frac{g(a)+g(b)}{2}-g(t)\right)^{\alpha}dt$$
$$-\frac{1}{2\Gamma(\alpha+1)}\frac{f(b)-f(M_g(a,b))}{b-M_g(a,b)}\int_{M_g(a,b)}^b\left(g(t)-\frac{g(a)+g(b)}{2}\right)^{\alpha}dt \tag{3.5}$$

and

$$\frac{1}{2^{\alpha}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}$$
$$\times f\left(M_g(a,b)+\frac{\int_{M_g(a,b)}^b(g(b)-g(t))^{\alpha}dt-\int_a^{M_g(a,b)}(g(t)-g(a))^{\alpha}dt}{2^{1-\alpha}(g(b)-g(a))}\right)$$
$$\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}\left[f\left(M_g(a,b)-\frac{1}{\left(\frac{g(b)-g(a)}{2}\right)^{\alpha}}\int_a^{M_g(a,b)}(g(t)-g(a))^{\alpha}dt\right)\right.$$
$$\left.+f\left(M_g(a,b)+\frac{1}{\left(\frac{g(b)-g(a)}{2}\right)^{\alpha}}\int_{M_g(a,b)}^b(g(b)-g(t))^{\alpha}dt\right)\right](g(b)-g(a))^{\alpha}$$
$$\leq \breve{I}^{\alpha}_{g,a+,b-}f(M_g(a,b))$$
$$\leq \frac{1}{2^{\alpha}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}f(M_g(a,b))$$

$$+\frac{1}{2\Gamma(\alpha+1)}\frac{f(b)-f\left(M_g(a,b)\right)}{b-M_g(a,b)}\int_{M_g(a,b)}^{b}\left(g(b)-g(t)\right)^{\alpha}dt$$
$$-\frac{1}{2\Gamma(\alpha+1)}\frac{f\left(M_g(a,b)\right)-f(a)}{M_g(a,b)-a}\int_{a}^{M_g(a,b)}\left(g(t)-g(a)\right)^{\alpha}dt. \tag{3.6}$$

If we take $g(t)=t,\ t\in[a,b]$ in (3.3) and (3.4), then we get

$$\begin{aligned}
&\frac{1}{2\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]f\left(\frac{\frac{a\alpha+x}{\alpha+1}(x-a)^{\alpha}+\frac{x+\alpha b}{\alpha+1}(b-x)^{\alpha}}{(x-a)^{\alpha}+(b-x)^{\alpha}}\right)\\
&\le\frac{1}{2\Gamma(\alpha+1)}\left[f\left(\frac{a\alpha+x}{\alpha+1}\right)(x-a)^{\alpha}+f\left(\frac{x+\alpha b}{\alpha+1}\right)(b-x)^{\alpha}\right]\\
&\le J_{a+,b-}^{\alpha}f(x)\\
&\le\frac{1}{2\Gamma(\alpha+1)}\left[(x-a)^{\alpha}f(a)+(b-x)^{\alpha}f(b)\right]\\
&\quad+\frac{1}{2\Gamma(\alpha+2)}\left[\left(f(x)-f(a)\right)(x-a)^{\alpha}-\left(f(b)-f(x)\right)(b-x)^{\alpha}\right]
\end{aligned}\tag{3.7}$$

while from (2.12) we get

$$\begin{aligned}
&\frac{1}{2\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]f\left(\frac{\frac{\alpha x+a}{\alpha+1}(x-a)^{\alpha}+\frac{b+\alpha x}{\alpha+1}(b-x)^{\alpha}}{(x-a)^{\alpha}+(b-x)^{\alpha}}\right)\\
&\le\frac{1}{2\Gamma(\alpha+1)}\left[f\left(\frac{\alpha x+a}{\alpha+1}\right)(x-a)^{\alpha}+f\left(\frac{b+\alpha x}{\alpha+1}\right)(b-x)^{\alpha}\right]\\
&\le \breve{J}_{a+,b-}^{\alpha}f(x)\\
&\le\frac{1}{2\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]f(x)\\
&\quad+\frac{1}{2\Gamma(\alpha+2)}\left[\left(f(b)-f(x)\right)(b-x)^{\alpha}-\left(f(x)-f(a)\right)(x-a)^{\alpha}\right]
\end{aligned}\tag{3.8}$$

for $x\in(a,b)$, where

$$J_{a+,b-}^{\alpha}f(x):=\frac{1}{2}\left[J_{a+}^{\alpha}f(x)+J_{b-}^{\alpha}f(x)\right]$$

and

$$\breve{J}_{a+,b-}^{\alpha}f(x):=\frac{1}{2}\left[J_{x+}^{\alpha}f(b)+J_{x-}^{\alpha}f(a)\right]$$

for $x\in(a,b)$.

If we take $x=\frac{a+b}{2}$ in (3.7) and (3.8), then we get, after required calculations

$$\begin{aligned}
\frac{(b-a)^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)&\le\frac{(b-a)^{\alpha}}{2^{\alpha+1}\Gamma(\alpha+1)}\left[f\left(\frac{(2\alpha+1)a+b}{2(\alpha+1)}\right)+f\left(\frac{a+(2\alpha+1)b}{2(\alpha+1)}\right)\right]\\
&\le J_{a+,b-}^{\alpha}f\left(\frac{a+b}{2}\right)\\
&\le\frac{(b-a)^{\alpha}}{2^{\alpha}\Gamma(\alpha+2)}f\left(\frac{a+b}{2}\right)+\frac{\alpha(b-a)^{\alpha}}{2^{\alpha}\Gamma(\alpha+2)}\frac{f(a)+f(b)}{2}
\end{aligned}$$

$$\leq \frac{(b-a)^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)} \frac{f(a)+f(b)}{2} \tag{3.9}$$

and

$$\begin{aligned}\frac{(b-a)^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) &\leq \frac{(b-a)^{\alpha}}{2^{\alpha+1}\Gamma(\alpha+1)}\left[f\left(\frac{(\alpha+2)a+\alpha b}{2(\alpha+1)}\right)+f\left(\frac{\alpha a+(\alpha+2)b}{2(\alpha+1)}\right)\right] \\ &\leq \breve{J}^{\alpha}_{a+,b-} f\left(\frac{a+b}{2}\right) \\ &\leq \frac{\alpha(b-a)^{\alpha}}{2^{\alpha}\Gamma(\alpha+2)} f\left(\frac{a+b}{2}\right)+\frac{(b-a)^{\alpha}}{2^{\alpha}\Gamma(\alpha+2)} \frac{f(b)+f(a)}{2} \\ &\leq \frac{(b-a)^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)} \frac{f(a)+f(b)}{2}. \end{aligned} \tag{3.10}$$

The last inequalities follow by the fact that

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(b)+f(a)}{2}.$$

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Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa.

E-mail: sever.dragomir@vu.edu.au