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NONLINEAR BOUNDARY VALUE PROBLEM FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH ADVANCED ARGUMENTS UNDER INTEGRAL BOUNDARY CONDITIONS

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Abstract. In this paper, we prove the existence and uniqueness of solutions of fractional differential equation involving Riemann-Liouville differential operator of order $\alpha \in (0,1)$, with advanced argument under integral boundary conditions. The uniqueness of the solution is obtained by using a Banach fixed point theorem with weighted norm. By using the comparison result and applying monotone iterative technique, the existence and uniqueness results are obtained.

1. Introduction

In this paper, we investigate the existence and uniqueness of the solution of Riemann-Liouville fractional differential equation with advanced argument under integral boundary conditions:

$$\begin{cases} D^{\alpha}_{0^+} u(t) = f\left(t, u(t), u(\theta(t))\right), \, t \in J = (0, T], \quad T > 0, \\ \quad u(0) = \lambda \displaystyle\int_0^T u(s) ds + d, \, d \in \mathbb{R}, \end{cases} \tag{1.1}$$

where λ is 1 or -1 and $f\left(t, u(t), u(\theta(t))\right) \in C\left(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\right)$, $\theta \in C\left(J, J\right)$, $t \leq \theta(t) \leq T$, $t \in J$ and $D^{\alpha}_{0^+}$ is the Riemann-Liouville fractional derivative of order α $(0 < \alpha < 1)$.

Recently in 2008, Wang and Xie [17] have studied the problem (1.1) without advanced argument and with strong assumption on a function f that f is Hölder continuous and obtained existence uniqueness results. Further Nanware and Dhaigude [14], have studied the problem of Wang and Xie without assuming the strong condition of locally Hölder continuity on a function f and developed the monotone iterative method for proving existence and uniqueness results.

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Many researchers have paid an attention and published series of papers to study the existence and uniqueness of solution of nonlinear fractional differential equations with advanced and deviating arguments (see [1, 3, 5, 7, 8, 16]). The monotone technique is an interesting and powerful tool to deal with existence results for fractional differential equations. Furthermore, the monotone technique combined with the notion of upper and lower solutions provides an effective mechanism to prove existence results for nonlinear differential equations. For details (see [4, 6, 11, 12, 13, 14]).

The paper is organized as follows: In Section 2, we present some useful definitions and lemmas. In Section 3, we prove the uniqueness of solution for the problem (1.1) by using Banach fixed point theorem. In Section 4, we develop the monotone iterative technique and then apply it to obtain existence and uniqueness results for the problem (1.1). In Section 5, we study the weakly coupled lower and upper solutions of the problem (1.1) and obtain existence and uniqueness of solution for the problem (1.1) when λ is 1 or -1.

2. Preliminaries

For the convenience of the readers, we present some definitions and basic Lemmas from the theory of fractional calculus. Let $C_{1-\alpha}(J,\mathbb{R}) = \left\{u \in C((0,T],\mathbb{R}) : t^{1-\alpha}u(t) \in C(J,\mathbb{R})\right\}$ with the norm $\|u\|_{C_{1-\alpha}} = \max_{t\in J}\left|t^{1-\alpha}u(t)\right|$. Obviously $C_{1-\alpha}(J,\mathbb{R})$ is a Banach space.

Definition 2.1. [9, 15] The Riemann-Liouville fractional integral of order $\alpha > 0$ for a continuous function $u(t) \in C([0,T])$ is defined as

$$I_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\,ds,$$

provided the integral exists. $\Gamma(\alpha)$ denotes Euler's Gamma function.

Definition 2.2 ([9, 15])**.** For function $I_{0^+}^{n-\alpha} u(t) \in AC^n[0,T]$ the Riemann-Liouville derivative of order α $(n-1 < \alpha \le n)$ can be written as

$$D_{0^+}^{\alpha} u(t) = \left(\frac{d}{dt}\right)^n \left(I_{0^+}^{n-\alpha} u(t)\right) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s)\,ds,\ t > 0.$$

Lemma 2.1 ([9])**.** *Let $u(t) \in C^n[0,T], \alpha \in (n-1,n)$, $n \in \mathbb{N}$. Then for $t \in J$,*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} u^{(k)}(0).$$

Now we state the following lemma without proof.

Lemma 2.2 ([2])**.** *Let $m \in C_{1-\alpha}(J,\mathbb{R})$ and for any $t_1 \in (0,T]$, we have*

$$m(t_1) = 0 \text{ and } m(t) \le 0 \text{ for } 0 \le t \le t_1.$$

Then it follows that,

$$D_{0^+}^{\alpha} m(t_1) \ge 0.$$

Lemma 2.3. *Function $u(t) \in C_{1-\alpha}(J,\mathbb{R})$ is a solution of the problem (1.1) if and only if $u(t)$ is a solution of the integral equation*

$$u(t) = \lambda \int_0^T u(s)ds + d + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{-\alpha} f(s, u(s), u(\theta(s)))ds.$$

Proof. The proof is easy, so we omit it. □

Lemma 2.4. *Let $\{u_\epsilon(t)\}$ be a family of continuous functions defined on J, for each $\epsilon > 0$, which satisfies*

$$\begin{cases} D_{0^+}^{\alpha} u_\epsilon(t) = f(t, u_\epsilon(t), u_\epsilon(\theta(t))), \\ u_\epsilon(0) = \lambda \int_0^T u_\epsilon(s)ds + d, \end{cases} \tag{2.1}$$

where $|f(t, u_\epsilon(t), u_\epsilon(\theta(t)))| \le M$ for $t \in J$. Then the family $\{u_\epsilon(t)\}$ is equicontinuous on J.

Proof. We need to show that the family $\{u_\epsilon(t)\}$ is equicontinuous on J. Let $0 \le t_1 < t_2 \le T$. Note that

$$\int_{t_1}^{t_2} (t-s)^{\alpha-1} ds = \int_0^{t_2-t_1} (t_2 - t_1 - s)^{\alpha-1} ds = (t_2 - t_1)^{\alpha} \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)}. \tag{2.2}$$

Then, in view of (2.2), we have

$$\begin{aligned} |u_\epsilon(t_1) - u_\epsilon(t_2)| &\le \frac{M}{\Gamma(\alpha)} \left(\int_0^{t_1} [(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}]ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds \right) \\ &\le \frac{M}{\Gamma(\alpha+1)} \left[t_1^{\alpha} - t_2^{\alpha} + 2(t_2-t_1)^{\alpha} \right] \\ &\le \frac{2M}{\Gamma(\alpha+1)} (t_2-t_1)^{\alpha} < \epsilon \end{aligned}$$

provided that $|t_2 - t_1| < \delta = \left[\frac{\epsilon\Gamma(\alpha+1)}{2M} \right]^{\frac{1}{\alpha}}$, proving the result. □

3. Uniqueness of solution

In this section, we discuss the uniqueness of solution of the problem (1.1) for Riemann-Liouville fractional differential equation with advanced argument under integral boundary conditions. We introduce the following assumption for later use.

(H_1) There exists nonnegative constants M, N such that

$$\left| f(t, u_1, u_2) - f(t, v_1, v_2) \right| \le M|u_1 - v_1| + N|u_2 - v_2|, \ \forall t \in J, \ u_i, v_i \in \mathbb{R}, i = 1, 2$$

Theorem 3.1. *Let (H_1) hold and $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. If $\lambda < \frac{\Gamma(2\alpha) - \Gamma(\alpha)T^{\alpha}(M+N)}{T\Gamma(2\alpha)}$, then the problem (1.1) has a unique solution.*

Proof. Define an operator $T : C_{1-\alpha}(J,\mathbb{R}) \to C_{1-\alpha}(J,\mathbb{R})$ as

$$Tu(t) = \lambda \int_0^T u(s)ds + d + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,u(s),u(\theta(s)))\,ds.$$

Using assumption (H_1), for any $u,v \in C_{1-\alpha}(J,\mathbb{R})$, we have

$$\begin{aligned}
\|Tu - Tv\|_{C_{1-\alpha}} &= \max_{t\in J} \left| t^{1-\alpha}\left[(Tu)(t) - (Tv)(t)\right]\right| \\
&\le \max_{t\in J} t^{1-\alpha}\lambda \int_0^T |u(s)-v(s)|\,ds + \max_{t\in J} \tfrac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
&\qquad \times \left| f(s,u(s),u(\theta(s))) - f(s,v(s),v(\theta(s)))\right| ds \\
&\le \lambda \int_0^T ds\, \|u-v\|_{C_{1-\alpha}} + \max_{t\in J} \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
&\qquad \times \left[|M(u(s)-v(s))| + |N(u(\theta(s))-v(\theta(s)))|\right] ds \\
&\le \lambda T \|u-v\|_{C_{1-\alpha}} + \max_{t\in J} \frac{t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \\
&\qquad \times \left[M s^{\alpha-1} \left\|u-y\right\|_{C_{1-\alpha}} + N(\theta(s))^{\alpha-1} \left\|u-y\right\|_{C_{1-\alpha}}\right] ds \\
&\le \lambda T \|u-v\|_{C_{1-\alpha}} + \max_{t\in J} \frac{(M+N)t^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{\alpha-1} \|u-v\|_{C_{1-\alpha}}\, ds \\
&\le \lambda T \|u-v\|_{C_{1-\alpha}} + \max_{t\in J} \frac{(M+N)t^{\alpha}}{\Gamma(\alpha)} \int_0^1 (1-\eta)^{\alpha-1}\eta^{\alpha-1} d\eta\, \|u-v\|_{C_{1-\alpha}} \\
&\le \left[\lambda T + \frac{\Gamma(\alpha)T^{\alpha}}{\Gamma(2\alpha)}(M+N)\right] \|u-v\|_{C_{1-\alpha}}.
\end{aligned}$$

Therefore, $\|Tu - Tv\|_{C_{1-\alpha}} \le \|u-v\|_{C_{1-\alpha}}$ and T is a contraction operator on $C_{1-\alpha}(J,\mathbb{R})$. Consequently, by the contraction mapping theorem T has a unique fixed point $u(t)$, i.e. $u(t)$ is a unique solution of the problem (1.1). The proof is complete. □

Lemma 3.1. *Suppose that M, N are constants and $\sigma \in C_{1-\alpha}(J,\mathbb{R})$. Function $u \in C_{1-\alpha}(J,\mathbb{R})$ is a unique solution of the following linear problem*

$$\begin{cases} D^{\alpha}_{0^+} u(t) + Mu(t) + Nu(\theta(s)) = \sigma(t), & t \in J,\ 0<\alpha<1, \\ u(0) = \lambda \int_0^T u(s)ds + d, & d \in \mathbb{R}, \end{cases} \tag{3.1}$$

if u is a unique solution of the following integral equation

$$u(t) = \lambda \int_0^T u(s)ds + d + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\left[-Mu(s) - Nu(\theta(s)) + \sigma(s)\right] ds.$$

Proof. By the proof of Theorem 3.1, we see the solving (3.1) is equivalent to solving a fixed point problem with operator T_σ defined by

$$T_\sigma u(t) = \lambda \int_0^T u(s)ds + d + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\left[-Mu(s) - Nu(\theta(s)) + \sigma(s)\right] ds.$$

For any $\sigma \in C_{1-\alpha}(J,\mathbb{R})$. Then the operator T_σ has a unique fixed point. □

4. Monotone iterative technique

In this section, we mainly investigate the existence and uniqueness of solution of the problem (1.1) for fractional differential equation with advanced argument by monotone iterative technique. We need the following comparison result which play a very important role in further discussion.

Lemma 4.1. *Let $\alpha \in (0,1)$, $\theta(t) \in C(J,J)$ and $t \le \theta(t)$ on J. Suppose that $p \in C_{1-\alpha}(J,\mathbb{R})$ satisfies the inequalities*

$$\begin{cases} D^{\alpha}_{0^+} p(t) \le -Mp(t) - Np(\theta(t)) \equiv Fp(t), \ t \in J \\ \quad p(0) \le 0, \end{cases} \tag{4.1}$$

where M and $N \ge 0$. If

$$-T^{\alpha}(M+N)\Gamma(1-\alpha) < 1,$$

then $p(t) \le 0$ for all $t \in J$.

Proof. Put $p_\varepsilon(t) = p(t) - \varepsilon$, $\varepsilon > 0$. Then

$$\begin{aligned} D^{\alpha}_{0^+} p_\varepsilon(t) &= D^{\alpha}_{0^+} p(t) - D^{\alpha}_{0^+}\varepsilon \\ &\le Fp(t) - \frac{\varepsilon}{t^{\alpha}\Gamma(1-\alpha)} \\ &\le -Mp_\varepsilon(t) - Np_\varepsilon(\theta(t)) + \varepsilon[-(M+N) - (1/(t^{\alpha}\Gamma(1-\alpha)))] \\ &< Fp_\varepsilon(t), \end{aligned}$$

and

$$p_\varepsilon(0) = p(0) - \varepsilon < 0.$$

We prove that $p_\varepsilon(t) < 0$ on J. Assume that it is not true. It means there exists $t_1 \in (0,T]$ such that $p_\varepsilon(t_1) = 0$ and $p_\varepsilon(t) < 0$, $t \in (0,t_1)$. In view of Lemma 2.2 we have $D^{\alpha}_{0^+} p_\varepsilon(t_1) \ge 0$. It follows that

$$0 < Fp_\varepsilon(t_1) = -Np_\varepsilon(\theta(t_1)).$$

If $N = 0$, then $0 < 0$, so it is a contradiction. If $-N < 0$, then $p_\varepsilon(\theta(t_1)) < 0$, it is a contradiction too. This proves that $p_\varepsilon(t) < 0$ on J. So $p(t) - \varepsilon < 0$ on J. Now, if $\varepsilon \to 0$, we get required result. □

Definition 4.1. A pair of functions $(v_0(t), w_0(t))$ in $C_{1-\alpha}(J,\mathbb{R})$ is called lower and upper solutions of the problem (1.1) for $\lambda = 1$ if

$$\begin{aligned} &D^{\alpha}_{0^+} v_0(t) \le f(t, v_0(t), v_0(\theta(t))), \qquad v_0(0) \le \int_0^T v_0(s)ds + d, \\ &D^{\alpha}_{0^+} w_0(t) \ge f(t, w_0(t), w_0(\theta(t))), \quad w_0(0) \ge \int_0^T w_0(s)ds + d. \end{aligned}$$

Theorem 4.1. *Assume that:*

(i) *functions $v_0(t)$ and $w_0(t)$ in $C_{1-\alpha}(J,\mathbb{R})$ are lower and upper solutions of the problem (1.1) such that $v_0(t) \le w_0(t)$ on J,*

(ii) $f(t,u(t),u(\theta(t))) \in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $\theta \in C(J,J)$, $t \le \theta(t) \le T$, $t \in J$,

(iii) *there exists nonnegative constants M, N such that function f satisfies the condition*

$$f(t,v_1,v_2) - f(t,u_1,u_2) \ge -M(v_1-u_1) - N(v_2-u_2),$$

for $v_0(t) \le u_1 \le v_1 \le w_0(t)$, $v_0(\theta(t)) \le u_2 \le v_2 \le w_0(\theta(t))$.

Then there exists monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ in $C_{1-\alpha}(J,\mathbb{R})$ such that

$$\{v_n(t)\} \to v(t) \text{ and } \{w_n(t)\} \to w(t) \text{ as } n \to \infty$$

where $v(t)$ and $w(t)$ are minimal and maximal solutions of the problem (1.1) respectively, and $v(t) \le u(t) \le w(t)$ on J.

Proof. We consider the following linear problem:

$$\begin{cases} D^{\alpha}_{0^+}u(t) = -Mu(t) - Nu(\theta(t)) + \sigma(t), \\ \quad u(0) = \displaystyle\int_0^T u(s)ds + d, \end{cases} \tag{4.2}$$

where $\sigma(t) = f\left(t,\eta(t),\eta(\theta(t))\right) + M\eta(t) + N\eta(\theta(t))$ and $\eta \in C_{1-\alpha}(J,\mathbb{R})$.

Obviously, by Lemma 3.1, the linear problem (4.2) has a unique solution $u(t)$.

We next define the iterates as follows:

$$\begin{cases} D^{\alpha}_{0^+}v_{n+1}(t) = f(t,v_n(t),v_n(\theta(t))) - M[v_{n+1}(t) - v_n(t)] - N[v_{n+1}(\theta(t)) - v_n(\theta(t))], \\ \quad v_{n+1}(0) = \int_0^T v_n(s)ds + d, \end{cases} \tag{4.3}$$

and

$$\begin{cases} D^{\alpha}_{0^+}w_{n+1}(t) = f(t,w_n(t),w_n(\theta(t))) - M[w_{n+1}(t) - w_n(t)] - N[w_{n+1}(\theta(t)) - w_n(\theta(t))], \\ \quad w_{n+1}(0) = \int_0^T w_n(s)ds + d, \end{cases} \tag{4.4}$$

Obviously, the above arguments imply the existence of the unique solutions $v_{n+1}(t)$ and $w_{n+1}(t)$ of the problems (4.3), (4.4). By putting $n = 0$ in the problems (4.3), (4.4), we get the existence of solutions $v_1(t)$ and $w_1(t)$. We show that $v_0(t) \le v_1(t) \le w_1(t) \le w_0(t)$. For this, consider $p(t) = v_0(t) - v_1(t)$ on J, and $v_0(t)$ is the lower solution of the problem (1.1). Then

$$\begin{aligned} D^{\alpha}_{0^+}p(t) &= D^{\alpha}_{0^+}v_0(t) - D^{\alpha}_{0^+}v_1(t) \\ &\le -M[v_0(t) - v_1(t)] - N[v_0(\theta(t)) - v_1(\theta(t))] \\ &\le -Mp(t) - Np(\theta(t)), \end{aligned}$$

and

$$p(0) = v_0(0) - v_1(0) \le \int_0^T v_0(s)ds + d - \int_0^T v_0(s)ds - d = 0.$$

By Lemma 4.1, we get $p(t) \le 0$, implies that $v_0(t) \le v_1(t)$ on J. Similarly, we can prove $w_1 \le w_0$ and $v_1(t) \le w_1(t)$ on J. Thus $v_0(t) \le v_1(t) \le w_1(t) \le w_0(t)$. Assume that for some $k > 1$,

$$v_{k-1}(t) \le v_k(t) \le w_k(t) \le w_{k-1}(t) \text{ on } J.$$

We claim that $v_k(t) \le v_{k+1}(t) \le w_{k+1}(t) \le w_k(t)$ on J. To prove the claim, set $p(t) = v_k(t) - v_{k+1}(t)$, we have

$$\begin{aligned} D_{0^+}^{\alpha} p(t) &= D_{0^+}^{\alpha} v_k(t) - D_{0^+}^{\alpha} v_{k+1}(t) \\ &\le -M[v_k(t) - v_{k+1}(t)] - N[v_n(\theta(t)) - v_{k+1}(\theta(t))] \\ &\le -Mp(t) - Np(\theta(t)), \end{aligned}$$

and

$$\begin{aligned} p(0) &= v_k(0) - v_{k+1}(0) = \int_0^T v_{k-1}(s)ds - \int_0^T v_k(s)ds \\ &\le \int_0^T v_k(s)ds - \int_0^T v_k(s)ds = 0. \end{aligned}$$

By Lemma 4.1, we get $p(t) \le 0$, implies that $v_k(t) \le v_{k+1}(t)$ on J. Similarly, we can prove that $w_{k+1}(t) \le w_k(t)$ and $v_{k+1}(t) \le w_{k+1}(t)$ on J. By the principle of mathematical induction, we have

$$v_0 \le v_1 \le v_2 \le \cdots \le v_k \le w_k \le \cdots \le w_2 \le w_1 \le w_0 \text{ on } J. \tag{4.5}$$

Obviously, the sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ are uniformly bounded. We observe that $\{D_{0^+}^{\alpha} v_n\}$ and $\{D_{0^+}^{\alpha} w_n\}$ are also uniformly bounded on J, in view of the relations (4.3) and (4.4). Then using Lemma 2.4 we can conclude that sequences $\{v_n(t)\}$,$\{w_n(t)\}$ are equicontinuous. Hence by the Ascoli-Arzela theorem, the sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ converge uniformly to v and w, respectively on J. If $n \to \infty$, then we see that v, w are continuous solutions of the problem (1.1).

Now, we prove that $v(t)$ and $w(t)$ are the minimal and maximal solutions of the problem (1.1). Let $u(t)$ be any solution of the problem (1.1) different from $v(t)$ and $w(t)$, so that there exists k such that $v_k(t) \le u(t) \le w_k(t)$ on J. Set $p(t) = v_{k+1}(t) - u(t)$. we have

$$\begin{aligned} D_{0^+}^{\alpha} p(t) &= D_{0^+}^{\alpha} v_{k+1}(t) - D_{0^+}^{\alpha} u(t) \\ &\le -M[v_{k+1}(t) - u(t)] - N[v_{k+1}(\theta(t)) - u(\theta(t))] \\ &\le -Mp(t) - Np(\theta(t)), \end{aligned}$$

and

$$p(0) = v_{k+1}(0) - u(0) = \int_0^T [v_k(s) - u(s)]\, ds \le 0.$$

By Lemma 4.1, we get $p(t) \le 0$, implies that $v_{k+1}(t) \le u(t)$ for all k on J. Similarly we can prove $u(t) \le w_{k+1}(t)$ for all k on J. Since $v_0(t) \le u(t) \le u_0(t)$ on J. By induction it follows that $v_k(t) \le u(t)$ and $u(t) \le w_k(t)$ for all k. Thus $v_k(t) \le u(t) \le w_k(t)$ on J. Taking limit as $k \to \infty$, we get $v(t) \le u(t) \le w(t)$ on J. The functions $v(t)$ and $w(t)$ are the minimal and maximal solutions to the problem (1.1). The proof is complete. □

Next, we prove the uniqueness of solution of the problem (1.1) as follows:

Theorem 4.2. *Assume that:*

(i) *functions $v_0(t)$ and $w_0(t)$ in $C_{1-\alpha}(J,\mathbb{R})$ are lower and upper solutions of the problem* (1.1) *such that $v_0(t) \le w_0(t)$ on J,*

(ii) $f(t,u(t),u(\theta(t))) \in C(J\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $\theta \in C(J,J)$, $t \le \theta(t) \le T$, $t \in J$,

(iii) *there exists nonnegative constants M, N such that function f satisfies the condition*

$$f(t,v_1,v_2) - f(t,u_1,u_2) \le M(v_1-u_1) + N(v_2-u_2), \tag{4.6}$$

for $v_0(t) \le u_1 \le v_1 \le w_0(t)$, $v_0(\theta(t)) \le u_2 \le v_2 \le w_0(\theta(t))$.

(iv) $\lim_{n\to\infty} \|w_n(t) - v_n(t)\| = 0$, *where the norm is defined by $\|f\| = \int_0^T |f(s)|\,ds$ then the problem* (1.1) *has a unique solution.*

Proof. Since $v(t) \le w(t)$, it is sufficient to prove $v(t) \ge w(t)$. Consider $p(t) = w(t) - v(t)$, then

$$\begin{aligned} D^\alpha_{0^+} p(t) &= D^\alpha_{0^+} w(t) - D^\alpha_{0^+} v(t) \\ &\le M[w(t)-v(t)] + N[w(\theta(t)) - v(\theta(t))] \\ &\le Mp(t) + Np(\theta(t)), \end{aligned}$$

and

$$\begin{aligned} p(0) &= w(0) - v(0) = \int_0^T [w(s) - v(s)]\,ds \\ &= \|w(0) - v(0)\| = \lim_{n\to\infty} \|w_n(0) - v_n(0)\| = 0. \end{aligned}$$

By Lemma 4.1, we get $p(t) \le 0$, implies that $w(t) \le v(t)$. Hence $v(t) = w(t)$ is the unique solution of the problem (1.1) on J. □

5. Weakly coupled lower and upper solutions

In this section, we investigate the existence and uniqueness of solution of the problem (1.1) by weakly coupled lower and upper solutions.

Definition 5.1. A pair of functions $(v_0(t), w_0(t))$ in $C_{1-\alpha}(J,\mathbb{R})$ is called weakly coupled lower and upper solutions of *the problem* (1.1) for $\lambda = -1$ if

$$D^\alpha_{0^+} v_0(t) \le f(t, v_0(t), v_0(\theta(t))), \qquad v_0(0) \le -\int_0^T w_0(s)ds + d,$$

$$D^{\alpha}_{0^+} w_0(t) \geq f(t, w_0(t), w_0(\theta(t))), \quad w_0(0) \geq -\int_0^T v_0(s)ds + d.$$

Theorem 5.1. *Assume that:*

(i) *functions $v_0(t)$ and $w_0(t)$ in $C_{1-\alpha}(J,\mathbb{R})$ are weakly coupled lower and upper solutions of the problem (1.1) with $\lambda = -1$ such that $v_0(t) \leq w_0(t)$ on J,*

(ii) $f(t, u(t), u(\theta(t))) \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\theta \in C(J, J)$, $t \leq \theta(t) \leq T$, $t \in J$,

(iii) *there exists nonnegative constants M, N such that function f satisfies the condition*

$$f(t, v_1, v_2) - f(t, u_1, u_2) \geq -M(v_1 - u_1) - N(v_2 - u_2),$$

for $v_0(t) \leq u_1 \leq v_1 \leq w_0(t)$, $v_0(\theta(t)) \leq u_2 \leq v_2 \leq w_0(\theta(t))$.
Then there exists monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ in $C_{1-\alpha}(J,\mathbb{R})$ such that

$$\{v_n(t)\} \to v(t) \text{ and } \{w_n(t)\} \to w(t) \text{ as } n \to \infty$$

where $v(t)$ and $w(t)$ are minimal and maximal solutions of the problem (1.1) with $\lambda = -1$, respectively; and $v(t) \leq u(t) \leq w(t)$ on J.

Proof. We consider the following linear problem:

$$\begin{cases} D^{\alpha}_{0^+} u(t) = -Mu(t) - Nu(\theta(t)) + \sigma(t) \\ u(0) = -\displaystyle\int_0^T u(s)ds + d, \end{cases} \tag{5.1}$$

where $\sigma(t) = f\left(t, \eta(t), \eta(\theta(t))\right) - M\eta(t) - N\eta(\theta(t))$ and $\eta \in C_{1-\alpha}(J,\mathbb{R})$.

The unique of solution of the linear problem (5.1) can be proved as in Lemma 3.1.

Define the iterates as follows:

$$\begin{cases} D^{\alpha}_{0^+} v_{n+1}(t) = f(t, v_n(t), v_n(\theta(t))) - M[v_{n+1}(t) - v_n(t)] - N[v_{n+1}(\theta(t)) - v_n(\theta(t))], \\ v_{n+1}(0) = -\int_0^T w_n(s)ds + d, \end{cases} \tag{5.2}$$

and

$$\begin{cases} D^{\alpha}_{0^+} w_{n+1}(t) = f(t, w_n(t), w_n(\theta(t))) - M[w_{n+1}(t) - w_n(t)] - N[w_{n+1}(\theta(t)) - w_n(\theta(t))], \\ w_{n+1}(0) = -\int_0^T v_n(s)ds + d, \end{cases} \tag{5.3}$$

Obviously, the above arguments imply the existence of the unique solutions $v_{n+1}(t)$ and $w_{n+1}(t)$ for the problems (5.2), (5.3). By setting $n = 0$ in the problems (5.2), (5.3), we get the existence of solutions $v_1(t)$ and $w_1(t)$. We show that $v_0(t) \leq v_1(t) \leq w_1(t) \leq w_0(t)$. For this, consider $p(t) = v_0(t) - v_1(t)$ on J, and $v_0(t)$ is the lower solution of the problem (1.1). Then

$$D^{\alpha}_{0^+} p(t) = D^{\alpha}_{0^+} v_0(t) - D^{\alpha}_{0^+} v_1(t)$$

$$\leq -M[v_0(t)-v_1(t)]-N[v_0(\theta(t))-v_1(\theta(t))]$$
$$\leq -Mp(t)-Np(\theta(t)),$$

and

$$p(0) = v_0(0)-v_1(0) \leq -\int_0^T w_0(s)ds + d + \int_0^T w_0(s)ds - d = 0.$$

By Lemma 4.1, we get $p(t) \leq 0$, implies that $v_0(t) \leq v_1(t)$ on J. Similarly, we can prove $w_1 \leq w_0$ and $v_1(t) \leq w_1(t)$ on J. Thus $v_0(t) \leq v_1(t) \leq w_1(t) \leq w_0(t)$. Assume that for some $k > 1$,

$$v_{k-1}(t) \leq v_k(t) \leq w_k(t) \leq w_{k-1}(t) \text{ on } J.$$

We claim that $v_k(t) \leq v_{k+1}(t) \leq w_{k+1}(t) \leq w_k(t)$ on J. To prove the claim, set $p(t) = v_k(t) - v_{k+1}(t)$, we have

$$\begin{aligned} D^\alpha_{0^+} p(t) &= D^\alpha_{0^+} v_k(t) - D^\alpha_{0^+} v_{k+1}(t) \\ &\leq -M[v_k(t)-v_{k+1}(t)]-N[v_n(\theta(t))-v_{k+1}(\theta(t))] \\ &\leq -Mp(t)-Np(\theta(t)). \end{aligned}$$

and

$$\begin{aligned} p(0) &= v_k(0)-v_{k+1}(0) = -\int_0^T w_{k-1}(s)ds + \int_0^T w_k(s)ds \\ &\leq -\int_0^T w_k(s)ds + \int_0^T w_k(s)ds = 0. \end{aligned}$$

By Lemma 4.1, we get $p(t) \leq 0$, implies that $v_k(t) \leq v_{k+1}(t)$ on J. Similarly, we can prove that $v_{k+1}(t) \leq w_{k+1}(t)$ and $w_{k+1}(t) \leq w_k(t)$ on J. By the principle of mathematical induction, we have

$$v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_k \leq w_k \leq \cdots \leq w_2 \leq w_1 \leq w_0 \text{ on } J. \tag{5.4}$$

Obviously, the sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ are uniformly bounded. We observe that $\{D^\alpha_{0^+} v_n\}$ and $\{D^\alpha_{0^+} w_n\}$ are also uniformly bounded on J, in view of the relations (5.2) and (5.3). Then using Lemma 2.4 we can conclude the equicontinuous of the sequences $\{v_n(t)\}, \{w_n(t)\}$. Hence by the Ascoli-Arzela theorem, the sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ converge uniformly to v and w, respectively on J. If $n \to \infty$, then we see that v, w are continuous solutions of the problem (1.1) with $\lambda = -1$. Now, we prove that $v(t)$ and $w(t)$ are the minimal and maximal solutions of the problem (1.1) with $\lambda = -1$. Let $u(t)$ be any solution of the problem (1.1) different from $v(t)$ and $w(t)$, so that there exists k such that $v_k(t) \leq u(t) \leq w_k(t)$ on J. Set $p(t) = v_{k+1}(t) - u(t)$. we have

$$\begin{aligned} D^\alpha_{0^+} p(t) &= D^\alpha_{0^+} v_{k+1}(t) - D^\alpha_{0+} u(t) \\ &\leq -M[v_{k+1}(t)-u(t)] - N[v_{k+1}(\theta(t))-u(\theta(t))] \\ &\leq -Mp(t)-Np(\theta(t)), \end{aligned}$$

and

$$p(0) = v_{k+1}(0) - u(0) = -\int_0^T [w_k(s) - u(s)]\, ds \leq 0.$$

By Lemma 4.1, we get $p(t) \leq 0$, implies that $v_{k+1}(t) \leq u(t)$ for all k on J. Similarly we can prove $u(t) \leq w_{k+1}(t)$ for all k on J. Since $v_0(t) \leq u(t) \leq u_0(t)$ on J. By induction it follows that $v_k(t) \leq u(t)$ and $u(t) \leq w_k(t)$ for all k. Thus $v_k(t) \leq u(t) \leq w_k(t)$ on J. Taking limit as $k \to \infty$, it follows that $v(t) \leq u(t) \leq w(t)$ on J. The functions $v(t)$ and $w(t)$ are the minimal and maximal solutions to the problem (1.1) with $\lambda = -1$. The proof is complete. □

Next, we prove the uniqueness of solutions of the problem (1.1) as follows:

Theorem 5.2. *Assume that:*

(i) *functions $v_0(t)$ and $w_0(t)$ in $C_{1-\alpha}(J,\mathbb{R})$ are weakly coupled lower and upper solutions of the problem (1.1) with $\lambda = -1$ such that $v_0(t) \leq w_0(t)$ on J,*

(ii) $f(t, u(t), u(\theta(t))) \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\theta \in C(J, J)$, $t \leq \theta(t) \leq T$, $t \in J$,

(iii) *there exists nonnegative constants M, N such that function f satisfies the condition*

$$f(t, v_1, v_2) - f(t, u_1, u_2) \leq M(v_1 - u_1) + N(v_2 - u_2),$$

for $v_0(t) \leq u_1 \leq v_1 \leq w_0(t)$, $v_0(\theta(t)) \leq u_2 \leq v_2 \leq w_0(\theta(t))$.

(iv) $\lim\limits_{n\to\infty} \|w_n(t) - v_n(t)\| = 0$, *where the norm is defined by $\|f\| = \int_0^T |f(s)|\, ds$ then the problem* (1.1) *has a unique solution with $\lambda = -1$.*

Proof. It is as in Theorem 4.2. □

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