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ON THE PLANARITY AND PERFECTNESS OF ANNIHILATOR IDEAL GRAPHS

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Abstract. Let R be a commutative ring with unity. The annihilator ideal graph of R, denoted by $\Gamma_{\mathrm{Ann}}(R)$, is a graph whose vertices are all non-trivial ideals of R and two distinct vertices I and J are adjacent if and only if $I \cap \mathrm{Ann}_R(J) \neq \{0\}$ or $J \cap \mathrm{Ann}_R(I) \neq \{0\}$. In this paper, all rings with planar annihilator ideal graphs are classified. Furthermore, we show that all annihilator ideal graphs are perfect. Among other results, it is proved that if $\Gamma_{\mathrm{Ann}}(R)$ is a tree, then $\Gamma_{\mathrm{Ann}}(R)$ is star.

1. Introduction

First we recall some necessary notation and terminology from ring theory and graph theory.

Throughout this paper, R denotes a unitary commutative ring which is not an integral domain. The sets of all zero-divisors, nilpotent elements, non-trivial ideals, non-trivial nilpotent ideals and minimal prime ideals of R are denoted by $Z(R)$, $\mathrm{Nil}(R)$, $\mathbb{I}(R)$, $\mathbb{I}_N(R)$ and $\mathrm{Min}(R)$, respectively. For a subset T of a ring R we let $T^* = T \setminus \{0\}$. A non-zero ideal of R is called *essential*, if I has a non-zero intersection with any non-zero ideal of R. The ring R is said to be *reduced* if it has no non-zero nilpotent element. We say that $\mathrm{depth}(R) = 0$, whenever every non-unit element of R is a zero-divisor. Also, $x \in R$ is called *regular*, if it is neither unit nor zero-divisor. For any undefined notation or terminology in ring theory, we refer the reader to [3, 11].

Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. If x, y are adjacent vertices, then we write $x - y$. By $\overline{G}$, $\mathrm{diam}(G)$ and $\mathrm{gr}(G)$, we mean the complement, the diameter and the girth of G, respectively. A complete bipartite graph with part sizes m and n is denoted by $K_{m,n}$. If the size of one of the parts is 1, then the graph is said to be a *star graph*. Also, a complete graph and a cycle of order n are denoted by K_n and C_n, respectively. For any $x \in V(G)$, $\deg(x)$ represents the degree of x. The graph $H = (V_0, E_0)$ is a

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subgraph of G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph by* V_0, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \mid u, v \in V_0\}$. The *subdivision* of G is a graph obtained from G by subdividing some of the edges, that is, by replacing the edges by paths having at most their endvertices in common. A graph G is said to *null* if it has no edge. A *clique* of G is a complete subgraph of G and the number of vertices in a largest clique of G denoted by $\omega(G)$, is called the *clique number* of G. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. A graph G is said to be *weakly perfect* if $\omega(G) = \chi(G)$. Moreover G is called *perfect* if every induced subgraph of G is weakly perfect. Let G and H be two arbitrary graphs. By $G \vee H$, we denote the *join* of G and H. A graph G is said to be *planar* if it can be drawn on the plane in such a way that its edges intersect only at their endpoints. For any undefined notation or terminology in graph theory, we refer the reader to [6, 7].

Assigning a graph to a ring gives us the ability to translate algebraic properties of rings into graph theory language and vice versa. It leads to arising interesting algebraic and combinatorics problems. Therefore, the study of graphs associated with rings has attracted many researches. There are a lot of papers which apply combinatorial methods to obtain algebraic results in ring theory (see for instance [1], [4], [12] and [13]). Moreover, for the most recent study in this direction see [8] and [13]. The *annihilator ideal graph* of a ring R, denoted by $\Gamma_{\mathrm{Ann}}(R)$, is a simple graph with the vertex set $\mathbb{I}(R)$ and two distinct vertices I and J are adjacent if and only if $I \cap \mathrm{Ann}_R(J) \neq \{0\}$ or $J \cap \mathrm{Ann}_R(I) \neq \{0\}$. This graph was first introduced and studied in [2]. Some of basic properties of $\Gamma_{\mathrm{Ann}}(R)$ and conditions under which $\Gamma_{\mathrm{Ann}}(R)$ is complete or complete bipartite may be found in [2]. After that authors in [9] studied the clique number and chromatic number of $\Gamma_{\mathrm{Ann}}(R)$. In this paper, first we complete the study of basic properties of $\Gamma_{\mathrm{Ann}}(R)$. And then, the planarity and perfectness of $\Gamma_{\mathrm{Ann}}(R)$ are investigated.

2. Complete bipartite annihilator ideal graphs

Let R be a ring. In this section, we show that $\Gamma_{\mathrm{Ann}}(R)$ is tree if and only if it is complete bipartite if and only if it is star.

We first recall the following useful lemma from [9].

Lemma 2.1. [9, Lemma 2.1] *Let R be a ring and $I, J \in \mathbb{I}(R)$. Then the following statements hold.*

(i) *If $I — J$ is not an edge of $\Gamma_{\mathrm{Ann}}(R)$, then $\mathrm{Ann}_R(I) = \mathrm{Ann}_R(J)$. Moreover, if R is a reduced ring, then the converse is also true.*

(ii) *If $I \cap \mathrm{Ann}_R(I) \neq \{0\}$, then I is adjacent to every other vertex.*

(iii) *If $\mathrm{Ann}_R(I) = \{0\}$ and $\mathrm{Ann}_R(J) \neq \{0\}$, then $I — J$ is an edge of $\Gamma_{\mathrm{Ann}}(R)$.*

To prove Theorem 2.1, we state the following lemma.

Lemma 2.2. *Let R be a ring. If I and J are adjacent in $\Gamma_{\mathrm{Ann}}(R)$, then the following statements hold.*

(i) *If $\Gamma_{\mathrm{Ann}}(R)$ is triangle free, then $\mathrm{Ann}_R(I) \neq \mathrm{Ann}_R(J)$.*

(ii) *Either $I - K$ or $J - K$, for every $K \in \mathbb{I}(R) \setminus \{I, J\}$.*

Proof. (i) If $\Gamma_{\mathrm{Ann}}(R) = K_2$, then it is not hard to check that $IJ = 0$ and so $\mathrm{Ann}_R(I) \neq \mathrm{Ann}_R(J)$. Thus assume that $\Gamma_{\mathrm{Ann}}(R)$ has at least three vertices. Since I and J are adjacent, without loss of generality we may suppose that $I \cap \mathrm{Ann}_R(J) \neq \{0\}$. Thus $\mathrm{Ann}_R(J) \neq \{0\}$. Suppose to the contrary, $\mathrm{Ann}_R(I) = \mathrm{Ann}_R(J)$ and consider the following cases:

Case 1. $I \neq \mathrm{Ann}_R(I)$. Since $I\mathrm{Ann}_R(I) = 0$ and $J\mathrm{Ann}_R(I) = 0$, $I - \mathrm{Ann}_R(I) - J - I$ is a triangle in $\Gamma_{\mathrm{Ann}}(R)$, a contradiction. If $J = \mathrm{Ann}_R(I)$, then $J = \mathrm{Ann}_R(J)$ and this is similar to Case 2.

Case 2. $I = \mathrm{Ann}_R(I)$. If there exists an ideal K with $\mathrm{Ann}_R(K) = 0$, then by part (iii) of Lemma 2.1, $I - L - J - I$ is a triangle, which is impossible. Thus suppose that $\mathrm{Ann}_R(K) \neq 0$, for every ideal K of R. By part (ii) of Lemma 2.1, I is adjacent to every other vertex of $\Gamma_{\mathrm{Ann}}(R)$. Therefore, if $K \in \mathbb{I}(R) \setminus \{I, J\}$, then $I \cap \mathrm{Ann}_R(K) \neq \{0\}$ or $K \cap \mathrm{Ann}_R(I) \neq \{0\}$. If $K \cap \mathrm{Ann}_R(I) \neq \{0\}$, then $I - K - - J - I$ is a triangle, a contradiction. So let $I \cap \mathrm{Ann}_R(K) \neq \{0\}$. Hence $\mathrm{Ann}_R(J) \cap \mathrm{Ann}_R(K) \neq \{0\}$, i.e., $I - \mathrm{Ann}_R(K) - J - I$ is a cycle of order three, a contradiction. If $I = \mathrm{Ann}_R(K)$, for every vertex $K \neq I$, then [5, Theorem 2.2] implies that $Z(R) = \mathrm{Ann}_R(x) = \mathrm{Ann}_R(I)$, for some $x \in R$. Thus $I - K - J - I$ is a triangle, a contradiction.

(ii) Suppose that $I - J$ is an edge of $\Gamma_{\mathrm{Ann}}(R)$. If $I - K$ is not an edge of $\Gamma_{\mathrm{Ann}}(R)$, then by part (1) of Lemma 2.1, $\mathrm{Ann}_R(I) = \mathrm{Ann}_R(K)$. If $\mathrm{Ann}_R(J) \neq \mathrm{Ann}_R(K)$, then $J - K$ is an edge of $\Gamma_{\mathrm{Ann}}(R)$ by part (1) of Lemma 2.1, and if $\mathrm{Ann}_R(J) = \mathrm{Ann}_R(K)$, then $J - K$ is again an edge of $\Gamma_{\mathrm{Ann}}(R)$. □

Theorem 2.1. *Let R be a ring and $\Gamma_{\mathrm{Ann}}(R)$ is a tree. If I and J are adjacent in $\Gamma_{\mathrm{Ann}}(R)$, then* $\deg(I) = 1$ *or* $\deg(J) = 1$

Proof. Suppose that I is adjacent to J, $\deg(I) \geq 2$ and $\deg(J) \geq 2$. Let $I_1 \neq J$ be adjacent to I and $J_1 \neq I$ be adjacent to J. Since $\Gamma_{\mathrm{Ann}}(R)$ is a tree, I_1 is not adjacent to J and J_1 in $\Gamma_{\mathrm{Ann}}(R)$. By part (i) of Lemma 2.1, $\mathrm{Ann}_R(I_1) = \mathrm{Ann}_R(J)$, and $\mathrm{Ann}_R(I_1) = \mathrm{Ann}_R(J_1)$. Thus $\mathrm{Ann}_R(J) = \mathrm{Ann}_R(J_1)$, which is impossible by part 1 of Lemma 2.2. □

Now, by Theorem 2.1, one can easily deduce the next result.

Corollary 2.1. *Let R be a ring, which is not an integral domain. Then $\Gamma_{\mathrm{Ann}}(R)$ is a star graph if and only if $\Gamma_{\mathrm{Ann}}(R)$ is a tree.*

Theorem 2.2. *Let R be a ring, which is not an integral domain. Then $\Gamma_{\mathrm{Ann}}(R)$ is a complete bipartite graph if and only if $\Gamma_{\mathrm{Ann}}(R)$ is a star graph.*

Proof. One side is clear. To prove the other side, suppose that $\Gamma_{\mathrm{Ann}}(R)$ is a complete bipartite graph and V_1, V_2 are two parts of $\Gamma_{\mathrm{Ann}}(R)$. We claim that $|V_i| = 1$ for some $1 \le i \le 2$. Suppose to the contrary $|V_i| \ge 2$ for every $1 \le i \le 2$. Thus $\Gamma_{\mathrm{Ann}}(R)$ contains a cycle. But $\mathrm{gr}(\Gamma_{\mathrm{Ann}}(R)) \in \{3, \infty\}$ (see [2, Theorem 5]), which is a contradiction. □

Corollary 2.2. *Let R be a ring, which is not an integral domain. Then the following statements are equivalent:*

(i) $\Gamma_{\mathrm{Ann}}(R)$ *is a complete bipartite graph;*

(ii) $\Gamma_{\mathrm{Ann}}(R)$ *is a star graph;*

(iii) $\Gamma_{\mathrm{Ann}}(R)$ *is a tree.*

By Corollary 2.2, we observed that $\mathrm{gr}(\Gamma_{\mathrm{Ann}}(R)) = \infty$ if and only if $\Gamma_{\mathrm{Ann}}(R)$ is a star graph.

3. Planar annihilator ideal graphs

One of the most of important invariant in graph theory is the planarity. Our focus in this section is on the planarity of the annihilator ideal graphs. First, we need a celebrated theorem due to Kuratowski.

Theorem 3.1. ([6, Theorem 10.30]) *A graph is planar if and only if it contains no subdivision of either K_5 or $K_{3,3}$.*

In what follows, we first study reduced rings.

Proposition 3.1. *Let R be a reduced ring with* $|\mathrm{Min}(R)| \ge 3$. *Then* $\mathrm{gr}(\Gamma_{\mathrm{Ann}}(R)) = 3$.

Proof. Suppose that R is a reduced ring with at least three minimal ideal. Assume to the contrary, $\mathrm{gr}(\Gamma_{\mathrm{Ann}}(R)) \neq 3$. Thus $\Gamma_{\mathrm{Ann}}(R)$ is a star graph and so $\omega(\Gamma_{\mathrm{Ann}}(R)) < \infty$. By the proof of [2, Proposition 20], $\mathrm{Min}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$. By using of [11, Lemma 3.55], $\mathfrak{p}_2\mathfrak{p}_3 \ldots \mathfrak{p}_k - \mathfrak{p}_1\mathfrak{p}_3 \ldots \mathfrak{p}_k - -\mathfrak{p}_2\mathfrak{p}_3 \cdots \mathfrak{p}_{k-1} - \mathfrak{p}_2\mathfrak{p}_3 \ldots \mathfrak{p}_k$ is a triangle in $\Gamma_{\mathrm{Ann}}(R)$, a contradiction. □

Corollary 3.1. *Let R be a reduced ring with* $|\mathrm{Min}(R)| \ge 3$. *Then* $\Gamma_{\mathrm{Ann}}(R)$ *is not a star graph.*

Proposition 3.2. *Let R be reduced ring with* $|\mathrm{Min}(R)| = 2$. *Then the following statements hold.*

(i) *If* $\mathrm{depth}(R) \neq (0)$, *then* $\mathrm{gr}(\Gamma_{\mathrm{Ann}}(R)) = 3$.

(ii) *If* $\mathrm{depth}(R) = (0)$, *then* $\mathrm{gr}(\Gamma_{\mathrm{Ann}}(R)) = \infty$.

Proof. (i) Suppose that R is a reduced ring, $\mathrm{Min}(R) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$ and $\mathrm{depth}(R) \neq (0)$. Thus $\mathfrak{p}_1 - -Rx - \mathfrak{p}_2 - \mathfrak{p}_1$ is a triangle in $\Gamma_{\mathrm{Ann}}(R)$, where x is an regular element of R, by part (3) of Lemma 2.1, which is a contradiction.

(ii) It is clearly by [9, Theorem 4.3]. □

In [9, Theorem 4.3], it is prove that, if R is a reduced ring with $|\mathrm{Min}(R)| = 2$ and $\mathrm{depth}(R) = 0$, then $\Gamma_{\mathrm{Ann}}(R) \cong K_2$. It is easy to see that in this case $R \cong F_1 \times F_2$ by [2, Theorem 24].

By using this fact and Corollary 2.2, we obtain the following result.

Theorem 3.2. *Let R be a reduced ring. Then $\Gamma_{\mathrm{Ann}}(R)$ is a star graph if and only if $R \cong F_1 \times F_2$, where F_1 and F_2 are fields.*

Remark 3.1. Let R be a reduced ring. By Theorem 3.2, if $\Gamma_{\mathrm{Ann}}(R)$ is a star graph, then R must be an Artinian ring. If R is an Artinian ring, then $\Gamma_{\mathrm{Ann}}(R)$ is a complete graph (see [2, Theorem 10 (1)]), and it is planar if and only if $|\mathbb{I}(R)| \leq 4$, by Kuratowski's Theorem. Therefore, from now on, we may assume that R is not an Artinian ring.

Theorem 3.3. *Let R be a reduced ring. Then $\Gamma_{\mathrm{Ann}}(R)$ is not planar.*

Proof. Suppose that R is a reduced ring. By [9, Theorem 2.1], $\Gamma_{\mathrm{Ann}}(R)$ is a complete n-partite graph. Assume that $\Gamma_{\mathrm{Ann}}(R)$ is not complete graph and $n \geq 2$. We consider the following cases:

case1. If $n = 2$, then $\Gamma_{\mathrm{Ann}}(R)$ is a complete bipartite graph. By Corollary 2.2, $\Gamma_{\mathrm{Ann}}(R)$ is a star graph. Remark 3.1 leads to a contradiction.

case2. If $n \geq 3$, then $\Gamma_{\mathrm{Ann}}(R)$ contains $K_{3,3}$ as a subgraph, and hence it is not planar. □

Example 3.1. Let $R \cong F \times D$, where F is a field and D is an integral domain which is not a field. It is not hard to see that $\Gamma_{\mathrm{Ann}}(R)$ is a complete 3-partite graph and contains $K_{3,3}$ as a subgraph. Thus $\Gamma_{\mathrm{Ann}}(R)$ is not planar.

In the rest of this section, we assume that R is a non-reduced ring. Note that by [2, Lemma 8], the subgraph induced by nilpotent ideals of R is complete in $\Gamma_{\mathrm{Ann}}(R)$. Therefore, we study non-reduced rings with at most four nilpotent ideals.

Theorem 3.4. *Let R be a non-reduced ring. If $3 \leq |\mathbb{I}_N(R)| \leq 4$, then $\Gamma_{\mathrm{Ann}}(R)$ is not planar.*

Proof. Assume to the contrary $\Gamma_{\mathrm{Ann}}(R)$ is a planar. Thus by part (2) of Lemma 2.1, $\Gamma_{\mathrm{Ann}}(R)$ is a finite graph and R is an Artinian ring, a contradiction. □

Corollary 3.2. *Let R be a non-reduced ring. If $|\mathbb{I}_N(R)| \geq 3$, then $\Gamma_{\mathrm{Ann}}(R)$ is not planar.*

By Corollary 3.2, we need only to study non-reduced rings, where $|\mathbb{I}_N(R)| \leq 2$.

To prove Theorem 3.5, the following lemma is needed.

Lemma 3.1. *Let R be a non-reduced ring and $\Gamma_{\mathrm{Ann}}(R)$ be planar. If $\Gamma_{\mathrm{Ann}}(R)$ is an infinite graph, then R is indecomposable.*

Proof. Suppose to the contrary, $R \cong R_1 \times R_2$, for some rings R_1 and R_2. With no loss of generality, we may assume that $I \in \mathbb{I}_N(R_1)$. If $|\mathbb{I}_N(R_2)| = \infty$, then the vertices of the set $\{R_1 \times (0), I \times (0), I \times R_2\}$ and $\{(0) \times R_2, (0) \times J_1, (0) \times J_2\}$ form $K_{3,3}$ in $\Gamma_{\mathrm{Ann}}(R)$, where $J_1, J_2 \in \mathbb{I}(R_2)$, a contradiction. If $|\mathbb{I}_N(R_1)| = \infty$, then the vertices of the set $\{(0) \times R_2, I \times (0), I \times R_2\}$ and $\{R_1 \times (0), K_1 \times (0), K_2 \times (0)\}$ form $K_{3,3}$, where $K_1, K_2 \in \mathbb{I}(R_1)$, a contradiction. □

Recall that the annihilating-ideal graph of a ring R, denoted by $\mathbb{AG}(R)$, is a graph with the vertex set $\mathbb{A}(R)^* := \mathbb{A}(R) \setminus \{(0)\}$, where $\mathbb{A}(R)$ be the set of ideals of R with non-zero annihilator and two distinct vertices I, J are adjacent if and only if $IJ = 0$. By [5, Theorem 2.1], for every ring R, the annihilating-ideal graph $\mathbb{AG}(R)$ is a connected graph and $\mathrm{diam}(\mathbb{AG}(R)) \leq 3$. Moreover, if $\mathbb{AG}(R)$ contains a cycle, then $\mathrm{gr}(\mathbb{AG}(R)) \leq 4$. By using this fact the following theorem is proved.

Theorem 3.5. *Let R be a Noetherian non-reduced ring with* $\mathrm{depth}(R) = 0$ *such that* $|\mathbb{I}_N(R)| = 1$. *Then the following statements are equivalent:*

(1) $\mathrm{Ann}_R(Z(R))$ *is a prime ideal, $I \not\in \mathbb{I}_N(R)$ and $I \cap \mathrm{Ann}_R(Z(R)) = (0)$, for every $I \neq Z(R)$;*

(2) $\Gamma_{\mathrm{Ann}}(R)$ *is planar.*

Proof.

(1) ⇒ (2) It is clear, by [9, Theorem 4.7 (2)].

(2) ⇒ (1) Suppose that $\Gamma_{\mathrm{Ann}}(R)$ is planar. Assume that R has exactly one nilpotent ideal. It is easy to see that, $\mathrm{Nil}(R)$ is the only nilpotent ideal of R, $(\mathrm{Nil}(R))^2 = (0)$ and $\mathrm{Nil}(R)$ is adjacent to every other vertex of $\Gamma_{\mathrm{Ann}}(R)$ (see [2, Lemma 4]). We show that $IJ \neq (0)$, for every $I, J \neq \mathrm{Nil}(R)$, where $I, J \in \mathbb{I}(R)$. Assume to the contrary, $IJ = (0)$. Thus there exists $x \in I \setminus \mathrm{Nil}(R)$ and $y \in J \setminus \mathrm{Nil}(R)$, where $xy = 0$. By Lemma 3.1, R is indecomposable and so $x^i \neq x^{i+1}$ and $y^i \neq y^{i+1}$, for $0 \leq i \leq 2$. Hence the vertices of the set $\{Rx, Rx^2, Rx^3\}$ and $\{Ry, Ry^2, Ry^3\}$ form $K_{3,3}$ in $\Gamma_{\mathrm{Ann}}(R)$, a contradiction. Therefore, $\mathrm{Ann}_R(Z(R)) = \mathrm{Nil}(R)$ and $\mathbb{AG}(R) \cong K_1 \vee \overline{K_\infty}$. Thus $\mathrm{Ann}_R(Z(R)) \neq (0)$ and it is a prime ideal of R (see [9, Theorem 4.7 (1)]). So $Z(R)$ is a vertex of $\Gamma_{\mathrm{Ann}}(R)$. Finally, we show that $\Gamma_{\mathrm{Ann}}(R) \cong K_2 \vee \overline{K_\infty}$. It is clear that $Z(R)$ is adjacent to every other vertex. Let $I — J$ be an edge of $\Gamma_{\mathrm{Ann}}(R)$, where $I, J \neq Z(R)$ and $I, J \neq \mathrm{Nil}(R)$. Then either $I \cap \mathrm{Nil}(R) \neq \{0\}$ or $J \cap \mathrm{Nil}(R) \neq \{0\}$. With loss of generality, we may assume that $I \cap \mathrm{Nil}(R) \neq \{0\}$. Then $\Gamma_{\mathrm{Ann}}(R)$ contains $K_{3,,3}$ as a subgraph, a contradiction. Thus $\Gamma_{\mathrm{Ann}}(R) \cong K_2 \vee \overline{K_\infty}$. Now, [9, Theorem 4.7 (2)] completes the proof. □

Remark 3.2. We note that $|\mathbb{I}_N(R)| \neq 2$, in Theorem 3.5. By part (1) of Theorem 3.5 and [9, Theorem 4.7 (2)], it is clear that if $|\mathbb{I}_N(R)| = 2$, then $\Gamma_{\mathrm{Ann}}(R) \cong K_3 \vee \overline{K_\infty}$. Thus $\Gamma_{\mathrm{Ann}}(R)$ contains $K_{3,3}$ as a subgraph, and so $\Gamma_{\mathrm{Ann}}(R)$ is not planar.

Theorem 3.6. *If R is not a Noetherian ring and* $\mathrm{depth}(R) = 0$, *then* $\Gamma_{\mathrm{Ann}}(R)$ *is not planar.*

Proof. Let $I = \mathrm{Ann}(I)$, for every $I \subseteq Z(R)$. Then $Z(R) = \mathrm{Nil}(R)$. and hence $\omega(\Gamma_{\mathrm{Ann}}(R)) = \infty$. Thus $\Gamma_{\mathrm{Ann}}(R)$ is not planar. Suppose that there exists $I \in \mathbb{I}(R)$ such that $I \subseteq Z(R)$ and $I \neq \mathrm{Ann}(I)$. Since R is not Noetherian, it is enough to consider the following cases:

Case1. Suppose that $\mathrm{Ann}(I) = (0)$. Since R is a non-reduced ring, by [2, Lemma 4], there exists $J \in \mathbb{I}(R)$, where $J^2 = (0)$ and J is adjacent to every other vertex in $\Gamma_{\mathrm{Ann}}(R)$. Assume that K, L are two ideals of R, where $KL = (0)$ (we can suppose that, $x, y \in Z(R)$ such that $xy = 0$ and put $K = Rx, L = Ry$). Since $J^2 = (0)$, then by part (2) of [10, Lemma 2.1], $\mathrm{Ann}(J)$ is an essential ideal of R. If $\mathrm{Ann}(J) \neq J$, then the vertices $\{I, J, \mathrm{Ann}(J), K, L\}$ form K_5 in $\Gamma_{\mathrm{Ann}}(R)$. Thus $\Gamma_{\mathrm{Ann}}(R)$ is not planar. If $\mathrm{Ann}(J) = J$, then the vertices of the set $\{I, J, I+J\}$ and $\{K, L, J+K\}$ form $K_{3,3}$. Then $\Gamma_{\mathrm{Ann}}(R)$ is not planar.

Case2. Suppose that $\mathrm{Ann}(I) \neq (0)$. By [10, Lemma 21 (1)], $I + \mathrm{Ann}(I)$ is an essential ideal of R. If $\mathrm{Ann}(J) \neq J$, then the vertices $\{I, J, \mathrm{Ann}(J), I + \mathrm{Ann}(I), \mathrm{Ann}(I)\}$ form K_5. Thus $\Gamma_{\mathrm{Ann}}(R)$ is not planar. If $\mathrm{Ann}(J) = J$, then $\{I, J, I+J+\mathrm{Ann}(I), I+\mathrm{Ann}(I), \mathrm{Ann}(I)\}$ form K_5 in $\Gamma_{\mathrm{Ann}}(R)$. Therefore $\Gamma_{\mathrm{Ann}}(R)$ is not planar. (Note that by [14, Theorem 17.3], $I + J + \mathrm{Ann}(I)$ is an essential ideal of R). □

Example 3.2.

(1) Let $R = \mathbb{Z}_2[X, Y]/(X^2, XY)$ and let $x = X + (X^2 + XY)$, $y = Y + (X^2 + XY) \in R$. Then $Z(R) = (x, y)R$, $\mathrm{Ann}_R(Z(R)) = \mathrm{Nil}(R) = (x)R = \{0, x\}$ is a prime ideal of R. It is not hard to check that $\Gamma_{\mathrm{Ann}}(R) \cong K_2 \vee \overline{K_\infty}$ and $\Gamma_{\mathrm{Ann}}(R)$ is planar.

(2) Let $D = \mathbb{Z}_2[X, Y, Z]$, $I = (X^2, Y^2, XY, XZ, YZ)D$ be an ideal of D, and let $R = D/I$. Let $x = X + I$, $y = Y + I$ and $z = Z + I$ be elements of R. Then $\mathrm{Nil}(R) = R(x, y)$ and $Z(R) = R(x, y, z)$. It is not hard to see that $\Gamma_{\mathrm{Ann}}(R) \cong K_4 \vee \overline{K_\infty}$ and K_5 is a subgraph of $\Gamma_{\mathrm{Ann}}(R)$. Thus $\Gamma_{\mathrm{Ann}}(R)$ is not planar.

Theorem 3.7. *Let R be a non-reduced ring with* $\mathrm{depth}(R) \neq 0$. *Then $\Gamma_{\mathrm{Ann}}(R)$ is not planar.*

Proof. If $\mathrm{Nil}(R) = Z(R)$, then by [9, Theorem 2.3], $\omega(\Gamma_{\mathrm{Ann}}(R)) = \infty$. Thus $\Gamma_{\mathrm{Ann}}(R)$ contains K_5 as a subgraph and so $\Gamma_{\mathrm{Ann}}(R)$ is not planar. We may suppose that $\mathrm{Nil}(R) \neq Z(R)$. By Theorem 3.4, it is sufficient to study rings R with at most two nilpotent ideals. First assume that R has exactly two nilpotent ideal, say I and J. Let $z, y \in Z(R) \setminus \mathrm{Nil}(R)$ such that $zy = 0$. Thus the vertices of $\{I, J, Rx, Ry, Rz\}$ form K_5 in $\Gamma_{\mathrm{Ann}}(R)$, where x is an regular element of R. Hence $\Gamma_{\mathrm{Ann}}(R)$ is not planar. Now, suppose that R has exactly one nilpotent ideal. It is easy to see that $\mathrm{Nil}(R)$ is the only nilpotent ideal of R, $(\mathrm{Nil}(R))^2 = (0)$ and $\mathrm{Nil}(R)$ is adjacent to every other vertex of $\Gamma_{\mathrm{Ann}}(R)$. If $\mathrm{Nil}(R) \neq \mathrm{Ann}(\mathrm{Nil}(R))$, then the vertices of $\{\mathrm{Nil}(R), \mathrm{Ann}(\mathrm{Nil}(R)), Rx, Ry, Rz\}$ form K_5, where x is an regular element of R. If $\mathrm{Nil}(R) = \mathrm{Ann}(\mathrm{Nil}(R))$, then by [14, Theorem 17.3], $\mathrm{Nil}(R) + Rx$ is an essential ideal of R. Thus the vertices of $\{Rx^2, Rx + \mathrm{Nil}(R), Rx\}$ and $\{\mathrm{Nil}(R), Ry, Rz\}$ form $K_{3,3}$ in $\Gamma_{\mathrm{Ann}}(R)$, for some regular element x in R. Therefore $\Gamma_{\mathrm{Ann}}(R)$ is not planar. □

We are now in a position to characterize all rings with planar annihilator graphs.

Corollary 3.3. *Let R be a ring. Then the following statements hold.*

(1) *Suppose that R is an Artinian ring. Then $\Gamma_{\mathrm{Ann}}(R)$ is planar if and only if R has at most four non-trivial ideals.*

(2) *Suppose that R is not an Artinian ring. Then*

 (i) *If R is a reduced ring, then $\Gamma_{\mathrm{Ann}}(R)$ is not planar.*

 (ii) *If R is a non-reduced ring, then $\Gamma_{\mathrm{Ann}}(R)$ is planar if and only if the following statements hold.*

  (a) *R is Noetherian ring.*

  (b) $\mathrm{depth}(R) = 0$.

  (c) *R has exactly one nilpotent ideal.*

  (d) *$\mathrm{Ann}_R(Z(R))$ is a prime ideal of R and for every $I \neq Z(R)$ and $I \not\in \mathbb{I}_N(R)$, $I \cap \mathrm{Ann}_R(Z(R)) = (0)$.*

4. Every annihilator ideal graph is perfect

Let R be a ring. In this section, we show that $\Gamma_{\mathrm{Ann}}(R)$ is a perfect graph.

We first recall the strong perfect graph theorem.

Lemma 4.1 ([7])**.** *A graph G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length ≥ 5.*

Lemma 4.2. *Let R be a ring. Then $\Gamma_{\mathrm{Ann}}(R)$ does not contain C_n $(n > 4)$ as an induced subgraph.*

Proof. Assume to contrary, $\Gamma_{\mathrm{Ann}}(R)$ contains an induced cycle $C_n : I_1 - I_2 - \cdots - I_n - I_1$, for $n > 4$. Since C_n is an induced cycle, neither I_1 and I_3 nor I_1 and I_4 are adjacent. By part (1) of Lemma 2.1, $\mathrm{Ann}(I_1) = \mathrm{Ann}(I_3)$, $\mathrm{Ann}(I_1) = \mathrm{Ann}(I_4)$. As I_3 and I_4 are adjacent, $I_3 \cap \mathrm{Ann}(I_4) \neq \{0\}$ or $I_4 \cap \mathrm{Ann}(I_3) \neq \{0\}$. We consider the following two cases:

Case 1. If $I_3 \cap \mathrm{Ann}(I_4) \neq \{0\}$, then $I_3 \cap \mathrm{Ann}(I_1) \neq \{0\}$ and so I_1 is adjacent to I_3, a contradiction.

Case 2. If $I_4 \cap \mathrm{Ann}(I_3) \neq \{0\}$, then $I_4 \cap \mathrm{Ann}(I_1) \neq \{0\}$ and so I_1 is adjacent to I_4, a contradiction. □

Lemma 4.3. *Let R be a ring. Then $\overline{\Gamma_{\mathrm{Ann}}(R)}$ does not contain C_{2n+1} $(n \geq 2)$ as an induced subgraph.*

Proof. Assume to the contrary, $\overline{\Gamma_{\mathrm{Ann}}(R)}$ contains an induced cycle $I_1 - I_2 - \cdots - I_{2n+1} - I_1$, for $n \geq 2$. Since I_1 is adjacent to I_3 and also I_2 is adjacent to I_4 in $\overline{\Gamma_{\mathrm{Ann}}(R)}$, by part (1) of Lemma 2.1, $\mathrm{Ann}(I_1) = \mathrm{Ann}(I_2) = \mathrm{Ann}(I_3)$. By a similar argument, we conclude:

$$\mathrm{Ann}(I_1) = \cdots = \mathrm{Ann}(I_{2n+1}) \tag{4.1}$$

As I_3 and I_4 are adjacent in $\overline{\Gamma_{\text{Ann}}(R)}$, we deduce that $I_3 \cap \text{Ann}(I_4) = \{0\}$. Thus by the equality (1), $I_3 \cap \text{Ann}(I_1) = \{0\}$. Also I_1 and I_2 are adjacent in $\overline{\Gamma_{\text{Ann}}(R)}$. Hence $I_1 \cap \text{Ann}(I_2) = \{0\}$. Again by the equality (1), $I_1 \cap \text{Ann}(I_3) = \{0\}$. Therefore, I_1 and I_3 are adjacent in $\overline{\Gamma_{\text{Ann}}(R)}$ and $I_1 - I_2 - I_3 - I_1$ is triangle, a contradiction. □

We close this paper with the following result.

Theorem 4.1. *Let R be a ring. Then $\Gamma_{\text{Ann}}(R)$ is a perfect graph.*

Proof. The result follows from Lemmas 4.1, 4.2 and 4.3. □

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