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FURTHER PROPERTIES OF SOME MAPPINGS ASSOCIATED WITH HERMITE-HADAMARD INEQUALITIES

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Abstract. New properties of some functionals associated to the Hermite-Hadamard integral inequality for convex functions are given.

1. Introduction

Now for a given convex function $f : [a, b] \to \mathbb{R}$, let $H : [0, 1] \to \mathbb{R}$ be defined by

$$H(t) := \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx.$$

The following theorem holds (see also [11], [13], [19] and [28]):

Theorem 1. *With the above assumptions, we have:*

(i) *H is convex on $[0, 1]$;*

(ii) *One has the bounds:*

$$\inf_{t\in[0,1]} H(t) = H(0) = f\left(\frac{a+b}{2}\right)$$

and

$$\sup_{t\in[0,1]} H(t) = H(1) = \frac{1}{b-a}\int_a^b f(x)dx;$$

(iii) *H increases monotonically on $[0, 1]$;*

(iv) *The following inequalities*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\le \frac{2}{b-a}\int_{\frac{(3a+b)}{4}}^{\frac{(a+3b)}{4}} f(x)dx \\ &\le \int_0^1 H(t)dt \\ &\le \frac{1}{2}\left(f\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^b f(x)dx\right) \end{aligned} \tag{1.1}$$

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hold.

Now, we shall introduce another mapping which is connected with H and the $H.-H.$ result.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Define the mapping $G : [0,1] \to \mathbb{R}$, given by

$$G(t) := \frac{1}{2}\left[f\left(ta + (1-t)\frac{a+b}{2}\right) + f\left((1-t)\frac{a+b}{2} + tb\right)\right].$$

The following theorem contains some properties of this mapping [28]:

Theorem 2. *Let f and G be as above. Then*

(i) *G is convex and monotonically increasing on $[0,1]$;*

(ii) *We have the bounds:*

$$\inf_{t\in[0,1]} G(t) = G(0) = f\left(\frac{a+b}{2}\right)$$

and

$$\sup_{t\in[0,1]} G(t) = G(1) = \frac{f(a)+f(b)}{2};$$

(iii) *One has the inequality*

$$H(t) \le G(t) \quad \textit{for all} \quad t \in [0,1];$$

(iv) *One has the inequalities*

$$\begin{aligned} \frac{2}{b-a}\int_{\frac{(3a+b)}{4}}^{\frac{(a+3b)}{4}} f(x)dx &\le \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \\ &\le \int_0^1 G(t)dt \\ &\le \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right]. \end{aligned} \tag{1.2}$$

Now, we shall consider another mapping associated with the Hermite-Hadamard inequality given by $L : [0,1] \to \mathbb{R}$,

$$L(t) := \frac{1}{2(b-a)}\int_a^b [f(ta + (1-t)x) + f((1-t)x + tb)]dx$$

where $f : I \subseteq \mathbb{R} \to \mathbb{R}$ and $a, b \in I$ with $a < b$.

The following theorem also holds [28]:

Theorem 3. *With the above assumptions one has:*

(i) *L is convex on* $[0,1]$*;*
(ii) *We have the inequalities:*

$$G(t) \leq L(t) \leq \frac{1-t}{b-a} \cdot \int_a^b f(x)dx + t \cdot \frac{f(a)+f(b)}{2} \leq \frac{f(a)+f(b)}{2} \tag{1.3}$$

for all $t \in [0,1]$ *and the bound:*

$$\sup_{t\in[0,1]} L(t) = \frac{f(a)+f(b)}{2}; \tag{1.4}$$

(iii) *One has the inequalities:*

$$H(1-t) \leq L(t) \quad \textit{and} \quad \frac{H(t)+H(1-t)}{2} \leq L(t)$$

for all $t \in [0,1]$.

Now, we shall introduce another mapping defined by a double integral in connection with the Hermite-Hadamard inequalities:

$$F : [0,1] \to \mathbb{R}, \quad F(t) := \frac{1}{(b-a)^2} \int_a^b \int_a^b f(tx + (1-t)y)dxdy$$

The following theorem holds [19] (see also [13]):

Theorem 4. *Let* $f : [a,b] \to \mathbb{R}$ *be as above. Then*
(i) $F(\tau + \frac{1}{2}) = F(\frac{1}{2} - \tau)$ *for all* $\tau \in [0, \frac{1}{2}]$ *and* $F(t) = F(1-t)$ *for all* $t \in [0,1]$*;*
(ii) *F is convex on* $[0,1]$*;*
(iii) *We have the bounds:*

$$\sup_{t\in[0,1]} F(t) = F(0) = F(1) = \frac{1}{b-a}\int_a^b f(x)dx$$

and

$$\inf_{t\in[0,1]} F(t) = F\left(\frac{1}{2}\right) = F(1) = \frac{1}{(b-a)^2}\int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dxdy;$$

(iv) *The following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq F\left(\frac{1}{2}\right)$$

(v) *F decreases monotonically on* $[0, \frac{1}{2}]$ *and increases monotonically on* $[\frac{1}{2}, 1]$*;*
(vi) *We have the inequality:*

$$H(t) \leq F(t) \quad \text{for all} \quad t \in [0,1].$$

In what follows, we shall point out some reverse inequalities for the mappings H, G, L and F considered above.
We shall start with the following result [17] (see also [58]).

Theorem 5. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping on I and $a, b \in \mathring{I}$ with $a < b$. Then we have the inequality:*

$$0 \le \frac{1}{b-a}\int_a^b f(x)dx - H(t)$$
$$\le (1-t)\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right]. \tag{1.5}$$

for all $t \in [0,1]$.

Corollary 1. *With the above assumptions, one has*

$$0 \le \frac{1}{b-a}\int_a^b f(x)dx - \frac{2}{b-a}\int_{\frac{(3a+b)}{4}}^{\frac{(a+3b)}{4}} f(x)dx$$
$$\le \frac{1}{2}\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right].$$

Remark 1. If in (1.5) we choose $t = 0$, we obtain

$$0 \le \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx,$$

which is the well-known Bullen result [62, p.140].

Another theorem of this type in which the mapping G defined above is involved, is the following one:

Theorem 6. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping on I and $a, b \in \mathring{I}$ with $a < b$. Then we have the inequality:*

$$0 \le H(t) - f\left(\frac{a+b}{2}\right) \le G(t) - H(t) \tag{1.6}$$

for all $t \in [0,1]$.

Proof. It is sufficient to prove the above inequality for differentiable convex functions. By the convexity of f, we have that

$$f\left(\frac{a+b}{2}\right) - f\left(tx + (1-t)\frac{a+b}{2}\right) \le t\left(\frac{a+b}{2} - x\right) f'\left(tx + (1-t)\frac{a+b}{2}\right)$$

for all x in (a,b) and $t \in [0,1]$.
Integrating this inequality over x on $[a,b]$ one gets

$$f\left(\frac{a+b}{2}\right) - H(t) \geq \frac{t}{b-a}\int_a^b \left(\frac{a+b}{2} - x\right) f'\left(tx + (1-t)\frac{a+b}{2}\right) dx.$$

As a simple calculation (an integration by parts) yields that

$$\frac{t}{b-a}\int_a^b \left(\frac{a+b}{2} - x\right) f'\left(tx + (1-t)\frac{a+b}{2}\right) dx = H(t) - G(t), \quad t \in [0,1],$$

then, the above inequality gives us the desired result (1.6).

Remark 2. If in the above inequality we choose $t = 1$, we also recapture Bullen's result [62, p.140].

2. Some New Results

Now, we shall investigate the case of the mapping F defined by the use of double integrals ([17] and [58])

Theorem 7. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping on I and $a, b \in \mathring{I}$ with $a < b$. Then we have the inequality:*

$$\begin{aligned} 0 &\leq \frac{1}{b-a}\int_a^b f(x)dx - F(t) \\ &\leq \min\{t, 1-t\}\left(\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right) \end{aligned} \tag{2.1}$$

for all $t \in [0,1]$.

Proof. As above, it is sufficient to prove the above inequality for differentiable convex functions.
Thus, for all $x, y \in (a,b)$ and $f \in [0,1]$ we have:

$$f(tx + (1-t)y) - f(y) \geq t(x-y)f'(y).$$

Integrating this inequality on $[a,b]^2$ over x and y, we obtain

$$\int_a^b\int_a^b f(tx + (1-t)y)dxdy - (b-a)\int_a^b f(x)dx \geq t\int_a^b\int_a^b (x-y)f'(y)dxdy$$

for all $t \in [0,1]$.
As a simple computation shows that

$$\int_a^b\int_a^b (x-y)f'(y)dxdy = (b-a)\int_a^b f(x)dx - (b-a)^2 \cdot \frac{f(a)+f(b)}{2},$$

the above inequality gives us that

$$\frac{1}{b-a}\int_a^b f(x)dx - F(t) \le t\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right]$$

for all $t \in [0,1]$.
As $F(t) = F(1-t)$ for all $t \in [0,1]$, if we replace in the above inequality t with $1-t$ we get the desired result (2.1).

Corollary 2. *With the above assumptions, one has:*

$$\begin{aligned} 0 &\le \frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy \\ &\le \frac{1}{2}\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right]. \end{aligned}$$

Now, let us define the mapping $J : [0,1] \to \mathbb{R}$, $J(t) := L(1-t)$, i.e.,

$$J(t) = \frac{1}{2(b-a)}\int_a^b [f(tx+(1-t)a) + f(tx+(1-t)b)]dx,$$

where $t \in [0,1]$.
We have the following result:

Theorem 8. *Let f and $a, b \in \mathring{I}$ be as above. Then we have the inequality:*

$$0 \le F(t) - H(t) \le J(t) - F(t) \tag{2.2}$$

for all $t \in [0,1]$.

Proof. As above, it is sufficient to prove the above inequality for differentiable convex functions.
By the convexity of f on $[a,b]$ we have that

$$f\left(tx+(1-t)\frac{a+b}{2}\right) - f(tx+(1-t)y) \ge (1-t)f'(tx+(1-t)y)\left(\frac{a+b}{2} - y\right)$$

for all $x, y \in (a,b)$ and $t \in [0,1]$.
If we integrate over x and y on $[a,b]^2$, we get that:

$$\begin{aligned} &\int_a^b\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dxdy - \int_a^b\int_a^b f(tx+(1-t)y)dxdy \\ &\ge (1-t)\int_a^b\int_a^b f'(tx+(1-t)y)\left(\frac{a+b}{2} - y\right)dxdy \end{aligned}$$

which gives us that:

$$\begin{aligned} 0 &\le F(t) - H(t) \\ &\le \frac{1-t}{(b-a)^2} \int_a^b \int_a^b f'(tx + (1-t)y) \left(y - \frac{a+b}{2} \right) dxdy =: A(t) \end{aligned}$$

for all $t \in [0,1]$.
Define

$$I_1(t) := \frac{1-t}{(b-a)^2} \int_a^b \int_a^b f'(tx + (1-t)y) y dxdy$$

and

$$I_2(t) := \frac{1-t}{(b-a)^2} \int_a^b \int_a^b f'(tx + (1-t)y) dxdy.$$

Note that, for $t = 1$, the inequality (2.2) is obvious. Assume that $t \in [0,1)$. Integrating by parts, we get that:

$$\int_a^b f'(tx+(1-t)y) y dy = \frac{f((1-t)b+tx)b - f((1-t)a+tx)a}{1-t} - \frac{1}{1-t} \int_a^b f((1-t)y+tx)dy.$$

Thus, we deduce that

$$I_1(t) = \frac{b \int_a^b f(tx + (1-t)b)dx - a \int_a^b f(tx + (1-t)a)dx}{(b-a)^2} - F(t).$$

We also have

$$\int_a^b f'(tx + (1-t)y)dy = \frac{f(tx + (1-t)b) - f(tx + (1-t)a)}{1-t},$$

and thus

$$I_2(t) = \frac{\int_a^b f(tx + (1-t)b)dx - \int_a^b f(tx + (1-t)a)dx}{(b-a)^2}.$$

Now, we get that

$$\begin{aligned} A(t) &= \frac{b \int_a^b f(tx + (1-t)b)dx - a \int_a^b f(tx + (1-t)a)dx}{(b-a)^2} - F(t) \\ &\quad - \frac{a+b}{2} \cdot \frac{\int_a^b f(tx + (1-t)b)dx - \int_a^b f(tx + (1-t)a)dx}{(b-a)^2} \\ &= \frac{\frac{b-a}{2} \int_a^b f(tx + (1-t)b)dx + \frac{b-a}{2} \int_a^b f(tx + (1-t)a)dx}{(b-a)^2} - F(t) \\ &= J(t) - F(t) \end{aligned}$$

and the theorem is proved.

Corollary 3. *With the above assumptions, we have:*

$$0 \leq F(t) - \frac{H(t) + H(1-t)}{2} \leq \frac{L(t) + L(1-t)}{2} - F(t)$$

for all $t \in [0,1]$.

Finally, the following theorem holds.

Theorem 9. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a convex function on* I *and* $a, b \in \mathring{I}$ *with* $a < b$. *Then one has the inequality*

$$\begin{aligned} 0 &\leq F(t) - F\left(\frac{1}{2}\right) \\ &\leq \frac{1}{2t(1-t)}\left[(1-2t)^2 F(t) - \frac{1-2t}{b-a} \cdot \int_{(1-t)a+tb}^{ta+(1-t)b} f(x)dx\right] \end{aligned} \tag{2.3}$$

for all $t \in (0,1)$.

Proof. As above, we can prove the inequality (2.3) only for the case where f is a differentiable convex function.
By the convexity of f we have that:

$$\begin{aligned} & f\left(\frac{x+y}{2}\right) - f(tx + (1-t)y) \\ \geq & \left[\frac{x+y}{2} - (tx + (1-t)y)\right] f'(tx + (1-t)y) \\ = & \frac{1-2t}{2}(x-y) f'(tx + (1-t)y) \end{aligned} \tag{2.4}$$

for all $x, y \in (a,b)$ and $t \in [0,1]$.
If we integrate the inequality (2.4) over x, y on $[a,b]^2$ we can deduce

$$F\left(\frac{1}{2}\right) - F(t) \geq \frac{1-2t}{2} \cdot \frac{1}{(b-a)^2} \int_a^b \int_a^b (x-y) f'(tx + (1-t)y)dxdy.$$

Denote

$$I(t) := \frac{1}{(b-a)^2} \int_a^b \int_a^b (x-y) f'(tx + (1-t)y)dxdy,$$

$$I_1(t) := \frac{1}{(b-a)^2} \int_a^b \int_a^b x f'(tx + (1-t)y)dxdy$$

and

$$I_2(t) := \frac{1}{(b-a)^2} \int_a^b \int_a^b y f'(tx + (1-t)y)dxdy.$$

Then we have $I(t) = I_1(t) - I_2(t)$ for all $t \in [0,1]$.
An integration by parts gives us that

$$\int_a^b x f'(tx + (1-t)y)dx = \frac{xf(tx + (1-t)y)}{t}\Big|_a^b - \frac{1}{t}\int_a^b f(tx + (1-t)y)dx$$

then

$$I_1(t) = \frac{1}{(b-a)^2}\int_a^b \left[\frac{f(tb+(1-t)y)b - f(ta+(1-t)y)a}{t} - \frac{1}{t}\int_a^b f(tx + (1-t)y)dx\right]dy$$
$$= \frac{1}{(b-a)^2}\cdot\frac{1}{t}\left[b\int_a^b f(tb + (1-t)y)dy - a\int_a^b f(ta + (1-t)y)dy\right] - \frac{1}{t}F(t).$$

Also, by an integration by parts, we have:

$$\int_a^b y f'(tx + (1-t)y)dx = \frac{f(tx + (1-t)y)y}{1-t}\Big|_a^b - \frac{1}{1-t}\int_a^b f(tx + (1-t)y)dy$$

then we obtain:

$$I_2(t) = \frac{1}{(b-a)^2}\int_a^b \left[\frac{f(tx+(1-t)b)b - f(tx+(1-t)a)a}{1-t} - \frac{1}{1-t}\int_a^b f(tx+(1-t)y)dy\right]dx$$
$$= \frac{1}{(b-a)^2}\cdot\frac{1}{1-t}\left[b\int_a^b f(tx + (1-t)b)dx - a\int_a^b f(tx + (1-t)a)dx\right] - \frac{1}{1-t}F(t).$$

Thus, we have

$$I(t) = \frac{1}{t(b-a)^2}\cdot\left[b\int_a^b f(tb + (1-t)y)dy - a\int_a^b f(ta + (1-t)y)dy\right] - \frac{1}{t}F(t)$$
$$-\frac{1}{(1-t)(b-a)^2}\cdot\left[b\int_a^b f(tx + (1-t)b)dx - a\int_a^b f(tx + (1-t)a)dx\right] + \frac{1}{1-t}F(t)$$
$$= \frac{2t-1}{t(1-t)}\cdot F(t) + \frac{1}{t(1-t)(b-a)^2}\cdot V(t),$$

where

$$V(t) = (1-t)b\int_a^b f(tb + (1-t)y)dy - (1-t)a\int_a^b f(ta + (1-t)y)dy$$
$$-tb\int_a^b f(tx + (1-t)b)dx + ta\int_a^b f(tx + (1-t)a)dx$$
$$= b\int_{(1-t)a+tb}^b f(u)du - a\int_a^{(1-t)b+ta} f(u)du - b\int_{ta+(1-t)b}^b f(u)du + a\int_a^{tb+(1-t)a} f(u)du$$

$$= b\int_{(1-t)a+tb}^{ta+(1-t)b} f(u)du - a\int_{(1-t)a+tb}^{ta+(1-t)b} f(u)du$$
$$= (b-a)\int_{(1-t)a+tb}^{ta+(1-t)b} f(x)dx.$$

Consequently, we have

$$\begin{aligned} F\left(\frac{1}{2}\right) - F(t) &\geq \frac{1-2t}{2}\cdot I(t) \\ &= \frac{1-2t}{2}\left[\frac{2t-1}{t(1-t)}\cdot F(t) + \frac{1}{(b-a)t(1-t)}\int_{(1-t)a+tb}^{ta+(1-t)b} f(x)dx\right] \\ &= \frac{1-2t}{2(b-a)t(1-t)}\int_{(1-t)a+tb}^{ta+(1-t)b} f(x)dx - \frac{(2t-1)^2}{2t(1-t)}F(t) \end{aligned}$$

for all $t \in (0,1)$, which is equivalent with the desired inequality (2.3).

References

[1] G. Allasia, C. Giordano, J. Pečarić, *Hadamard-type inequalities for* $(2r)$*-convex functions with applications*, Atti Acad. Sci. Torino-Cl. Sc. Fis. **133**(1999), 1-14.

[2] H. Alzer, *A note on Hadamard's inequalities*, C. R. Math. Rep. Acad. Sci. Canada **11**(1989), 255-258.

[3] H. Alzer, *On an integral inequality*, Math. Rev. Anal. Numer. Theor. Approx. **18**(1989), 101-103.

[4] A. G. Azpeitia, *Convex functions and the Hadamard inequality*, Rev.-Colombiana-Mat. **28**(1994), 7-12.

[5] D. Barbu, S. S. Dragomir and C. Buşe, *A probabilistic argument for the convergence of some sequences associated to Hadamard's inequality*, Studia Univ. Babeş-Bolyai, Math. **38**(1)(1993), 29-33.

[6] E. F. Beckenbach, *Convex functions*, Bull. Amer. Math. Soc. **54**(1948), 439-460.

[7] C. Borell, *Integral inequalities for generalized concave and convex functions*, J. Math. Anal. Appl. **43**(1973), 419-440.

[8] C. Buşe, S. S. Dragomir and D. Barbu, *The convergence of some sequences connected to Hadamard's inequality*, Demostratio Math. **29**(1996), 53-59.

[9] L. J. Dedić, C. E. M. Pearce and J. Pečarić, *The Euler formulae and convex functions*, Math. Ineq. and Appl. **2**(2000), 211-221.

[10] L. J. Dedić, C. E. M. Pearce and J. Pečarić, *Hadamard and Dragomir-Argarwai inequalities, high-order convexity and the Euler Formula*, submitted.

[11] S. S. Dragomir, *A mapping in connvection to Hadamard's inequalities*, An. Öster. Akad. Wiss. Math.-Natur. **128**(1991), 17-20. MR 934:26032. ZBL No.747:26015.

[12] S. S. Dragomir, *A refinement of Hadamard's inequality for isotonic linear functionals*, Tamkang J. of Math. **24**(1993), 101-106. MR 94a: 26043. 2BL No.799:26016.

[13] S. S. Dragomir, *On Hadamard's inequality for convex functions*, Mat. Balkanica **6**(1992), 215-222. MR: 934:26033.

[14] S. S. Dragomir, *On Hadamard's inequality for the convex mappings defined on a ball in the space and applications*, Math. Ineq. and Appl. **3**(2000), 177-187.

[15] S. S. Dragomir, *On Hadamard's inequality on a disk*, Journal of Ineq. in Pure and Appl. Maht. **1**(2000), Article 2, http://jipam.vu.edu.au/ No.1

[16] S. S. Dragomir, *On some integral inequalities for convex functions*, Zb.-Rad. (Kragujevac)(1996), 21-25.

[17] S. S. Dragomir, *Some integral inequalities for differentiable convex functions*, Contributions, Macedonian Acad. of Sci. and Arts **13**(1992), 13-17.

[18] S. S. Dragomir, *Some remarks on Hadamard's inequalities for convex functions*, Extracta Math. **9**(1994), 88-94.

[19] S. S. Dragomir, *Two mappings in connection to Hadamard's inequalities*, J. Math. Anal. Appl. **167**(1992), 49-56. MR:934:26038, ZBL No.758:26014.

[20] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Mat. Lett. **11**(1998), 91-95.

[21] S. S. Dragomir and R. P. Agarwal, *Two new mappings associated with Hadamard's inequalities for convex functions*, Appl. Math. Lett. **11**(1998), 33-38.

[22] S. S. Dragomir and C. Buşe, *Refinements of Hadamard's inequality for multiple integrals*, Utilitas Math **47**(1995), 193-195.

[23] S. S. Dragomir, Y. J. Cho and S. S. Kim, *Inequalities of Hadamard's type for Lipschitzian mappings and their applications*, J. of Math. Anal. Appl. **245**(2000), 489-501.

[24] S. S. Dragomir and S. Fitzpatrick, *The Hadamard's inequality for $s-$convex functions in the first sense*, Demonstratio Math. **31**(1998), 633-642.

[25] S. S. Dragomir and S. Fitzpatrick, *The Hadamard's inequality for $s-$convex functions in the second sense*, Demonstratio Math. **32**(1999), 687-696.

[26] S. S. Dragomir and N. M. Ionescu, *On some inequalities for convex-dominated functions*, Anal. Num. Theor. Approx. **19**(1990), 21-28. MR 936:26014 ZBL No.733:26010.

[27] S. S. Dragomir and N. M. Ionescu, *Some integral inequalities for differentiable convex functions*, Coll. Pap. of the Fac. of Sci. Kragujevac (Yugoslavia) **13**(1992), 11-16, ZBL No.770.

[28] S. S. Dragomir, D. S. Milośević and J. Sándor, *On some refinements of Hadamard's inequalities and applications*, Univ. Belgrad, Publ. Elek. Fak. Sci. Math. **4**(1993), 21-24.

[29] S. S. Dragomir and B. Mond, *On Hadamard's inequality for a class of functions of Godunova and Levin*, Indian J. Math. **39**(1997), 1-9.

[30] S. S. Dragomir and C. E. M. Pearce, *Quasi-convex functions and Hadamard's inequality*, Bull. Austral. Math. Soc. **57**(1998), 377-385.

[31] S. S. Dragomir, C. E. M. Pearce and J. E. Pečarić, *On Jessen's and related inequalities for isotonic sublinear functionals*, Acta Math. Sci. (Szeged) **61**(1995), 373-382.

[32] S. S. Dragomir, J. E. Pečarić and L. E. Persson, *Some inequalities of Hadamard type*, Soochow J. of Math. **21**(1995), 335-341.

[33] S. S. Dragomir, J. E. Pečarić and J. Sándor, *A note on the Jensen-Hadamard inequality*, Anal. Num. Theor. Approx. **19**(1990), 21-28. MR 93b: 260 14. ZBL No.733:26010.

[34] S. S. Dragomir and G. H. Toader, *Some inequalities for $m-$convex functions*, Studia Univ. Babeş-Bolyai, Math. **38**(1993), 21-28.

[35] A. M. Fink, *A best possible Hadamard inequality*, Math. Ineq. and Appl. **2**(1998), 223-230.

[36] A. M. Fink, *Hadamard inequalities for logarithmic concave functions*, Boundary value problems and related topics, Math. Comput. Modeling, **32**(2000), 625-629.

[37] A. M. Fink, *Toward a theory of best possible inequalities*, Nieuw Archief von Wiskunde **12**(1994), 19-29.

[38] A. M. Fink, *Two inequalities*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. **6**(1995), 48-49.

[39] B. Gavrea, *On Hadamard's inequality for the convex mappings defined on a convex domain in the space*, Journal of Ineq. in Pure and Appl. Math. **1**(2000), Article 9, http://jipam.vu.edu.au/

[40] P. M. Gill, C. E. M. Pearce and J. Pečarić, *Hadamard's inequality for $r-$convex functions*, J. of Math. Anal. and Appl. **215**(1997), 461-470.

[41] G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, 2nd Ed., Cambridge University Press, 1952.

[42] K.-C. Lee and K.-L. Tseng, *On a weighted generalisation of Hadamard's Inequality for $G-$convex functions*, Tamsui Oxford Journal of Math. Sci. **16**(2000), 91-104.

[43] A. Lupaş, *The Jensen-Hadamard inequality for convex functions of higher order*, Octogon Math. Mag. **5**(1997), 8-9.

[44] A. Lupaş, *A generalisation of Hadamard inequalities for convex functions*, Online: (http://rgmia.vu.edu.au/authors/ALupas.htm).

[45] A. Lupaş, *Jensen-Hadamard inequality for convex functions of higher order*, Online: (http://rgmia.vu.edu.au/authors/ALupas.htm).

[46] A. Lupaş, *A generalisation of Hadamard's inequality for convex functions*, Univ. Beograd. Publ. Elek. Fak. Ser. Mat. Fiz. No.544-576(1976), 115-121.

[47] D. M. Makisimović, *A short proof of generalized Hadamard's inequalities*, Univ. Begrad. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No.634-677(1979), 126-128.

[48] D. S. Mitrinović and I. Lacković, *Hermite and convexity*, Aequat. Math. **28**(1985), 229-232.

[49] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, Dordrecht/Boston/London.

[50] B. Mond and J. E. Pečarić, *A companion to Fink's inequality*, Octogon Math. Mag. **5**(1997), 12-18.

[51] E. Neuman, *Inequalities involving generalized symmetric means*, J. Math. Anal. Appl. **120**(1986), 315-320.

[52] E. Neuman and J. E. Pečarić, *Inequalities involving multivariate convex functions*, J. Math. Anal. Appl. **137**(1989), 514-549.

[53] E. Neuman, *Inequalities involving multivariate convex functions II*, Proc. Amer. Math. Soc. **109**(1990), 965-974.

[54] C. P. Niculescu, *A Note on the Dual Hermite-Hadamard Inequality*, The Math. Gazette, July, 2000.

[55] C. P. Niculescu, *Convexity according to the geometric mean*, Math. Ineq. and Appl. **3**(2000), 155-167.

[56] C. E. M. Pearce, J. Pečarić and V. Šimić, *Stolarsky means and Hadamard's inequality*, J. Math. Anal. Appl. **220**(1998), 99-109.

[57] C. E. M. Pearce and A. M. Rubinov, *$P-$functions, quasi-convex functions and Hadamard-type inequalities*, J. Math. Anal. Appl. **240**(1999), 92-104.

[58] J. E. Pečarić, *Remarks on two interpolations of Hadamard's inequalities*, Contributions, Macedonian Acad. of Sci. and Arts, Sect. of Math. and Technical Sciences, (Scopje) **13**(1992), 9-12.

[59] J. Pečarić and V. Čuljak, *On Hadamard inequalities for logarithmic convex functions*, submitted.

[60] J. Pečarić, V. Čuljak and A. M. Fink, *On some inequalities for convex functions of higher order*, Nonlinear Stud. **6**(1999), 131-140.

[61] J. Pečarić, V. and S. S. Dragomir, *A generalization of Hadamard's integral inequality for isotonic linear functionals*, Rudovi Mat. (Sarajevo) **7**(1991), 103-107. MR 924:26026. 2BL No.738:26006.

[62] J. Pečarić, F. Proschan and Y. L. Tong, Convex Functions, Partial Orderings and Statistical Applications, Academic Press, Inc., 1992.

[63] F. Qi and Q.-M. Luo, *Refinements and extensions of an inequality*, Mathematics and Informatics Quarterly **9**(1999), 23-25.

[64] F. Qi, S.-L. Xu, and L. Debnath, *A new proof of monotonicity for extended mean values*, Intern. J. Math. Sci. **22**(1999), 415-420.

[65] A. W. Roberts and P. E. Varberg, Convex Functions, Academic Press, 1973.

[66] F. Saidi and R. Younis, Hadamard and Fejer-type Inequalities, Archiv der Mathematik. **74**(2000), 30-39.

[67] J. Sándor, *A note on the Jensen-Hadamard inequality*, Anal. Numer. Theor. Approx **19**(1990), 29-34.

[68] J. Sándor, *An application of the Jensen-Hadamard inequality*, Nieuw-Arch.-Wisk. **8**(1990), 63-66.

[69] J. Sándor, *On the Jensen-Hadamard inequality*, Studia Univ. Babes-Bolyai, Math. **36**(1991), 9-15.

[70] G. H. Toader, *Some generalisations of the convexity*, Proc. Colloq. Approx. Optim, Cluj-Napoca (Romania), 1984, 329-338.

[71] P. M. Vasić, I. B. Lacković and D. M. Maksimović, *Note on convex functions IV: On-Hadamard's inequality for weighted arithmetic means*, Univ. Beograd Publ. Elek. Fak., Ser. Mat. Fiz. No.678-715(1980), 199-205.

[72] G. S. Yang and M. C. Hong, *A note on Hadamard's inequality*, Tamkang J. Math., **28**(1997), 33-37.

[73] G. S. Yang and K. L. Tseng, *On certain integral inequalities related to Hermite-Hadamard inequalities*, J. Math. Anal. Appl. **239**(1999), 180-187.

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