



## THE RECIPROCAL COMPLEMENTARY WIENER NUMBER OF GRAPHS

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**Abstract.** The reciprocal complementary Wiener number (*RCW*) of a connected graph  $G$  is defined as the sum of weights  $\frac{1}{D+1-d_G(x,y)}$  over all unordered vertex pairs in a graph  $G$ , where  $D$  is the diameter of  $G$  and  $d_G(x, y)$  is the distance between vertices  $x$  and  $y$ . In this paper, we find new bounds for *RCW* of graphs, and study this invariant of two important types of graphs, named the Bar-Polyhex and the Mycielskian graphs.

### 1. Introduction

Throughout this work, we are concerned with undirected simple graphs. The vertex and edge sets of a graph  $G$  are denoted by  $V(G)$  and  $E(G)$ , respectively. For  $x, y \in V(G)$ , the *distance*  $d_G(x, y)$  between the vertices  $x$  and  $y$  is equal to the length of a shortest path that connects  $x$  and  $y$ . The *diameter*  $D = D(G)$  of  $G$  is the greatest distance between any pair of vertices of  $G$ . Also, the *girth* of each graph is the length of a shortest cycle contained in it.

Let  $G$  be a  $n$ -vertex graph with the vertex-set  $V(G) = \{v_1, v_2, \dots, v_n\}$  and diameter  $D$ . The *reciprocal complementary distance matrix*  $RCD = [rc_{ij}]$  of  $G$  is an  $n \times n$  matrix such that  $rc_{ij} = \frac{1}{D+1-d_G(v_i, v_j)}$  if  $i \neq j$ , and 0 otherwise ([11]c). Ivanciuc et al. [9, 10] defined the *reciprocal complementary Wiener number* of the graph  $G$  as:

$$RCW(G) = \sum_{1 \leq i < j \leq n} rc_{ij} = \sum_{\{v_i, v_j\} \subseteq V(G)} \frac{1}{D+1-d_G(v_i, v_j)}. \quad (1)$$

The reciprocal complementary Wiener number has been successfully applied in the structure-property modeling of the molar heat capacity, standard Gibbs energy of formation and vaporization enthalpy of 134 alkanes  $C_6$ - $C_{10}$  [9]. Cai and Zhou [5] determined the trees with the smallest, the second smallest and the third smallest *RCW*, and the unicyclic and bicyclic graphs with the smallest and the second smallest *RCW*. Zhou et al. [20] gave some properties, especially various upper and lower bounds and the Nordhaus-Gaddum-type result of

2010 *Mathematics Subject Classification*. .

*Key words and phrases*. Reciprocal complementary Wiener number, distance, the Bar-Polyhex graph, The Mycielskian graph, Cactus chains..

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this invariant. Moreover, the unique tree with  $4 \leq D \leq n - 3$  and minimum  $RCW$ , the non-caterpillars with the smallest, the second smallest and the third smallest value of the reciprocal complementary Wiener number are characterized [21]. In [16], some bounds for  $RCW$  of line graphs are presented. Recently, we study the reciprocal complementary Wiener number of various graph operations like join, cartesian product, composition, strong product, disjunction, symmetric difference, corona product, splice and link of graphs [14].

The *Wiener index* is one of the oldest distance-based invariants and of the most studied topological quantities, both from a theoretical point of view and applications. This number is the sum of all distances between pairs of vertices of a graph  $G$  [18]. We refer the reader to [7, 13, 19] for more information and results on this quantity.

For a graph  $G$ , recall that the first Zagreb index  $M_1(G)$  equals to the sum of squares of the vertex degrees of  $G$ , and the second Zagreb index  $M_2(G)$  equals to the sum of product of degree of pairs of adjacent vertices of  $G$ , i.e.

$$M_1(G) = \sum_{x \in V(G)} d_G^2(x), \quad M_2(G) = \sum_{xy \in E(G)} d_G(x)d_G(y).$$

Let  $d(G, k)$  be the number of vertex pairs at distance  $k$  in a graph  $G$ . Brückler et al. [3] introduced a general distance-based topological index as

$$Q(G) = \sum_{k \geq 0} f(k)d(G, k), \quad (2)$$

where  $f$  is a function such that  $f(0) = 0$ .  $Q$  is an additive function of increments associated with pairs of vertices of  $G$ . By choosing  $f(k) = k$ ,  $\frac{k^2}{2} + \frac{k}{2}$ ,  $\frac{1}{k}$ ,  $\frac{k^3}{6} + \frac{k^2}{2} + \frac{k}{3}$ , and  $\frac{1}{D+1-k}$  the  $Q$ -index is equal to the Wiener, hyper-Wiener, Harary, Tratch-Stankevich-Zefirov indices and the reciprocal complementary Wiener number, respectively. In other words, we have:

$$RCW(G) = \sum_{k=1}^D \frac{d(G, k)}{D+1-k}. \quad (3)$$

In this work, we first present some new bounds for  $RCW$  of graphs. Then, we study the reciprocal complementary Wiener number of the Bar-Polyhex and the Mycielskian graphs.

## 2. Preliminaries and Lemmas

In this section, we first recall some basic analytical inequalities will be used in this research. Then, we prove an effective lemma which will be used for establishing various bounds on the reciprocal complementary Wiener number of graphs.

**Lemma 2.1** ([2]). *If  $a_i$  and  $b_i$  are real numbers such that there exist the real numbers  $m_1, M_1, m_2$  and  $M_2$  with  $m_1 \leq a_i \leq M_1, m_2 \leq b_i \leq M_2, i = 1, 2, \dots, l$ , then*

$$\left| \frac{1}{l} \sum_{i=1}^l a_i b_i - \frac{1}{l^2} \sum_{i=1}^l a_i \cdot \sum_{i=1}^l b_i \right| \leq \frac{1}{l^2} \left[ \frac{l^2}{4} \right] (M_1 - m_1)(M_2 - m_2),$$

where  $[x]$  is the largest integer equal to or less than  $x$ .

**Lemma 2.2** ([6]). *If  $a_i$  and  $b_i$  are real numbers such that  $a_i \neq 0$ , and  $m \leq \frac{b_i}{a_i} \leq M, i = 1, 2, \dots, l$ , then*

$$\sum_{i=1}^l b_i^2 + mM \sum_{i=1}^l a_i^2 \leq (M + m) \sum_{i=1}^l a_i b_i \quad (\text{Diaz-Metcalf inequality}),$$

with equality if and only if for all  $i, 1 \leq i \leq l$ , either  $b_i = ma_i$  or  $b_i = Ma_i$ .

**Lemma 2.3** ([4, 15]). *If  $a_i$  and  $b_i$  are positive real numbers such that  $m_1 \leq a_i \leq M_1, m_2 \leq b_i \leq M_2, i = 1, 2, \dots, l$ , then*

$$\frac{\sum_{i=1}^l a_i^2 \sum_{i=1}^l b_i^2}{(\sum_{i=1}^l a_i b_i)^2} \leq \frac{1}{4} \left( \sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}} \right)^2 \quad (\text{Pólya-Szegő inequality}),$$

with equality if and only if  $v = \frac{nM_1 m_2}{M_1 m_2 + M_2 m_1}$  is an integer and if  $v$  of the  $a_i$  are equal to  $m_1$  and the others equal to  $M_1$ , with the corresponding  $b_i$  being  $M_2, m_2$  respectively.

**Lemma 2.4** ([17]). *If  $a_i$  and  $b_i$  are positive real numbers such that  $m_1 \leq a_i \leq M_1, m_2 \leq b_i \leq M_2, i = 1, 2, \dots, l$ , then*

$$\frac{\sum_{i=1}^l a_i^2}{\sum_{i=1}^l a_i b_i} - \frac{\sum_{i=1}^l a_i b_i}{\sum_{i=1}^l b_i^2} \leq \left( \sqrt{\frac{M_1}{m_2}} - \sqrt{\frac{m_1}{M_2}} \right)^2 \quad (\text{Shisha-Mond inequality}).$$

For an  $(n, m)$ -graph  $G$ , it is clear that  $d(G, 0)$  represents the number of vertices of  $G$ . Also, the number of vertex pairs at unit distance is equal to the number of edges. Thus,  $d(G, 0) = n$  and  $d(G, 1) = m$ . In the following lemma, the numbers of pairs of vertices at distance two and three are specified.

**Lemma 2.5** ([1, 8]). *Let  $G$  be an  $(n, m)$ -graph with  $h$  hexagons and  $g(G) > 4$ . Then*

- (i)  $d(G, 2) = \frac{M_1(G)}{2} - m,$
- (ii)  $d(G, 3) = M_2(G) - M_1(G) + m - 3h.$

**Lemma 2.6.** *Let  $G$  be a connected  $n$ -vertex graph with diameter  $D$ . Also, let  $\mathcal{D} \subseteq \{1, 2, \dots, D\}$ ,  $\mathcal{S} = \{\{x, y\} \subseteq V(G) \mid d_G(x, y) \in \mathcal{D}\}$  and  $\mathcal{S}^c$  be the complement of  $\mathcal{S}$ . Then*

$$RCW(G) \geq \frac{\binom{n}{2} - n_s}{\sum_{\{x,y\} \in \mathcal{S}} d_G(x, y) + (D+1) [\binom{n}{2} - n_s] - W(G)} + \sum_{\{x,y\} \in \mathcal{S}^c} \frac{1}{D+1 - d_G(x, y)},$$

where  $n_s = n(\mathcal{S})$ , and equality holds if and only if  $d_G(x, y) = D$ , for all  $\{x, y\} \in \mathcal{S}^c$ .

**Proof.** We prove the lemma by applying the Cauchy-Schwarz inequality. More precisely,

$$\begin{aligned} RCW(G) &= \sum_{\{x,y\} \in \mathcal{S}^c} \frac{1}{D+1 - d_G(x, y)} + \sum_{\{x,y\} \in \mathcal{S}} \frac{1}{D+1 - d_G(x, y)} \\ &\geq \frac{[\binom{n}{2} - n_s]^2}{(D+1) [\binom{n}{2} - n_s] - \sum_{\{x,y\} \in \mathcal{S}^c} d_G(x, y)} + \sum_{\{x,y\} \in \mathcal{S}} \frac{1}{D+1 - d_G(x, y)} \\ &= \frac{[\binom{n}{2} - n_s]^2}{\sum_{\{x,y\} \in \mathcal{S}} d_G(x, y) + (D+1) [\binom{n}{2} - n_s] - W(G)} + \sum_{\{x,y\} \in \mathcal{S}} \frac{1}{D+1 - d_G(x, y)}. \quad \square \end{aligned}$$

**3. Bounds on RCW**

For a given graph  $G$ , assume that  $V(G) = \{v_1, v_2, \dots, v_n\}$  is the vertex-set of graph  $G$ . For convenience, we may denote  $d_G(v_i, v_j)$  by  $d_{ij}$ . We are now ready to present some bounds for the reciprocal complementary Wiener number of graphs.

**Theorem 3.1.** *Let  $G$  be an  $(n, m)$ -graph with diameter  $D$  and having  $h$  hexagons, and  $g(G) > 4$ . Then*

$$\begin{aligned} \text{(i) } RCW(G) &\geq \frac{m}{D} + \frac{M_1(G) - 2m}{2(D-1)} + \frac{M_2(G) - M_1(G) + m - 3h}{D-2} + \frac{\binom{n}{2} + \frac{M_1(G)}{2} + 3h - M_2(G) - m}{D-3}, \\ \text{(ii) } RCW(G) &\leq \frac{m}{D} + \frac{M_1(G) - 2m}{2(D-1)} + \frac{M_2(G) - M_1(G) + m - 3h}{D-2} + \binom{n}{2} + \frac{M_1(G)}{2} + 3h - M_2(G) - m. \end{aligned}$$

**Proof.** By (3) and Lemma 2.5 and by noticing that  $d(G, 1) = m$ , we have

$$\begin{aligned} RCW(G) &= \frac{d(G, 1)}{D} + \frac{d(G, 2)}{D-1} + \frac{d(G, 3)}{D-2} + \sum_{k=4}^D \frac{d(G, k)}{D+1-k} \\ &= \frac{m}{D} + \frac{M_1(G) - 2m}{2(D-1)} + \frac{M_2(G) - M_1(G) + m - 3h}{D-2} + \sum_{k=4}^D \frac{d(G, k)}{D+1-k}. \end{aligned}$$

The proof is completed by considering the following relations.

$$\sum_{k=4}^D \frac{d(G, k)}{D-3} \leq \sum_{k=4}^D \frac{d(G, k)}{D+1-k} \leq \sum_{k=4}^D d(G, k),$$

and

$$\begin{aligned} \sum_{k=4}^D d(G, k) &= \sum_{k=1}^D d(G, k) - d(G, 1) - d(G, 2) - d(G, 3) \\ &= \binom{n}{2} + \frac{M_1(G)}{2} + 3h - M_2(G) - m. \end{aligned} \quad \square$$

**Theorem 3.2.** *Let  $G$  be an  $(n, m)$ -graph with diameter  $D$ . Then*

$$\begin{aligned} \text{(i) } RCW(G) &\geq \frac{\binom{n}{2}^2}{(D+1)\binom{n}{2} - W(G)}, \\ \text{(ii) } RCW(G) &\geq \frac{[\binom{n}{2} - m]^2}{(D+1)\binom{n}{2} - mD - W(G)} + \frac{m}{D}. \end{aligned}$$

*The equalities in parts (i) and (ii) hold if and only if  $D = 1$  and  $D \in \{1, 2\}$ , respectively.*

**Proof.** The proof of parts (i) and (ii) are completed by choosing  $\mathcal{D}$  as a empty set and  $\mathcal{D} = \{1\}$ , respectively, in Lemma 2.6. □

**Theorem 3.3.** *Let  $G$  be an  $(n, m)$ -graph with diameter  $D$  and having  $h$  hexagons, and  $g(G) > 4$ . Then*

$$\begin{aligned} \text{(i) } RCW(G) &> \frac{(n^2 - n - M_1(G))^2}{(2D+2)(n^2 - n - M_1(G)) + 4(M_1(G) - m - W(G))} + \frac{m}{D} + \frac{M_1(G) - 2m}{2D-2}, \\ \text{(ii) } RCW(G) &> \frac{\alpha^2}{(2D+2)\alpha + 4(3M_2(G) + 2m - 2M_1(G) - 9h - W(G))} + \frac{m}{D} + \frac{M_1(G) - 2m}{2D-2} \\ &\quad + \frac{M_2(G) - M_1(G) + m - 3h}{D-2}. \end{aligned}$$

Where  $\alpha = n^2 + M_1(G) + 6h - 2M_2(G) - 2m - n$ .

**Proof.** Similar to the proof of Theorem 3.2, we apply Lemma 2.6 to prove this proposition. By setting  $\mathcal{D} = \{1, 2\}$ , the inequality in the first part is gained. Also, to prove the second part of lemma, it is enough to consider the set  $\mathcal{D} = \{1, 2, 3\}$ . □

**Theorem 3.4.** *Let  $G$  be an  $(n, m)$ -graph with diameter  $D$ . Then*

$$RCW(G) \leq \lfloor \frac{\binom{n}{2}^2}{4} \rfloor \frac{(D-1)^2 + D\binom{n}{2}^2}{D[(D+1)\binom{n}{2} - W(G)]}.$$

**Proof.** Set  $a_{ij}^{-1} = b_{ij} = D + 1 - d_{ij}$ ,  $1 \leq i < j \leq n$ , and  $l = \binom{n}{2}$  in Lemma 2.1. So,  $M_1 = m_2 = 1$  and  $M_2 = m_1^{-1} = D$ . Also, define

$$C(\bar{a}, \bar{b}) := \frac{1}{l} \sum_{1 \leq i < j \leq n} a_{ij} b_{ij} - \frac{1}{l^2} \sum_{1 \leq i < j \leq n} a_{ij} \cdot \sum_{1 \leq i < j \leq n} b_{ij}.$$

Therefore,

$$C(\bar{a}, \bar{b}) = \frac{1}{\binom{n}{2}} \sum_{\{v_i, v_j\} \subseteq V(G)} 1 - \frac{1}{\binom{n}{2}^2} \sum_{\{v_i, v_j\} \subseteq V(G)} (D+1 - d_{ij}) \sum_{\{v_i, v_j\} \subseteq V(G)} \frac{1}{D+1 - d_{ij}}$$

$$= 1 - \frac{1}{\binom{n}{2}^2} RCW(G) \left[ (D+1) \binom{n}{2} - W(G) \right].$$

By applying Theorem 3.2 (i) one can see that the value of  $C(\bar{a}, \bar{b})$  is less than or equal to zero. Hence, by Lemma 2.1 we have

$$RCW(G) \left[ (D+1) \binom{n}{2} - W(G) \right] - \binom{n}{2}^2 \leq \lfloor \frac{\binom{n}{2}^2}{4} \rfloor \frac{(D-1)^2}{D},$$

as desired. □

**Theorem 3.5.** *Let  $G$  be an  $(n, m)$ -graph with diameter  $D$ . Then*

$$RCW(G) \leq \frac{W(G)}{D}.$$

*With equality if and only if  $G \cong K_n$ .*

**Proof.** By considering  $a_{ij}^{-1} = b_{ij} = \sqrt{D+1 - d_{ij}}$  in Lemma 2.2, we can consider  $m = 1$  and  $M = D$ . The inequality is achieved by applying Diaz-Metcalf inequality. Also, equality only occurs when either  $d_{ij} = D$  or  $d_{ij} = 1$ , for all  $i, 1 \leq i \leq l$ . Which in both cases we conclude that  $G$  must be a complete graph. □

**Theorem 3.6.** *Let  $G$  be an  $(n, m)$ -graph with diameter  $D$ . Then*

$$(i) RCW(G) \leq \frac{(D+1)^2 \binom{n}{2}^2}{4D \left[ (D+1) \binom{n}{2} - W(G) \right]},$$

$$(ii) RCW(G) \leq \binom{n}{2} \left[ \frac{(\sqrt{D}-1)^2}{D} + \frac{\binom{n}{2}}{(D+1) \binom{n}{2} - W(G)} \right].$$

*Each of the equalities holds if and only if  $G \cong K_n$ .*

**Proof.** By considering  $a_{ij}^{-1} = b_{ij} = \sqrt{D+1 - d_{ij}}$  in Lemmas 2.3 and 2.4, we can set  $M_1 = m_2 = 1$  and  $M_2 = m_1^{-1} = \sqrt{D}$ . Our main proof will consider two separate parts as follows.

1. Applying Pólya-Szegő inequality, we have

$$\binom{n}{2}^{-2} RCW(G) \left[ (D+1) \binom{n}{2} - W(G) \right] \leq \frac{(D+1)^2}{4D}.$$

Therefore,

$$RCW(G) \leq \frac{(D+1)^2 \binom{n}{2}^2}{4D \left[ (D+1) \binom{n}{2} - W(G) \right]}.$$

Which completes the proof of the first part.

2. And finally, by using Shisha-Mond inequality, we can see that

$$\frac{RCW(G)}{\binom{n}{2}} - \frac{\binom{n}{2}}{(D+1)\binom{n}{2} - W(G)} \leq \frac{(\sqrt{D}-1)^2}{D}.$$

Hence,

$$RCW(G) \leq \binom{n}{2} \left[ \frac{(\sqrt{D}-1)^2}{D} + \frac{\binom{n}{2}}{(D+1)\binom{n}{2} - W(G)} \right].$$

This completes the proof. □

#### 4. RCW of the Bar-Polyhex and Mycielski graphs

In this section, we study the reciprocal complementary Wiener number of the Bar-Polyhex and the Mycielski graphs.

The Bar-Polyhex graph, is composed of exclusively of hexagonal rings that are face bounded by six-membered cycles in the plane. Any two rings have either one common edge or have no common vertices. We denote by  $\mathcal{G}_n$ , the Bar-Polyhex with  $n$  hexagons (see Figure 1).

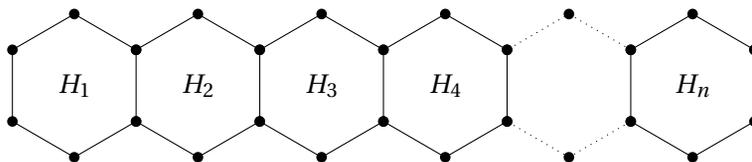


Figure 1: The Bar Polyhex graph  $\mathcal{G}_n$  with  $n$  hexagons.

For a simple graph  $G$  with vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$  and edge set  $E(G)$ , let  $U$  be a copy of  $V(G)$  (i.e.  $U = \{u_i : v_i \in V(G), i = 1, 2, \dots, n\}$ ) and let  $w$  be a new vertex. The *Mycielskian* or *Mycielski graph* of  $G$ , denoted by  $\mu(G)$ , is the graph with vertex set  $V(\mu(G)) = V(G) \cup U \cup \{w\}$  and the edge set

$$E(\mu(G)) = E(G) \cup \{v_i u_j : v_i v_j \in E(G)\} \cup \{w u_i : u_i \in U\}, \{i, j\} \subseteq \{1, 2, \dots, n\}.$$

If the Bar-Polyhex graph has only one hexagon ( $n = 1$ ), then by a simple calculation one can see that  $RCW(\mathcal{G}_1) = RCW(C_6) = 8$ . In the following result, we obtain the reciprocal complementary Wiener number of Bar-Polyhex graph  $\mathcal{G}_n$ , for  $n \geq 2$ .

**Proposition 4.1.** *Let  $\mathcal{G}_n$  be a Bar-Polyhex graph with  $n \geq 2$  hexagonals. Then*

$$\begin{aligned} RCW(\mathcal{G}_n) &= \frac{32n^4 + 12n^3 - 18n^2 - 3n + 1}{n(2n-1)(2n+1)} - 2 \sum_{k=1}^{2n-2} \frac{1}{k} \\ &> \frac{92n^3 - 18n^2 - 27n + 1}{n(2n-1)(2n+1)} - 8. \end{aligned}$$

**Proof.** For  $k = 0, 1, \dots, D = 2n + 1$ , the number of vertex pairs at distance  $k$  is obtained from the following (see [12]).

$$d(\mathcal{G}_n, k) = \begin{cases} 4n + 2 & k = 0, \\ 5n + 1 & k = 1, \\ 8n - 2 & k = 2, \\ 9n - 6 & k = 3, \\ 8n + 6 - 4k & k \geq 4. \end{cases}$$

Therefore, by relation (3) we have

$$\begin{aligned} RCW(\mathcal{G}_n) &= \sum_{k=1}^{D=2n+1} \frac{d(\mathcal{G}_n, k)}{D+1-k} \\ &= \frac{5n+1}{2n+1} + \frac{8n-2}{2n} + \frac{9n-6}{2n-1} + \sum_{k=4}^D \frac{8n+6-4k}{2n+2-k} \\ &= \frac{32n^4 + 12n^3 - 18n^2 - 3n + 1}{n(2n-1)(2n+1)} - 2 \sum_{k=1}^{2n-2} \frac{1}{k} \\ &> \frac{92n^3 - 18n^2 - 27n + 1}{n(2n-1)(2n+1)} - 8. \end{aligned} \quad \square$$

**Proposition 4.2.** Let  $G$  be an  $(n, m)$ -graph with diameter  $D$  and having  $h$  hexagons, and  $g(G) > 4$ . Then

$$RCW(\mu(G)) = \frac{1}{12} (14n^2 + 18h - 3n - 13m - 6M_1(G) - 6M_2(G)).$$

**Proof.** By the definitions of the Mycielski graph of  $G$ , we obtain that

$$d_{\mu(G)}(x, y) = \begin{cases} d_G(v_i, v_j) & x = v_i, y = v_j, d_G(v_i, v_j) \leq 3, \\ 4 & x = v_i, y = v_j, d_G(v_i, v_j) \geq 4, \\ 2 & x = v_i, y = u_i, \\ d_G(v_i, v_j) & x = v_i, y = u_j, i \neq j, d_G(v_i, v_j) \leq 2, \\ 3 & x = v_i, y = u_j, i \neq j, d_G(v_i, v_j) \geq 3, \\ 2 & x = v_i, y = w, \\ 2 & x = u_i, y = u_j, i \neq j, \\ 1 & x = u_i, y = w. \end{cases}$$

and

$$D(\mu(G)) = \begin{cases} 2 & D = 1 \text{ or } 2, \\ 3 & D = 3, \\ 4 & D \geq 4. \end{cases}$$

Hence,

$$\begin{aligned}
 RCW(\mu(G)) &= \sum_{\{x,y\} \subseteq V(\mu(G))} \frac{1}{D(\mu(G)) + 1 - d_{\mu(G)}(x,y)} \\
 &= \sum_{\{x,y\} \subseteq V(\mu(G))} \frac{1}{5 - d_{\mu(G)}(x,y)} \\
 &= \sum_{\substack{\{v_i,v_j\} \subseteq V(G) \\ d_G(v_i,v_j) \leq 3}} \frac{1}{5 - d_G(v_i,v_j)} + \sum_{\substack{\{v_i,v_j\} \subseteq V(G) \\ d_G(v_i,v_j) \geq 4}} 1 \\
 &\quad + \sum_{\substack{\{v_i,u_j\} \subseteq V(\mu(G)) \\ i \neq j, d_G(v_i,v_j) \leq 2}} \frac{1}{5 - d_G(v_i,v_j)} + \sum_{\substack{\{v_i,u_j\} \subseteq V(\mu(G)) \\ i \neq j, d_G(v_i,v_j) \geq 3}} \frac{1}{2} \\
 &\quad + \sum_{\{v_i,u_i\} \subseteq V(\mu(G))} \frac{1}{3} + \sum_{\{v_i,w\} \subseteq V(\mu(G))} \frac{1}{3} + \sum_{\{u_i,u_j\} \subseteq U} \frac{1}{3} + \sum_{\{u_i,w\} \subseteq V(\mu(G))} \frac{1}{4} \\
 &= \frac{1}{6}n^2 + \frac{3}{4}n + S_1 + S_2 + S_3 + S_4. \tag{4}
 \end{aligned}$$

Where,  $S_1$  to  $S_4$  are the sums of the first four above terms, in order. We shall calculate  $S_1$  to  $S_4$  separately. By Lemma 2.5 we have

$$S_1 = \sum_{\substack{\{v_i,v_j\} \subseteq V(G) \\ d_G(v_i,v_j) \leq 3}} \frac{1}{5 - d_G(v_i,v_j)} = \sum_{k=1}^3 \frac{d(G,k)}{5 - k} = \frac{m}{12} - \frac{M_1(G)}{3} + \frac{M_2(G)}{2} - \frac{3}{2}h, \tag{5}$$

$$S_2 = \sum_{\substack{\{v_i,v_j\} \subseteq V(G) \\ d_G(v_i,v_j) \geq 4}} 1 = \binom{n}{2} - \sum_{k=1}^3 d(G,k) = \binom{n}{2} - m + \frac{M_1(G)}{2} - M_2(G) + 3h, \tag{6}$$

$$\begin{aligned}
 S_3 &= \sum_{\substack{\{v_i,u_j\} \subseteq V(\mu(G)) \\ i \neq j, d_G(v_i,v_j) \leq 2}} \frac{1}{5 - d_G(v_i,v_j)} \\
 &= \sum_{\substack{\{v_i,u_j\} \subseteq V(\mu(G)) \\ v_i v_j \in E(G)}} \frac{1}{4} + \sum_{\substack{\{v_i,u_j\} \subseteq V(\mu(G)) \\ d_G(v_i,v_j)=2}} \frac{1}{3} \\
 &= \frac{m}{2} + \frac{2}{3}d(G,2) \\
 &= \frac{M_1(G)}{3} - \frac{m}{6}, \tag{7}
 \end{aligned}$$

$$S_4 = \sum_{\substack{\{v_i,u_j\} \subseteq V(\mu(G)) \\ i \neq j, d_G(v_i,v_j) \geq 3}} \frac{1}{2} = \binom{n}{2} - \sum_{k=1}^2 d(G,k) = \binom{n}{2} - \frac{M_1(G)}{2}. \tag{8}$$

Using (5), (6), (7) and (8) in (4), we conclude that

$$RCW(\mu(G)) = \frac{1}{12}(14n^2 + 18h - 3n - 13m - 6M_1(G) - 6M_2(G)). \quad \square$$

## 5. Concluding Remarks

In this paper, some new bounds for the reciprocal complementary Wiener number of graphs are presented. Also, this invariant for two types of graphs is studied. It would be of interest to study its behavior also on various classes of connected graphs with simple connectivity patterns and cycle structure.

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