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A NOTE ON COMMON FIXED POINTS BY ALTERING DISTANCES

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Abstract. The purpose of this paper is to provide answer to an open problem due to Sastry et al. [1].

1. Introduction

Recently, Sastry et al. [1] obtained conditions for the existence of unique common fixed point for weakly commuting pairs of self mappings in a complete metric space, by altering distances between points. Also, Sastry proved a common fixed point theorem, introducing the notion of certain control function in order to alter distance between the points. Sastry et al. [1] established a unique common fixed point theorem for four self mappings by applying the following notions:

Definition 1.1. A *control function* ψ is defined as $\psi : \Re^+ \to \Re^+$ which is continuous at zero, monotonically increasing, $\psi(2t) \leq 2\psi(t)$ and $\psi(t) = 0$ if, and only if $t = 0$. It is noted that this function ψ need not be sub-additive.

Definition 1.2. Two self mappings A and S of a metric space (X, d) are called *weakly commuting* if $d(ASx, SAx) \leq d(Ax, Sx)$ for each x in X. This condition implies that $ASx = SAx$ whenever $Ax = Sx$.

Definition 1.3. Two self mappings A and S of a metric space (X, d) are called ψ*-compatible* if $\mathrm{Lim}_n \psi(d(ASx_n, SAx_n)) = 0$ whenever $\{x_n\}$ is a sequence such that $\mathrm{Lim}_n Ax_n = \mathrm{Lim}_n Sx_n = t$ for some t in X.

Sastry et al. [1] proved the following theorem:

Theorem (2.4. of [1]) *Let (A, S) and (B, T) be weakly commuting pairs of self mappings of a complete metric space (X, d) and function ψ be as in definition* (1.1) *satisfying*

(i) *$AX \subset TX$, $BX \subset SX$ and*

(ii) *there exists h in $[0, 1)$ such that $\psi(d(Ax, By)) \leq hM_\psi(x, y)$ for all x, y in X.*

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Suppose that A and S are ψ-compatible and S is continuous. Then A, B, S and T have a unique common fixed point.

This theorem is valid if we assume B and T are ψ-compatible and T is continuous, instead of similar restrictions on A and S.

On the basis of theorem 2.4 of [1], Sastry posed the following open problem:

Is theorem 2.4 of [1] valid if we replace continuity of S by continuity of A or continuity of T by continuity of B?

In the present paper, we prove a common fixed point theorem which provides an affirmative answer to the above question on the existence of fixed point. Also, we state the following lemma (2.3 of [1]) which is used in the main theorem.

Lemma 1.4. *Let $f : \Re^+ \to \Re^+$ be increasing, continuous at the origin and vanishing only at zero. Then $\{t_n\} \subset \Re^+$ and $f(t_n) \to 0$ implies that $t_n \to 0$.*

2. Main Theorem

Theorem 2.1. *Let (A, S) and (B, T) be weakly commuting pairs of self mappings of a complete metric space (X, d) and function ψ be as in definition (1.1) satisfying*

(i) *$AX \subset TX$, $BX \subset SX$ and*

(ii) *there exists h in $[0, 1)$ such that $\psi(d(Ax, By)) \leq hM_\psi(x, y)$ for all x, y in X.*

Suppose that A and S are ψ-compatible and A is continuous. Then A, B, S and T have a unique common fixed point.

Proof. Let x_0 be any fixed point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \quad \text{and} \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}. \tag{2.1.1}$$

This can be done by virtue of (i). Then applying the same proof as that in the theorem 2.1 of Sastry et al. [1], we can show that $\{y_n\}$ is a Cauchy sequence. Since X is a complete metric space, there is a point z in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \to z \quad \text{and} \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+1} \to z. \tag{2.1.2}$$

Now, suppose that (A, S) is ψ-compatible then we have

$$Ax_{2n} \to z \ \text{ and } \ Sx_{2n} \to z \quad \text{implies that} \quad \operatorname{Lim}_n \psi(d(ASx_{2n}, SAx_{2n})) = 0. \tag{2.1.3}$$

Also, since A is continuous, so by (2.1.2), we get

$$AAx_{2n} \to Az \quad \text{and} \quad ASx_{2n} \to Az \quad \text{as} \quad n \to \infty. \tag{2.1.4}$$

We claim that $\operatorname{Lim}_n SAx_{2n} = Az$. Using (2.1.3), we get $\psi(d(SAx_{2n}, Az)) \leq \psi(d(SAx_{2n}, ASx_{2n}) + d(ASx_{2n}, Az)) \to 0$ as $n \to \infty$. Thus, using the above lemma (1.4), $d(SAx_{2n}, Az)$

$\to 0$ as $n \to \infty$, and so $\mathrm{Lim}_n SAx_{2n} = Az$. Also, since $AX \subset TX$, for each n, there exists w_{2n} in X such that $AAx_{2n} = Tw_{2n}$ and $AAx_{2n} = Tw_{2n} \to Az$. Thus, $AAx_{2n} \to Az$, $SAx_{2n} \to Az$, $ASx_{2n} \to Az$ and $Tw_{2n} \to Az$ as $n \to \infty$. Again, we claim that $\mathrm{Lim}_n Bw_{2n} \to Az$. If not, then there exists $\varepsilon > 0$ and a subsequence $\{n_k\}$ such that $d(AAx_{2n_k}, Bw_{2n_k}) > \varepsilon$ and $\psi(d(SAx_{2n_k}, ASx_{2n_k})) < \varepsilon$ for all n_k. Therefore,

$$\begin{aligned}\psi(\varepsilon) &\leq \psi(d(AAx_{2n_k}, Bw_{2n_k}))\\ &\leq hM_\psi(Ax_{2n_k}, w_{2n_k})\\ &= h\max\{\psi(d(SAx_{2n_k}, Tw_{2n_k})), \psi(d(AAx_{2n_k}, SAx_{2n_k})), \psi(d(Bw_{2n_k}, Tw_{2n_k})),\\ &\qquad [\psi(d(AAx_{2n_k}, Tw_{2n_k})) + \psi(d(Bw_{2n_k}, SAx_{2n_k}))]/2\}\\ &= h\max\{\psi(d(Bw_{2n_k}, Tw_{2n_k})), [\psi(d(Bw_{2n_k}, SAx_{2n_k}))]/2\},\\ &= h\psi(d(Bw_{2n_k}, AAx_{2n_k})),\\ &< \psi(d(Bw_{2n_k}, AAx_{2n_k})), \quad \text{a contradiction. Hence } \lim_n Bw_{2n} = Az.\end{aligned}$$

We claim that $Az = Sz$. For this, using (ii), we get

$$\begin{aligned}\psi(d(Sz, Bw_{2n})) &\leq hM_\psi(z, w_{2n})\\ &= h\max\{\psi(d(Sz, Tw_{2n})), \psi(d(Az, Sz)), \psi(d(Bw_{2n}, Tw_{2n})),\\ &\qquad [\psi(d(Az, Tw_{2n})) + \psi(d(Bz, Sw_{2n}))]/2\},\\ &= h\max\{\psi(d(Sz, Tw_{2n})), \psi(d(Az, Sz)), [\psi(d(Sz, Bw_{2n}))]/2\}.\end{aligned}$$

Letting $n \to \infty$, we get $\psi(d(Sz, Az)) \leq h\max\{\psi(d(Sz, Az)), [\psi(d(Sz, Az))]/2\} = h\psi(d(Sz, Az))$, a contradiction.

Thus we have
$$Az = Sz. \tag{2.1.5}$$

Since $AX \subset TX$, there exists some w in X such that $Az = Tw$. Therefore, we have

$$Az = Sz = Tw. \tag{2.1.6}$$

Moreover, we show that $Az = Bw$. Suppose on the contrary that $Az \neq Bw$. Then, using (ii), we get

$$\begin{aligned}&\psi(d(Az, Bw))\\ &\leq hM_\psi(z, w)\\ &= h\max\{\psi(d(Sz, Tw)), \psi(d(Az, Sz)), \psi(d(Bw, Tw)), [\psi(d(Az, Tw)) + \psi(d(Bz, Sw))]/2\},\\ &= h\max\{\psi(d(Bw, Az)), [\psi(d(Bw, Az))]/2\},\\ &= h\psi(d(Bw, Az)), \quad \text{a contradiction.}\end{aligned}$$

Therefore,
$$Az = Bw. \quad \text{Hence} \quad Az = Sz = Tw = Bw. \tag{2.1.7}$$

Since A and S are weakly commuting, we have by (2.1.7), $ASz = SAz$ and hence

$$AAz = ASz = SAz = SSz \tag{2.1.8}$$

and by the weakly commuting property of B and T, we get

$$BBw = BTw = TBw = TTw. \tag{2.1.9}$$

We now show that $AAz = Az$. Suppose that $AAz \neq Az$ then by (ii), we get

$$\psi(d(Az, AAz)) = \psi(d(Bw, AAz)) \leq hM_{\psi}(Az, w) = h\psi(d(Az, AAz)),$$

(using (2.1.7) and (2.1.8)), a contradiction. Hence $AAz = Az$. Also, we have $AAz = SAz$. Therefore, Az is a common fixed point of A and S. Again, suppose that $BBw \neq Bw$. Then using (ii), we get

$$\begin{aligned} \psi(d(Bw, BBw)) &= \psi(d(Az, BBw)) && \text{(by (2.1.6))} \\ &\leq hM_{\psi}(z, Bw) \\ &= h\psi(d(Bw, BBw)), && \text{(by using (2.1.7) and (2.1.9))} \\ &< \psi(d(Bw, BBw)), && \text{a contradiction.} \end{aligned}$$

Hence $BBw = Bw$ and since $TBw = BBw$, we have Bw being a common fixed point for B and T. Finally, since $Az = Bw$, we have Az as a common fixed point for A, B, S and T. Moreover, the uniqueness of a common fixed point follows from (ii).

This completes the proof of the theorem.

The proof is similar when the pair (B, T) is assumed ψ-compatible and B is continuous.

References

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