TAMKANG JOURNAL OF MATHEMATICS
Volume 34, Number 1, Spring 2003

ON NEW GENERALIZATIONS OF HILBERT-PACHPATTE TYPE INTEGRAL INEQUALITIES

LÜ ZHONGXUE

Abstract. In this paper, some new generalizations of Hilbert-Pachpatte type inequalities are given by introducing some parameters $r_i, i \in \{1, 2, \ldots, n\}$.

1. Introduction

Hilbert's double series inequality and its integral version [3, Theorem 316] have been generalized in several directions (see [1-6,8,9]). Pachpatte [10-12] considered inequalities similar to those of Hilbert. Recenty, G. D. Handley, J. I. Koliha and J. E. Pečarič [13] considered a new class of Hilbert-Pachpatte type integral inequalities by specializing the parameters and the functions Φ_i.

In this paper, we show some new generalizations of Hilbert-Pachpatte type integral inequalities in [13] by introducing some parameters.

2. Notation and Preliminaries

The symbols N, Z, R have their usual meaning; R_+ denotes the interval $[0, \infty)$. The following notation and hypotheses will be used throughout the paper:

$I = \{1, \ldots, n\}$	$n \in N$
$m_i, i \in I$	$m_i \in N$
$k_i, i \in I$	$k_i \in \{0, \ldots, m_i - 1\}$
$x_i, i \in I$	$x_i \in R,\ x_i > 0$
$p_i, q_i, r_i \in I$	$p_i, q_i, r_i \in R_+,\ 1/p_i + 1/q_i = 1 - 1/r_i$
p, q, r	$1/p = \sum_{i=1}^n (1/p_i), \quad 1/q + 1/r = \sum_{i=1}^n (1/q_i + 1/r_i)$
$a_i, b_i, i \in I$	$a_i, b_i \in R_+,\ a_i + b_i = 1$
$w_i, i \in I$	$w_i \in R,\ w_i > 0, \quad \sum_{i=1}^n w_i = 1$
$\alpha_i, i \in I$	$\alpha_i = (a_i + b_i \dfrac{q_i r_i}{q_i + r_i})(m_i - k_i - 1)$
$\beta_i, i \in I$	$\beta_i = a_i(m_i - k_i - 1)$
$u_i, i \in I$	$u_i \in C^{m_i'}([0, x_i])$ for some $m_i' \geq m_i$
$\Phi_i, i \in I$	$\Phi_i \in C^1([0, x_i]), \quad \Phi_i \geq 0.$

Received January 4, 2002; revised February 21, 2001.
2000 *Mathematics Subject Classification*. 26D15.
Key words and phrases. Inequalities, Hilbert-Pachpatte inequalities, Hölder inequality.

Here the u_i are given functions of sufficient smoothness, and the Φ_i are given continuous nonnegative functions.

3. Statement of Results

Our main result is given in the following theorem.

Theorem 3.1. *Let* $u_i \in C^{m_i}([0, x_i])$. *for* $i \in I$. *If*

$$|u_i^{(k_i)}(s_i)| \leq \int_0^{s_i} (s_i - \tau_i)^{m_i - k_i - 1} \Phi_i(\tau_i) d\tau_i \quad s_i \in [0, x_i], \quad i \in I, \tag{3.1}$$

Then

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)\frac{q_i+r_i}{q_i r_i w_i}}} ds_1 \cdots ds_n$$
$$\leq U \prod_{i=1}^n x_i^{\frac{q_i+r_i}{q_i r_i}} \prod_{i=1}^n \left(\int_0^{x_i} (x_i - s_i)^{\beta_i+1} \Phi_i(s_i)^{p_i} ds_i \right)^{1/p_i} \tag{3.2}$$

where

$$U = \frac{1}{\prod_{i=1}^n [(\alpha_i + 1)^{\frac{q_i+r_i}{q_i r_i}} (\beta_i + 1)^{1/p_i}]} \tag{3.3}$$

Proof. Factorize the integrand on the right side of (3.1) as

$$(s_i - \tau_i)^{(a_i \frac{p_i - 1}{p_i} + b_i)(m_i - k_i - 1)} \times (s_i - \tau_i)^{(a_i/p_i)(m_i - k_i - 1)} \Phi_i(\tau_i)$$

and apply Hölder's inequality [7, p.106]. Then

$$|u_i^{(k_i)}(s_i)| \leq \left(\int_0^{s_i} (s_i - \tau_i)^{(a_i + b_i \frac{q_i r_i}{q_i + r_i})(m_i - k_i - 1)} d\tau_i \right)^{1-1/p_i} \left(\int_0^{s_i} (s_i - \tau_i)^{a_i(m_i - k_i - 1)} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{1/p_i}$$
$$= \frac{s_i^{(\alpha_i+1)\frac{q_i+r_i}{q_i r_i}}}{(\alpha_i + 1)^{\frac{q_i+r_i}{q_i r_i}}} \left(\int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{1/p_i}$$

Using the inequality of means [7, p.15]

$$\prod_{i=1}^n s_i^{(\alpha_i+1)\frac{q_i+r_i}{q_i r_i}} \leq \sum_{i=1}^n w_i s_i^{(\alpha_i+1)\frac{q_i+r_i}{q_i r_i w_i}},$$

we get

$$\prod_{i=1}^n |u_i^{(k_i)}(s_i)| \leq W \sum_{i=1}^n w_i s_i^{(\alpha_i+1)\frac{q_i+r_i}{q_i r_i w_i}} \prod_{i=1}^n \left(\int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{1/p_i}$$

where
$$W = \frac{1}{\prod_{i=1}^{n} (\alpha_i + 1)^{\frac{q_i + r_i}{q_i r_i}}}$$

In the following estimate we apply Hölder's inequality and at the end, change the order of integration:

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^{n} w_i s_i^{(\alpha_i+1)\frac{q_i+r_i}{q_i r_i w_i}}} ds_1 \cdots ds_n$$
$$\le W \prod_{i=1}^{n} \left[\int_0^{x_i} \left(\int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{1/p_i} ds_i \right]$$
$$\le W \prod_{i=1}^{n} x_i^{\frac{q_i+r_i}{q_i r_i}} \left(\int_0^{x_i} \left(\int_0^{s_i} (s_i - \tau_i)^{\beta_i} \Phi_i(\tau_i)^{p_i} d\tau_i \right) ds_i \right)^{1/p_i}$$
$$= \frac{W}{\prod_{i=1}^{n} (\beta_i + 1)^{1/p_i}} \prod_{i=1}^{n} x_i^{\frac{q_i+r_i}{q_i r_i}} \prod_{i=1}^{n} \left(\int_0^{x_i} (x_i - \tau_i)^{\beta_i + 1} \Phi_i(\tau_i)^{p_i} d\tau_i \right)^{1/p_i}$$

This proves the theorem.

Corollary 3.2. *Under the assumptions of Theorem* 3.1*,*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^{n} w_i s_i^{(\alpha_i+1)\frac{q_i+r_i}{q_i r_i w_i}}} ds_1 \cdots ds_n$$
$$\le p^{1/p} U \prod_{i=1}^{n} x_i^{\frac{q_i+r_i}{q_i r_i}} \left(\sum_{i=1}^{n} \frac{1}{p_i} \int_0^{x_i} (x_i - s_i)^{\beta_i + 1} \Phi_i(s_i)^{p_i} ds_i \right)^{1/p_i} \tag{3.4}$$

where U is given by (3.3)*.*

Proof. By the inequality of means, for any $A_i \ge 0$,

$$\prod_{i=1}^{n} A_i^{1/p_i} \le p^{1/p} \left(\sum_{i=1}^{n} \frac{1}{p_i} A_i \right)^{1/p}$$

The corollary then follows from the preceding theorem.

In the following sections we discuss various choices of the functions Φ_i.

4. The First Inequality

Theorem 4.1. *Let* $u_i \in C^{m_i}([0, x_i])$ *be such that* $u_i^{(j)}(0) = 0$ *for* $j \in \{0, \cdots, m_i - 1\}$, $i \in I$. *Then*

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^{n} w_i s_i^{(\alpha_i+1)\frac{q_i+r_i}{q_i r_i w_i}}} ds_1 \cdots ds_n$$
$$\le U_1 \prod_{i=1}^{n} x_i^{\frac{q_i+r_i}{q_i r_i}} \prod_{i=1}^{n} \left(\int_0^{x_i} (x_i - s_i)^{\beta_i+1} |u_i^{(m_i)}(s_i)|^{p_i} ds_i \right)^{1/p_i} \tag{4.1}$$

where

$$U_1 = \frac{1}{\prod_{i=1}^{n} [(m_i - k_i - 1)!(\alpha_i + 1)^{\frac{q_i+r_i}{q_i r_i}} (\beta_i + 1)^{1/p_i}]} \tag{4.2}$$

Proof. By [10, Eq.(7)],

$$u_i^{(k_i)}(s_i) = \frac{1}{(m_i - k_i - 1)!} \int_0^{s_i} (s_i - \tau_i)^{m_i - k_i - 1} u_i^{(m_i)}(\tau_i) d\tau_i$$

Inequality (4.1) is proved when we set

$$\Phi_i(s_i) = \frac{|u_i^{(m_i)}(s_i)|}{(m_i - k_i - 1)!} \tag{4.3}$$

in Theorem 3.1.

Corollary 4.2. *Under the assumptions of Theorem* 4.1,

$$\int_0^{x_1} \cdots \int_0^{x_n} \frac{\prod_{i=1}^{n} |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^{n} w_i s_i^{(\alpha_i+1)\frac{q_i+r_i}{q_i r_i w_i}}} ds_1 \cdots ds_n$$
$$\le p^{1/p} U_1 \prod_{i=1}^{n} x_i^{\frac{q_i+r_i}{q_i r_i}} \left(\sum_{i=1}^{n} \frac{1}{p_i} \int_0^{x_i} (x_i - s_i)^{\beta_i+1} |u_i^{(m_i)}(s_i)|^{p_i} ds_i \right)^{1/p} \tag{4.4}$$

where U_1 *is given by* (4.2).

5. The Second Inequality

Theorem 5.1. *Let* $u_i \in C^{m_i+1}([0,x_i])$ *be such that* $u_i^{(j)}(0)=0$ *for* $j \in \{0,\dots,m_i\}$, $i \in I$, *and let* $\rho \in C^1([0,\infty))$. *Then*

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^{n}|u_i^{(k_i)}(s_i)|}{\sum_{i=1}^{n}w_i s_i^{(\alpha_i+1)\frac{q_i+r_i}{q_ir_iw_i}}}ds_1\cdots ds_n$$

$$\le U_1\prod_{i=1}^{n}x_i^{\frac{q_i+r_i}{q_ir_i}}\prod_{i=1}^{n}\left[\int_0^{x_i}(x_i-s_i)^{\beta_i+1}\frac{s_i^{p_i-1}}{\rho(s_i)^{p_i}}\left(\int_0^{s_i}|(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))'|^{p_i}d\sigma_i\right)ds_i\right]^{1/p_i} \tag{5.1}$$

where U_1 *is given by* (4.2).

Proof. By [10, Eq.(14)].

$$u_i^{(k_i)}(s_i)=\frac{1}{m_i-k_i-1}\int_0^{s_i}(s_i-\tau_i)^{(m_i-k_i-1)}\left(\frac{1}{\rho(\tau_i)}\int_0^{\tau_i}(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))'d\sigma_i\right)d\tau_i$$

By Hölder's inequality,

$$\int_0^{\tau_i}|(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))'|d\sigma_i \le \tau_i^{\frac{q_i+r_i}{q_ir_i}}\left(\int_0^{\tau_i}|(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))'|^{p_i}d\sigma_i\right)^{1/p_i},$$

and inequality (5.1) hold with

$$\Phi_i(\tau_i)=\frac{1}{(m_i-k_i-1)!}\frac{\tau_i^{\frac{q_i+r_i}{q_ir_i}}}{\rho(\tau_i)}\left(\int_0^{\tau_i}|(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i)')|^{p_i}d\sigma_i\right)^{1/p_i}.$$

in Theorem 3.1.

Corollary 5.2. *Under the assumptions of Theorem* 5.1.

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^{n}|u_i^{(k_i)}(s_i)|}{\sum_{i=1}^{n}w_i s_i^{(\alpha_i+1)\frac{q_i+r_i}{q_ir_iw_i}}}ds_1\cdots ds_n$$

$$\le P^{1/p}U_1\prod_{i=1}^{n}x_i^{\frac{q_i+r_i}{q_ir_i}}\prod_{i=1}^{n}\left[\int_0^{x_i}(x_i-s_i)^{\beta_i+1}\frac{s_i^{p_i-1}}{\rho(s_i)^{p_i}}\left(\int_0^{s_i}|(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))'|^{p_i}d\sigma_i\right)ds_i\right]^{1/p} \tag{5.2}$$

where U_1 *is given by* (4.2).

6. The Third Inequality

Theorem 6.1. *Let* $u_i \in C^{2m_i}([0,x_i])$, $\rho \in C^m([0,\infty))$ *with* $m = \max_i m_i$, $u_i^{(j)}(0) = 0$ *and* $(\rho(s_i)u_i^{(m_i)}(s_i))^{(j)} = 0$ *at* $s_i = 0$ *for* $j \in \{0,\ldots,m_i-1\}$, $i \in I$. *Then*

$$\int_0^{x_1}\cdots\int_0^{x_n} \frac{\prod_{i=1}^n |u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)\frac{q_i+r_i}{q_ir_iw_i}}} ds_1\cdots ds_n$$
$$\le U_2 \prod_{i=1}^n x_i^{\frac{q_i+r_i}{q_ir_i}} \prod_{i=1}^n \left[\int_0^{x_i}(x_i-s_i)^{\beta_i+1}\frac{s_i^{m_ip_i-1}}{\rho(s_i)^{p_i}}\left(\int_0^{s_i}|(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))^{(m_i)}|^{p_i}d\sigma_i\right)ds_i\right]^{1/p_i} \tag{6.1}$$

where

$$U_2 = \frac{1}{\prod_{i=1}^n [(m_i-1)!(m_i-k_i-1)!(\frac{p_im_i-1}{p_i-1})^{\frac{q_i+r_i}{q_ir_i}}(\alpha_i+1)^{\frac{q_i+r_i}{q_ir_i}}(\beta_i+1)^{1/p_i}]} \tag{6.2}$$

Proof. By [10, Eq.(21)].

$$u_i^{(k_i)}(s_i) = \frac{1}{(m_i-1)!(m_i-k_i-1)!}$$
$$\times\int_0^{s_i}(s_i-\tau_i)^{(m_i-k_i-1)}\left(\frac{1}{\rho(\tau_i)}\int_0^{\tau_i}(\tau_i-\sigma_i)^{m_i-1}(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))^{(m_i)}d\sigma_i\right)d\tau_i$$

For brevity write

$$F_i(\sigma_i) = |(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))^{(m_i)}|.$$

By Hölder's inequality,

$$\int_0^{\tau_i}(\tau_i-\sigma_i)^{m_i-1}F_i(\sigma_i)d\sigma_i \le \left(\int_0^{\tau_i}(\tau_i-\sigma_i)^{\frac{q_ir_i(m_i-1)}{q_i+r_i}}d\sigma_i\right)^{1-1/p_i}\left(\int_0^{\tau_i}F_i(\sigma_i)^{p_i}d\sigma_i\right)^{1/p_i}$$
$$= \frac{\tau_i^{m_i-1/p_i}}{(\frac{p_im_i-1}{p_i-1})^{\frac{q_i+r_i}{q_ir_i}}}\left(\int_0^{\tau_i}F_i(\sigma_i)^{p_i}d\sigma_i\right)^{1/p_i}$$

and inequality (6.1) hold with

$$\Phi_i(\tau_i) = W\frac{\tau_i^{m_i-1/p_i}}{\rho(\tau_i)}\left(\int_0^{\tau_i}|(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))^{(m_i)}|^{p_i}d\sigma_i\right)^{1/p_i}.$$

where

$$W = \frac{1}{(m_i-1)!(m_i-k_i-1)!(\frac{p_im_i-1}{p_i-1})^{\frac{q_i+r_i}{q_ir_i}}}$$

in Theorem 3.1.

Corollary 6.2. *Under the assumptions of Theorem* 6.1.

$$\int_0^{x_1}\cdots\int_0^{x_n}\frac{\prod_{i=1}^n|u_i^{(k_i)}(s_i)|}{\sum_{i=1}^n w_i s_i^{(\alpha_i+1)\frac{q_i+r_i}{q_ir_iw_i}}}ds_1\cdots ds_n$$
$$\leq p^{1/p}U_2\prod_{i=1}^n x_i^{\frac{q_i+r_i}{q_ir_i}}\left[\sum_{i=1}^n\int_0^{x_i}\frac{(x_i-s_i)^{\beta_i+1}s_i^{m_ip_i-1}}{p_i\rho(s_i)^{p_i}}\int_0^{s_i}|(\rho(\sigma_i)u_i^{(m_i)}(\sigma_i))^{(m_i)}|^{p_i}d\sigma_i ds_i\right]^{1/p},(6.3)$$

where U_2 *is given by* (6.2).

Remark. Let $r_i\to\infty$, $i\in\{1,\ldots,n\}$, (3.2), (3.4), (4.1), (4.4), (5.1), (5.2), (6.1) and (6.3) change into (3.2), (3.4), (4.1), (4.4), (5.1), (5.2), (6.1) and (6.3) in [13], respectively. Hence (3.2), (3.4), (4.1), (4.4), (5.1), (5.2), (6.1) and (6.3) are generalizations of Hilbert-Pachpatte inequalities in [13].

References

[1] Yang Bicheng, *On Hilbert's integral inequality*, J. Math. Anal. Appl. **220**(1998), 778-785.

[2] Y. C. Chow, *On inequalities of Hilbert and Widder*, J. London Math. Soc. **14**(1939), 151-154.

[3] G. H. Hary, J. E. Littlewood and G. Polya, Inequalities, Cambridge Univ. Press, London, 1952.

[4] Mingzhe Gao, *An improvement of Hardy-Riesz's extension of the Hilbert inequality*, J. Math. Res. Exposition **14**(1994), 255-259.

[5] Mingzhe Gao, *On Hilbert's inequality and its applications*, J. Math. Anal. Appl. **212**(1997).

[6] D. S. Mitrinović and J. E. Pečaricć, *On inequalities of Hilbert and Widder*, Proc. Edinburgh. Math. Soc. **34**(1991), 411-414.

[7] D. S. Mitrinović and J. E. Pečarić and A. M. Fink. Classial and New Inequalities in Analysis, Kluwer Academic, Dordrecht, 1993.

[8] B. G. Pachpatte, *A note on Hilbert type inequality*, Tamkang J. Math. **29**(1998), 293-298.

[9] B. G. Pachpatte, *On some new inequalities similar to Hilbert's inequality*, J. Math. Anal. Appl. **226**(1998), 166-179.

[10] B. G. Pachpatte, *Inequalities similar to the integral analogue of Hilbert's inequality*, Tamkang J. Math. **30**(1999), 136-146.

[11] B. G. Pachpatte, *Inequalities similar to certain extension of Hilbert's inequality*, J. Math. Anal. Appl. **243**(2000), 217-227.

[12] B. G. Pachpatte, *A note on inequality of Hilbert type*, Demonstratio Math, in press.

[13] G. D. Handley and J. J. Koliha, *New Hilbert-Pachpatte type integral inequalities*, J. Math. Anal. Appl. **257**(2001), 238-250.

Department of Basic Science of Technology College Xuzhou Normal University, 221011, People's Republic of China.

E-mail: lvzx1@163.net