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PROXIMINALITY IN ORLICZ-BOCHNER FUNCTION SPACES

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Abstract. A (closed) subspace Y of a Banach space X is called proximinal if for every $x \in X$ there exists some $y \in Y$ such that $\|x-y\| \leq \|x-z\|$ for $z \in Y$. It is the object of this paper is to study the proximinality of $L^{\Phi}(I,Y)$ in $L^{\Phi}(I,X)$ for some class of Young's functions Φ, where I is the unit interval. We prove (among other results) that if Y is a separable proximinal subspace of X, then $L^{\Phi}(I,Y)$ is proximinal in $L^{\Phi}(I,X)$.

1. Introduction

Let Φ be a Young's function, [10] and X be a Banach space. $L^{\Phi}(I,X)$ denotes the sapce of all strongly measurable functions for the unit interval I (with the Lebesgue measure) with values in X for which $\int_I \Phi(\alpha\|f(t)\|)dt < \infty$ for some $\alpha > 0$. It is known that $L^{\Phi}(I,X)$, [3], is a Banach space under the Luxemburg norm:

$$\|f\|_{\Phi} = \inf\left\{k,\ \int_I \Phi\left\|\frac{1}{k}f(t)\right\|dt \leq 1, k>0\right\},\quad f \in L^{\Phi}(I,X).$$

We refer to [4], [7] and [3] for the basic structure of $L^{\Phi}(I,X)$.

A subspace (closed) Y of the Banach space X is called proximinal in X if for every $x \in X$ there exists $y \in Y$ such that $\|x-y\| \leq \|x-z\|$ for all $z \in Y$. The element y is called a best approximant of x in Y. One of the interesting problems in best approximation in function spaces is :"If Y is proximinal in X must $L^p(I,Y)$ be proximinal in $L^p(I,X)$". We refer to [2], [5], [6] and [7] for the main results on that problem. It is the object of this paper to study the proximinality of $L^{\Phi}(I,Y)$ in $L^{\Phi}(I,X)$ for proximinal subspace Y in X. We prove that $L^{\Phi}(I,Y)$ is proximinal in $L^{\Phi}(I,X)$ if and only if $L^1(I,Y)$ is proximinal in $L^1(I,X)$, a result which has many consequences.

2. Proximinality in $L^{\Phi}(I,X)$

Throughout the rest of this paper Y is a closed subspace of X and the Young function Φ is continuous and finite valued.

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We say that the Young function Φ satisfies $\Delta_2 -$ *condition* (in short $\Phi \in \Delta_2$) if there exist $k > 1$ and $x_0 > 0$ such that $\Phi(2x) \leq k\Phi(x)$, for $x \geq x_0$, with $\Phi(x_0) < \infty$.

Lemma 1. *Let $\Phi \in \Delta_2$ and Y be a proximinal subspace of the Banach space X. Let $f \in L^{\Phi}(I, X)$ and suppose that g is a strongly measrable function such that $g(t)$ is a best approximant to $f(t)$ in Y for almost all t in I. Then g is a best approximant to f in $L^{\Phi}(I, Y)$.*

Proof. Since $g(t)$ is a best approximant to $f(t)$ in Y for almost all t in I and $0 \in Y$, then we have $\|g(t)\| \leq 2\|f(t)\|$ for almost all t in I [6]. Since $\Phi \in \Delta_2$, [4], it follows that

$$\int_I \Phi(\|g(t)\|)dt \leq \int_I \Phi(2\|f(t)\|)dt < \infty.$$

Hence $g \in L^{\Phi}(I, Y)$. Further for almost all t in I, $\|f(t) - g(t)\| \leq \|f(t) - z\|$ for all z in Y, and $\|f(t) - g(t)\| \leq \|f(t) - h(t)\|$, for all h in $L^{\Phi}(I, Y)$. From the monotonocity of the Luxemburg norm, [8], and the fact that $\|(\|f(\cdot)\|)\|_{\Phi} = \|f\|_{\Phi}$, we get that $\|f - g\|_{\Phi} \leq \|f - h\|_{\Phi}$ for all h in $L^{\Phi}(I, Y)$. Thus g is a best approximant to f in $L^{\Phi}(I, Y)$.

Theorem 2. *If $\Phi \in \Delta_2$ and $f \in L^{\Phi}(I, X)$. Then the function* $\operatorname{dist}(f(t), Y) \in L^{\Phi}(I)$ *and* $\operatorname{dist}(f, L^{\Phi}(I, Y)) = \|\operatorname{dist}(f(\cdot), Y)\|_{\Phi}$.

Proof. For $f \in L^{\Phi}(I, X)$, f is strongly measurable, and there exists a sequence of simple functions in $L^{\Phi}(I, X)$ such that $\lim_{n\to\infty} \|f_n(t) - f(t)\| = 0$ for almost all t in I. The continuity of $\operatorname{dist}(x, Y)$ implies that $\lim_{n\to\infty} |\operatorname{dist}(f_n(t), Y) - \operatorname{dist}(f(t), Y)| = 0$. Set $g_n(t) = \operatorname{dist}(f_n(t), Y)$. Then each g_n (measurable) is a simple function and so $\operatorname{dist}(f(\cdot), Y)$ is measurable. We have $\operatorname{dist}(f(t), Y) \leq \|f(t) - z\|$, for all $z \in Y$. Thus $\operatorname{dist}(f(t), Y) \leq \|f(t) - g(t)\|$, for almost all t in I and all g in $L^{\Phi}(I, Y)$. Therefore, $\|\operatorname{dist}(f(\cdot), Y)\|_{\Phi} \leq \|f - g\|_{\Phi}$ for all $g \in L^{\Phi}(I, Y)$. Consequently $\operatorname{dist}(f(\cdot), Y) \in L^{\Phi}(I)$ and

$$\|\operatorname{dist}(f(\cdot), Y)\|_{\Phi} \leq \operatorname{dist}(f, L^{\Phi}(I, Y)). \qquad (*)$$

Now, that $\Phi \in \Delta_2$, [3], implies that for a given $\varepsilon > 0$, there exists f' a simple function in $L^{\Phi}(I, X)$ such that $\|f - f'\|_{\Phi} < \varepsilon/3$. Assume f' has the form $f'(t) = \sum_{i=1}^{n} \chi_{B_i}(t) x_i$ with the $B_i's$ disjoint and measurable sets, $x_i \in X$ and $\bigcup_{i=1}^{n} B_i = I$. We can assume $\mu(B_i) > 0$ and $\Phi(1) \leq 1$. For each $i = 1, 2, \ldots, n$ take $y_i \in Y$ such that $\|x_i - y_i\| < \operatorname{dist}(x_i, Y) + \varepsilon/3$.
Set $g(t) = \sum_{i=1}^{n} \chi_{B_i}(t) y_i$ and $F(t) = \operatorname{dist}(f(t), Y) + \|f(t) - f'(t)\| + \varepsilon/3$.
Consider:

$$\int_I \Phi\Big(\frac{\|f'(t) - g(t)\|}{\|F\|_{\Phi}}\Big)dt = \sum_{i=1}^{n} \int_{B_i} \Phi\Big(\frac{\|x_i - y_i\|}{\|F\|_{\Phi}}\Big)dt$$

$$
\begin{aligned}
&< \sum_{i=1}^{n} \int_{B_i} \Phi\Big(\frac{\operatorname{dist}(x_i, Y) + \varepsilon/3}{\|F\|_\Phi}\Big) dt \\
&= \int_I \Phi\Big(\frac{\operatorname{dist}(f'(t), Y) + \varepsilon/3}{\|F\|_\Phi}\Big) d\mu \\
&\leq \int_I \Phi\Big(\frac{\|f(t) - f'(t)\| + \operatorname{dist}(f(t), Y) + \varepsilon/3}{\|F\|_\Phi}\Big) dt \\
&= \int_I \Phi\Big(\frac{F(t)}{\|F\|_\Phi}\Big) dt \leq 1.
\end{aligned}
$$

Consequently, $\|f' - g\|_\Phi \leq \|\operatorname{dist}(f(\cdot), Y) + \|f(\cdot) - f'(\cdot)\| + \varepsilon/3\|_\Phi$.
Using: $\Phi(1) \leq 1$ and the fact that $\mu(I) = 1$, we get:

$$
\begin{aligned}
\operatorname{dist}(f, L^\Phi(I, Y)) &\leq \|f - f'\|_\Phi + \operatorname{dist}(f', L^\Phi(I, Y)) \\
&\leq \varepsilon/3 + \|f' - g\|_\Phi \\
&\leq \varepsilon/3 + \|\operatorname{dist}(f(\cdot), Y) + \|f(\cdot) - f'(\cdot)\| + \varepsilon/3\|_\Phi \\
&\leq \|\operatorname{dist}(f(\cdot), Y)\|_\Phi + \|f - f'\|_\Phi + 2\varepsilon/3 \\
&\leq \|\operatorname{dist}(f(\cdot), Y)\|_\Phi + \varepsilon.
\end{aligned}
$$

Since ε is arbitary, then $\operatorname{dist}(f, L^\Phi(I, Y)) \leq \|\operatorname{dist}(f(\cdot), Y)\|_\Phi$. Together with $(*)$ we get the required result.

Corollary 3. *Let $\Phi \in \Delta_2$, X a Banach space and Y a closed subspace of X. For $f \in L^\Phi(I, X)$ and $g \in L^\Phi(I, Y), g$ is a best approximant of f in $L^\Phi(I, X)$ if and only if $g(t)$ is a best approximant of $f(t)$ in Y for almost all t in I.*

Now we introduce the main theorem of this paper:

Theorem 4. *Let X be a Banach space and Y be a closed subspace of X. For a strictly increasing function $\Phi \in \Delta_2$, the following are equivalent:*
(i) *$L^1(I, Y)$ is proximinal in $L^1(I, X)$.*
(ii) *$L^\Phi(I, Y)$ is proximinal in $L^\Phi(I, X)$.*

Proof of. (i)→(ii). Suppose that $L^1(I, Y)$ is proximinal in $L^1(I, X)$. Let $f \in L^\Phi(I, X)$. Then $f \in L^1(I, X)$, [3]. Therefore, there exists $g \in L^1(I, Y)$ such that $\|f - g\|_1 = \operatorname{dist}(f, L^1(I, Y))$. Hence $g(t)$ a best approximant to $f(t)$ for almost all t in I, [coroll. 2.11, 6]. Thus Lemma 1 gives (ii).

For (ii)→(i). Suppose that $L^\Phi(I, Y)$ is proximinal in $L^\Phi(I, X)$. Consider the map:

$$J : L^1(I, X) \to L^\Phi(I, X)$$

$$J(f)(t) = \frac{\Phi^{-1}(\|f(t)\|)}{\|f(t)\|} f(t)$$

if $f(t) \neq 0$ and $J(f)(t) = 0$ otherwise. Since $\|J(f)(t)\| = \Phi^{-1}(\|f(t)\|)$ and $f \in L^1(I, X)$, then $J(f) \in L^\Phi(I, X)$. Further, if $g \in L^\Phi(I, X)$, then $f(t) = \frac{\Phi(\|g(t)\|)}{\|g(t)\|} g(t) \in X$ and $\|f(t)\| = \Phi(\|g(t)\|)$. Thus $f \in L^1(I, X)$. Also

$$J(f)(t) = \frac{\Phi^{-1}(\|f(t)\|)}{\|f(t)\|} \cdot \frac{\Phi(\|g(t)\|)}{\|g(t)\|} g(t) = g(t).$$

Which implies that J is onto.

Now, let $f \in L^1(I, X)$ with no loss of generality we may assume that $f(t) \neq 0$ for almost all t in I, for otherwise we can restrict our measure to the support of f. Since $J(f) \in L^\Phi(I, X)$, then there exists some $w \in L^\Phi(I, Y)$ such that $\|J(f) - w\|_\Phi \leq \|J(f) - z\|_\Phi$, for all $z \in L^\Phi(I, Y)$. But J is onto. Hence $w = J(g)$ for some g in $L^1(I, Y)$ and $z = J(h)$ for some h in $L^1(I, X)$. Using Theorem 2, $\|J(f)(t) - J(g)(t)\| \leq \|J(f)(t) - y\|$, for almost all t in I and for all $y \in Y$. Hence for almost all t in I, we get

$$\|J(f)(t) - J(g)(t)\| \leq \left\| J(f)(t) - \frac{\Phi^{-1}(\|f(t)\|)}{\|f(t)\|} y \right\| \tag{1}$$

for all $y \in Y$. Multiplying both sides of inequality (1) by $\frac{\|f(t)\|}{\Phi^{-1}(\|f(t)\|)}$, we get: $\left\| f(t) - \frac{\|f(t)\|}{\Phi^{-1}(\|f(t)\|)} \frac{\Phi^{-1}(\|g(t)\|)}{\|g(t)\|} g(t) \right\| \leq \|f(t) - y\|$, for almost all t in I and for all $y \in Y$.

Set $w(t) = \frac{\|f(t)\|}{\Phi^{-1}(\|f(t)\|)} \frac{\Phi^{-1}(\|g(t)\|)}{\|g(t)\|} g(t) \in Y$. Since $w(t)$ is best approximant of $f(t)$ in Y for almost all t and $0 \in Y$, then $\|w(t)\| \leq 2\|f(t)\|$, for alomst all t in I. Thus $w \in L^1(I, Y)$. Consequently, for almost all t in I, we get

$$\|f(t) - w(t)\| \leq \|f(t) - h(t)\|,$$

for all $h \in L^1(I, Y)$. Thus w is a best approximtion of f in $L^1(I, Y)$, [6]. This ends the proof.

Remark 5. The monotonicity condition on Φ was not needed in the proof of (i)→(ii).

As corollaries to Theorem 4, we get:

Theorem 6. *Let $\Phi \in \Delta_2$ and Y be a reflexive subspace of a Banach space X. Then $L^\Phi(I,Y)$ is proximinal in $L^\Phi(I,X)$.*

Proof. Using Theorem 4 and Theorme 1.2 in [1], we get the result.

Theorem 7 [2, 9]. *Let Y be a reflexive subspace of the Banach space X. Then $L^p(I,Y)$ is proximinal in $L^p(I,X), 1 < p < \infty$.*

Theorem 8. *Let $\Phi \in \Delta_2$ and Y be a closed separable proximinal subspace of a Banach space X, then $L^\Phi(I,Y)$ is proximinal in $L^\Phi(I,X)$.*

Proof. Using Theorem 4 and Theorem 3.2 in [7], we get the result.

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