TAMKANG JOURNAL OF MATHEMATICS
Volume 34, Number 1, Spring 2003

ON THE MODULAR FUNCTIONS ARISING FROM THE THETA CONSTANTS

UĞUR S. KIRMACI

Abstract. Some modular functions arising from the theta constants $\vartheta_2(\tau)$, $\vartheta_3(\tau)$, $\vartheta_4(\tau)$ are investigated. Let n be an odd square-free positive integer as in [4,7]. It is obtained a necessary and sufficient condition that $\varphi_{\delta,\rho,3}(\tau)=\prod_{\delta|n,\rho|n}\left(\frac{\vartheta_3(\delta\tau)}{\vartheta_3(\rho\tau)}\right)^{r_\delta}$ is invariant with respect to transformations in $\theta(n)$. Also, It is deduced that $\varphi_{\delta,\rho,i}(\tau)$ is a modular function on $P^{-2}\theta(n)P^2$, $\theta(n)$, $P^{-1}\theta(n)P$, for $i=2,3,4$, respectively. Thus, the result of L. Wilson's paper [7] is generalized. Furthermore, let m and n denote positive integers. Let r, r_1, r_2 be integers such that $r(m-1)(n+1)\equiv 0(\text{mod } 8)$, $r_1(m-1)(n-1)\equiv 0(\text{mod } 8)$, $r_2^2(n-m)(nm-1)\equiv 0(\text{mod } 8)$, it is shown that $T^r_{m,n,i}(\tau)=\left(\frac{\vartheta_i(\tau)\vartheta_i(n\tau)}{\vartheta_i(m\tau)\vartheta_i(mn\tau)}\right)^r$, $H^{r_1}_{m,n,i}(\tau)=\left(\frac{\vartheta_i(m\tau)\vartheta_i(n\tau)}{\vartheta_i(\tau)\vartheta_i(mn\tau)}\right)^{r_1}$ and $\Phi^{r_2}_{m,n,i}(\tau)=\left(\frac{\vartheta_i(m\tau)}{\vartheta_i(n\tau)}\right)^{r_2}$ are modular functions on $\theta(mn)$, when $i=3$. Similar results are deduced for $P^{-2}\theta(mn)P^2$ and $P^{-1}\theta(mn)P$, the suffixes 3 being replaced by 2 and 4, respectively. Therefore, the modular functions used in B. C. Berndt's paper [1] is rewritten for theta constants.

1. Introduction

We shall use χ to denote the upper half-plane, Z for the set of rational integers and $\Gamma(1)$ for the modular group. Let be $U=\begin{pmatrix}1&1\\0&1\end{pmatrix}$, $V=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$, $W=\begin{pmatrix}1&0\\1&1\end{pmatrix}$, $P=\begin{pmatrix}0&-1\\1&1\end{pmatrix}$.

$\Gamma_u(2)$, $\Gamma_v(2)$ and $\Gamma_w(2)$ are defined by

$$\begin{aligned}
&\Gamma_u(2)=\{S\in\Gamma(1): S\equiv I \text{ or } S\equiv U(\text{mod } 2)\},\\
&\theta=\Gamma_v(2)=\{S\in\Gamma(1): S\equiv I \text{ or } S\equiv V(\text{mod } 2)\},\\
&\Gamma_W(2)=\{S\in\Gamma(1): S\equiv I \text{ or } S\equiv W(\text{mod } 2)\}
\end{aligned}$$

where I is the unit matrix. The three subgroups $\Gamma_u(2)$, $\Gamma_v(2)$ and $\Gamma_w(2)$ are conjugate [6].

Received January 21, 2002; revised April 22, 2002.
Key words and phrases. Theta constants, modular functions.

The subgroup $\theta = \Gamma_v(2)$ of $\Gamma(1)$ is generated by U^2 and V. For an odd positive integer n, the set of elements in θ of the form $\begin{pmatrix} a & b \\ cn & d \end{pmatrix}$ is a subgroup of θ which will be denoted $\theta(n)$. The subgroup $\Gamma_0(k)$ is defined to be the set of elements in $\Gamma(1)$ of the form $\begin{pmatrix} a & b \\ ck & d \end{pmatrix}$, where k is a positive integer. $\Gamma_0(2n)$ and $\theta(n)$ are conjugate subgroups of $\Gamma(1)$. That is, $\Gamma_0(2n) = W^{-n}\theta(n)W^n$. [7]

Let F denote the Ford fundamental region of $\theta(n)$, where n denotes an odd square-free positive integer. A complete set of non-equivalent parabolic points of $\theta(n)$ in $\overline{F}$ is given by

$$0, i\infty \text{ and } P(n) = \{2/\Delta : \Delta \mid n,\ \Delta > 0\} \cup \{1/\Delta : \Delta \mid n,\ \Delta > 0\}$$

[4], [7, Theorem 1].

We recall the theta constants $\vartheta_2(\tau)$, $\vartheta_3(\tau)$ and $\vartheta_4(\tau)$ defined by

$$\vartheta_2(\tau) = \sum_{n\in Z} q^{(n+\frac{1}{2})^2},\ \vartheta(\tau) = \vartheta_3(\tau) = \sum_{n\in Z} q^{n^2},\ \vartheta_4(\tau) = \sum_{n\in Z} (-1)^n q^{n^2}$$

for $\tau \in \chi$ and $q = e^{\pi i\tau}$. The Dedekind Eta function

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi in\tau})$$

is a cusp form of weight 1/2 on $\Gamma(1)$ and satisfies

$$\eta(M\tau) = \upsilon_\eta(M)(c\tau + d)^{1/2}\eta(\tau)$$

for all $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(1)$. An important connection between $\vartheta(\tau)$ and $\eta(\tau)$ is given by.

$$\vartheta(\tau) = \eta^2\left(\frac{\tau+1}{2}\right)/\eta(\tau+1) \tag{1}$$

[3].

Let Γ be a subgroup of $\Gamma(1)$. If $f(\tau)$ is a modular form of weight k for Γ with multiplier system υ, we write $f(\tau) \in M(\Gamma, k, \upsilon)$. If $f \in M(\Gamma, k, \upsilon)$ and $L \in \Gamma(1)$, the L-transform f_L of f is defined by

$$f_L(\tau) = f(\tau)|L = \{\mu(L, \tau)\}^{-1} f(L\tau)$$

Here, $\mu(L, \tau) = (c\tau + d)^k$, for $L = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$. If $f_1 \in M(\Gamma, k_1, \upsilon_1)$ and $f_2 \in M(\Gamma, k_2, \upsilon_2)$, then $f_1.f_2 \in M(\Gamma, k_1 + k_2, \upsilon_1.\upsilon_2)$ and $f_1/f_2 \in M(\Gamma, k_1 - k_2, \upsilon_1/\upsilon_2)$.

Lemma 1. *Suppose that* $f \in M(\Gamma, k, \upsilon)$ *and* $L \in \Gamma(1)$. *Then we have* $f_L \in M(L^{-1}\Gamma L, k, \upsilon^L)$. [6]

Lemma 2. *The functions* $\vartheta_2(\tau)$, $\vartheta_3(\tau)$ *and* $\vartheta_4(\tau)$ *are entire modular forms of weight* $1/2$ *for the groups* $\Gamma_u(2)$, $\Gamma_v(2)$ *and* $\Gamma_w(2)$, *respectively. Further,*

$$\vartheta_3(\tau)|P = e^{-\frac{1}{4}\pi i}\vartheta_4(\tau)$$
$$\vartheta_3(\tau)|P^2 = e^{-\frac{1}{4}\pi i}\vartheta_2(\tau) \quad (2)$$

Also, for $n \geq 0$, ϑ_2^n, ϑ_3^n *and* ϑ_4^n *are entire modular forms of weight* $n/2$ *for the groups* $\Gamma_u(2)$, $\Gamma_v(2)$ *and* $\Gamma_w(2)$, *respectively* [6].

Lemma 3. *Let* n *be an odd square-free positive integer. For each divisor* δ *of* n, *with* $\delta > 1$, *let* r_δ *be any integer and let* $r_1 = 0$. *Then*

$$f(\tau) = \prod_{\delta|n}\{\vartheta(\delta\tau)/\vartheta(\tau)\}^{r_\delta} \quad (3)$$

is invariant with respect to the transformations in $\theta(n)$ *if and only if the following conditions hold:*
(i) $\sum_{\delta|n}(\delta-1)r_\delta \equiv 0 (\mathrm{mod}\ 8)$
(ii) $\prod_{\delta|n}\delta^{r_\delta}$ *is the square of a rational number*

The set of functions given by (3) and satisfying (i) and (ii) will be denoted $F(n)$. The functions in $F(n)$ are modular functions on $\theta(n)$, where $\vartheta(\tau) = \vartheta_3(\tau)$ [7, Theorem 3].

For relatively prime integers x and y with $x \neq 0$ and y odd, define

$$\left(\frac{x}{y}\right)^* = \left(\frac{x}{|y|}\right) \quad \text{and} \quad \left(\frac{x}{y}\right)_* = \left(\frac{x}{|y|}\right)(-1)^{\frac{1}{4}\varepsilon(x)\varepsilon(y)}$$

where $\left(\frac{x}{|y|}\right)$ is the Jacobi symbol and $\varepsilon(x) = x/|x|$. We also define $\left(\frac{0}{\pm 1}\right)^* = \left(\frac{0}{1}\right)_* = 1$ and $\left(\frac{0}{-1}\right)_* = -1$.

The multiplier system for $\vartheta_3(\tau)$ is given by

$$\upsilon(A) = \begin{cases} \left(\frac{d}{c}\right)^* \exp\left(\frac{1}{4}\pi i c\right) & \text{if } A \equiv V (\mathrm{mod}\ 2) \\ \left(\frac{c}{d}\right)_* \exp\left(\frac{1}{4}\pi i(d-1)\right) & \text{if } A \equiv I (\mathrm{mod}\ 2) \end{cases} \quad (4)$$

where $A = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \theta$. Thus, $\upsilon(A)$ is an 8^{th} root of unity and

$$\vartheta(A\tau) = \upsilon(A)(c\tau + d)^{1/2}\vartheta(\tau), \quad \tau \in \chi, \quad [7].$$

This paper is a continuation of previous work [2]. We will develop some results in [1], [4], [5] and [7]. The valences of functions in this paper are the same as those constructed in [4] and [7]. Certain theorems given in [4] and [7] carry over to the present setting with only minor alterations in their proofs. The main results of this paper are the following theorems.

2. The Modular Functions $\varphi_{\delta,\rho,i}(\tau)$

Now, we define the functions $\varphi_{\delta,\rho,i}$ as follows:

$$\varphi_{\delta,\rho,i}(\tau) = \prod_{\delta|n,\rho|n} \{\vartheta_i(\delta\tau)/\vartheta_i(\rho\tau)\}^r, \quad i = 2, 3, 4$$

Theorem 1. *Let n be an odd square-free positive integer. For each divisor δ and ρ of n, with $\delta, \rho > 1$, $\delta \neq \rho$, let $r = r_\delta$ be an integer and let $r_1 = 0$. Then $\varphi_{\delta,\rho,3}$ is invariant with respect to the transformations in $\theta(n)$ if and only if the following conditions hold:*

$$\text{(i)} \quad \sum_{\delta|n,\rho|n} (\rho - \delta) r_\delta \equiv 0 \pmod 8$$

$$\text{(ii)} \quad \prod_{\delta|n,\rho|n} \left(\frac{n^2}{\delta\rho}\right)^r \text{ is the square of a rational number.} \tag{5}$$

Further $\varphi_{\delta,\rho,3}$ is a modular function on $\theta(n)$.

Proof. It is enough to consider only those matrices $A = \begin{pmatrix} a & b \\ nc & d \end{pmatrix} \in \theta\ (n)$ such that $c \geq 0$ (since A and -A represent the same transformation) and such that $A \equiv V \pmod 2$ (since these generate $\theta\ (n)$). Let $B = \begin{pmatrix} a & \delta b \\ \delta^1 c & d \end{pmatrix}$ and $C = \begin{pmatrix} a & \rho b \\ \rho' c & d \end{pmatrix}$, where $\delta\delta' = \rho\rho' = n$ so that B, $C \equiv V \pmod 2$ and $\delta A\tau = B\delta\tau$, $\rho A\tau = C\rho\tau$ Then using (4) and (5), we have

$$\begin{aligned} \varphi_{\delta,\rho,i}(A\tau) &= \prod_{\delta|n,\rho|n} \{\vartheta(\delta\tau)/\vartheta(\rho\tau)\}^{r_\delta} \\ &= \prod_{\delta|n,\rho|n} \{\nu(B)/\nu(C)\}^{r_\delta} \varphi_{\delta,\rho,i}(\tau) \\ &= \prod_{\delta|n,\rho|n} \left(\frac{d}{\delta'\rho'}\right)^{r_\delta} .k.\ \varphi_{\delta,\rho,i}(\tau) = \left(\frac{d}{\alpha}\right) \exp\left\{\frac{1}{4}\pi i c\beta\right\} \varphi_{\delta,\rho,i}(\tau) \end{aligned}$$

where, $k = \prod_{\delta|n,\rho|n} (\exp(1/4)\pi i c(\rho' - \delta'))r_\delta$, $\alpha = \prod_{\delta|n,\rho|n} (\delta'\rho')^{r_\delta}$ and $\beta = \sum_{\delta|n,\rho|n} (\rho' - \delta') r_\delta$

It remains to show that

(a) $\beta \equiv 0 \pmod 8$ if and only if (i) holds, and

(b) $\left(\frac{d}{\alpha}\right) = 1$ for all even integer d relatively prime to n if and only if (ii) holds.

Since n, δ' and ρ' are odd, $n^2 \equiv (\delta')^2 \equiv (\rho')^2 \equiv 1 \pmod 8$ and so $n\beta \equiv \sum_{\delta|n,\rho|n} (\rho - \delta)r_\delta \pmod 8$ and (a) follows. The "if" part of (b) is trivial. Assumming $\left(\dfrac{d}{\alpha}\right) = 1$ for all even integers d relatively prime to n implies the same for all integers d relatively prime to n. But if α is not a rational square, then by quadratic reciprocity and Dirichlet's Theorem there is a prime p with $p \nmid n$ and $\left(\dfrac{p}{\alpha}\right) = -1$; so α must be a square, giving (ii).

By (1), we obtain

$$\frac{\vartheta(\delta\tau)}{\vartheta(\rho\tau)} = \frac{\eta^2\left(\frac{\delta\tau+\delta}{2}\right)/\eta(\delta\tau+\delta)}{\eta^2(\frac{\rho\tau+\rho}{2})/\eta(\rho\tau+\rho)} = \frac{\Phi^2(\tau')\Psi(2\tau')}{\Phi(2\tau')\Psi^2(\tau')} \tag{6}$$

where $\tau' = \dfrac{\tau+1}{2}$, $\Phi(\tau') = \dfrac{\eta(\delta\tau')}{\eta(\tau')}$ and $\Psi(\tau') = \dfrac{\eta(\rho\tau')}{\eta(\tau')}$.

Finally, we consider the expansions of $\varphi_{\delta,\rho,3}(\tau)$ at the parabolic cusps ∞, 0, $2/\Delta$ and $1/\Delta$. We have

$$\Phi(\tau) = \exp\left\{\frac{\pi i(\delta-1)r}{12}\tau\right\}\left(1+\sum_{k=1}^{\infty} a_k e^{2\pi i k\tau}\right) \text{ and } \Psi(\tau) = \exp\left\{\frac{\pi i(\rho-1)r}{12}\tau\right\}\left(1+\sum_{k=1}^{\infty} b_k e^{2\pi i k\tau}\right)$$

as the Fourier expansions of Φ and ψ at ∞. [3, p.103]. Hence, by (6), $\varphi_{\delta,\rho,3}(\tau)$ has the Fourier expansion at ∞ of the form

$$\varphi_{\delta,\rho,3}(\tau) = 1 + \sum_{k=1}^{\infty} a'_k e^{2\pi i k\tau} \tag{7}$$

We have

$$\Phi(\tau) = \delta^{-r/2}\exp\left\{\frac{\pi i(\delta-1)r}{12\delta\tau}\right\}\left(1+\sum_{k=1}^{\infty} c_k e^{-2\pi i k/\delta\tau}\right)$$
$$\Psi(\tau) = \rho^{-r/2}\exp\left\{\frac{\pi i(\rho-1)r}{12\rho\tau}\right\}\left(1+\sum_{k=1}^{\infty} d_k e^{-2\pi i k/\rho\tau}\right)$$

as the Fourier expansion at 0 [3, p.103]. Hence, by (6), $\varphi_{\delta,\rho,3}(\tau)$ has the Fourier expansion at 0 of the form

$$\varphi_{\delta,\rho,3}(\tau) = \left(\frac{\rho}{\delta}\right)^{r/2}\exp\left(\frac{\pi i r}{8\tau}\left(\frac{\delta-\rho}{\delta\rho}\right)\right)\left(1+\sum_{k=1}^{\infty}\alpha_k e^{-\frac{2\pi i k}{\tau}\gamma(\delta,\rho)}\right) \tag{8}$$

where, $\gamma(\delta,\rho)$ is a rational function of δ and ρ. It is shown in [7] that $f(S\tau)$ has the form

$$f(S\tau) = \sum_{k=0}^{\infty} a_k \ \exp\{2\pi i k\tau/(2n/\Delta)\}$$

at the parabolic points $2/\Delta$. Hence, we have

$$\varphi_{\delta,\rho,3}(S\tau) = \sum_{k=0}^{\infty} a'_k \exp\{2\pi ik\tau/(2n/\Delta)\}$$

where the a'_k are complex numbers independent of τ with $a_0 \neq 0$ and $S = \begin{pmatrix} 2 & b \\ \Delta & d \end{pmatrix}$. Replacing τ by $S^{-1}\tau = \dfrac{d\tau - b}{-\Delta\tau + 2}$, we obtain

$$\varphi_{\delta,\rho,3}(\tau) = \sum_{k=0}^{\infty} a'_k \exp\{2\pi ikS^{-1}\tau/(2n/\Delta)\} \tag{9}$$

as the Fourier expansion at the parabolic points $2/\Delta$. From the function $\dfrac{\vartheta(\delta\tau)/\vartheta(\tau)}{\vartheta(\rho\tau)/\vartheta(\tau)}$ and the equation

$$\vartheta(\delta S\tau)/\vartheta(S\tau) = c' \prod_{m=1}^{\infty} (1-c^{2m}q^{2m\Delta k/n})(1+c^{2m-1}q^{(2m-1)\Delta k/n})^2(1-q^{2m})^{-1}(1+q^{2m-1})^{-2} \tag{10}$$

in [7, Theorem 4], it follows that the valence ν is 0. Where $k = ng/\Delta\delta_0$, $q = \exp(i\pi\tau)$ and c, c' are non-zero constants.

It is shown in [7] that $f(N\tau)$ has the form

$$f(N\tau) = \sum_{k=0}^{\infty} b_k \exp\{2\pi i(k+\nu)\tau/(n/\Delta)\}$$

at the parabolic points $1/\Delta$. Hence, we have

$$\varphi_{\delta,\rho,3}(N\tau) = \sum_{k=0}^{\infty} b'_k \exp\{2\pi i(k+\nu)\tau/(n/\Delta)\}$$

where the b'_k are complex numbers with $b_0 \neq 0$ and $N = \begin{pmatrix} 1 & 0 \\ \Delta & 1 \end{pmatrix}$. Hence, replacing τ by $N^{-1}\tau = \dfrac{\tau}{-\Delta\tau + 1}$, we obtain

$$\varphi_{\delta,\rho,3}(\tau) = \sum_{k=0}^{\infty} b'_k \exp\{2\pi i(k+\nu)N^{-1}\tau/(n/\Delta)\}, \tag{11}$$

as the Fourier expansion at the parabolic points $1/\Delta$. From the function $\dfrac{\vartheta(\delta\tau)/\vartheta(\tau)}{\vartheta(\rho\tau)/\vartheta(\tau)}$ and the equation

$$\vartheta(\delta N\tau)/\vartheta(N\tau) = c'' z^{(k-n/\Delta)/8} \prod_{m=1}^{\infty} (1-c^{2m}z^{2mk})(1+c^m z^{mk})(1-z^{2mn/\Delta})^{-1}(1+z^{mn/\Delta})^{-1} \tag{12}$$

in [7, Theorem 5], it is clear that the valence ν is

$$\frac{1}{8}\sum_{\delta|n,\rho|n}\{(ng/\Delta\delta_0)-(ng'/\Delta\rho_0)\}r_\delta$$

where $g=(\delta,\Delta)$, $\delta_0 g=\delta$, $\rho_0 g'=\rho$, $g'=(\rho,\Delta)$, $z=\exp(2\pi i\Delta\tau/n)$, $k=ng/\Delta\delta_0$, $c=\exp(-2\pi i\beta/\delta_0)$, $c''=c'\exp(-\pi i\beta/4\delta_0)$.

Theorem 2. *Let n, δ, ρ, r_δ be as in Theorem* 1. *If the conditions* (5) *hold, then $\varphi_{\delta,\rho,i}(\tau)$ are modular functions on $P^{-2}\theta(n)P^2$ and $P^{-1}\theta(n)P$, for $i=2,4$, respectively.*

Proof. By Lemma 2, using the equality $\vartheta_3(\tau)\mid P=e^{-\frac{1}{4}\pi i}\vartheta_4(\tau)$, we have

$$\varphi_{\delta,\rho,3}(\tau)\mid P=\varphi_{\delta,\rho,4}(\tau)$$

By Lemma 1, $\varphi_{\delta,\rho,4}(\tau)$ is a modular function on $P^{-1}\theta(n)P$. By Lemma 2, using the equality $\vartheta_3(\tau)\mid P^2=e^{-\frac{1}{4}\pi i}\vartheta_2(\tau)$, we obtain

$$\varphi_{\delta,\rho,3}(\tau)\mid P^2=\varphi_{\delta,\rho,2}(\tau)$$

By Lemma 1, $\varphi_{\delta,\rho,2}(\tau)$ is a modular function on $P^{-2}\theta(n)P^2$. This concludes the proof.

3. The Modular Functions $T_{m,n,i}(\tau)$, $H_{m,n,i}(\tau)$, $\Phi_{m,n,i}(\tau)$

Now we consider the following functions:

$$T_{m,n,i}(\tau)=\frac{\vartheta_i(\tau)\vartheta_i(n\tau)}{\vartheta_i(m\tau)\vartheta_i(mn\tau)},\quad H_{m,n,i}(\tau)=\frac{\vartheta_i(m\tau)\vartheta_i(n\tau)}{\vartheta_i(\tau)\vartheta_i(mn\tau)},\ \text{and}\ \Phi_{m,n,i}(\tau)=\frac{\vartheta_i(m\tau)}{\vartheta_i(n\tau)},\ \text{for}\ i=2,3,4$$

The following theorems show that, under appropriate conditions $T^r_{m,n,i}(\tau)$, $H^r_{m,n,i}(\tau)$ and $\Phi^r_{m,n,i}(\tau)$ are modular functions holomorphic on χ, with respect to the transformations of appropriate subgroups of finite index of the modular group, i.e., modular forms of weight 0.

Theorem 3. *Let m and n denote positive integers and suppose that r is an integer such that $r(m-1)(n+1)\equiv 0 \pmod 8$. Then $T^r_{m,n,3}(\tau)\in M(\theta(mn),0,1)$. Moreover, $T^r_{m,n,3}(\tau)$ is analytic on χ.*

Proof. The last assertion in Theorem 3 is obvious from the definition of $T_{m,n,3}(\tau)$. Let $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\theta(mn)$. Then, for $\tau\in\chi$,

$$\vartheta(A\tau)=\nu(A)(c\tau+d)^{1/2}\vartheta(\tau)$$

and for $s\mid c$,

$$\vartheta(sA\tau)=\vartheta\left(\frac{a(s\tau)+sb}{\frac{c}{s}(s\tau)+d}\right)=\nu\begin{pmatrix}a&sb\\c/s&d\end{pmatrix}(c\tau+d)^{1/2}\vartheta(s\tau)$$

where $\nu(A)$ is given by (4). Thus,

$$T_{m,n}(A\tau) = \nu_1(A)T_{m,n}(\tau)$$

where $\nu_1(A) = \frac{\nu\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\nu\left(\begin{smallmatrix} a & nb \\ c/n & d \end{smallmatrix}\right)}{\nu\left(\begin{smallmatrix} a & mb \\ c/m & d \end{smallmatrix}\right)\nu\left(\begin{smallmatrix} a & mnb \\ c/mn & d \end{smallmatrix}\right)}$. Suppose first that $A \equiv V \pmod 2$. Then from (4),

$$\nu_1(A) = \zeta_8^{-(c+\frac{c}{n}-\frac{c}{m}-\frac{c}{mn})} = \zeta_8^{-c(m-1)(n+1)/mn}$$

Since $r(m-1)(n+1) \equiv 0 \pmod 8$, $\nu_l^r(A) \equiv 1$. Where $\zeta_8 = \exp(2\pi i/8)$.

Secondly, suppose that $A \equiv I \pmod 2$. Then $\nu_l^r(A) \equiv 1$. Thus, in both instances,

$$T^r_{m,n,3}(A\tau) = T^r_{m,n,3}(\tau)$$

For $\delta = 1$, $\rho = m$ and $\delta = n$, $\rho = mn$ in the equation (8), we have the Fourier expansion of $T^r_{m,n,3}(\tau)$ at 0 of the form

$$T^r_{m,n,3}(\tau) = m^r.\exp\frac{\pi ir}{8\tau}\left(\frac{1-m}{m}+\frac{n-mn}{mn^2}\right)\left(1+\sum_{k=1}^{\infty}\alpha_k e^{-\frac{2\pi ik}{\tau}\gamma(m,n)}\right)$$

where, $\gamma(m,n)$ is a rational function of m and n. Similarly, from the equations (7), (9), (11), we obtain the Fourier expansions of $T^r_{m,n,3}(\tau)$ at the indicated other cusp points of the forms

$$T^r_{m,n,3}(\tau) = 1+\sum_{k=1}^{\infty} a'_k e^{2\pi ik\tau}, \quad \text{at } \infty$$

$$T^r_{m,n,3}(\tau) = \sum_{k=0}^{\infty} a'_k \exp\{2\pi ikS^{-1}\tau/(2mn/\Delta)\}, \quad \text{at } 2/\Delta$$

$$T^r_{m,n,3}(\tau) = \sum_{k=0}^{\infty} b'_k \exp\{2\pi i(k+\nu)N^{-1}\tau/(mn/\Delta)\}, \quad \text{at } 1/\Delta,$$

From the function $\frac{\vartheta(n\tau)/\vartheta(\tau)}{(\vartheta(m\tau)/\vartheta(\tau))(\vartheta(mn\tau)/\vartheta(\tau))}$ and the equation (10), we find that $T^r_{m,n,3}(\tau)$ has valence 0 at the parabolic points $2/\Delta$. From the same function and the equation (12), $T^r_{m,n,3}(\tau)$ has valence

$$\frac{1}{8}\{(mn/\Delta)+(mng'/\Delta n_0)-(mng''/\Delta m_0)-(mng'''/\Delta m_0n_0)\}r \tag{13}$$

at the parabolic points $1/\Delta$ $(\Delta \mid mn,\ \Delta > 0)$, where $g' = (n,\Delta)$, $g'' = (m,\Delta)$, $g''' = (mn,\Delta)$, $n_0g' = n$, $m_0g'' = m$, $m_0n_0g''' = mn$.

Theorem 4. *Let m,n,r be as in Theorem 3. Then, $T^r_{m,n,4}(\tau) \in M(P^{-1}\theta(mn)P,0,1)$ and $T^r_{m,n,2}(\tau) \in M(P^{-2}\theta(mn)P^2,0,1)$*

Proof. Using the equations (2), we have $T^r_{m,n,3}(\tau) \mid P = T^r_{m,n,4}(\tau)$ and $T^r_{m,n,3}(\tau) \mid P^2 = T^r_{m,n,2}(\tau)$. Hence, we have the conclusion.

Theorem 5. *Let m and n denote positive integers and suppose that r is an integer such that $r(m-1)(n-1) \equiv 0 \pmod 8$. Then, $H^r_{m,n,3}(\tau) \in M(\theta(mn), 0, 1)$. Moreover, $H^r_{m,n,3}(\tau)$ is analytic on χ.*

Proof. The proof is analogous to that of Theorme 3.

Theorem 6. *Let m, n, r be as in Theorem 5. Then, $H^r_{m,n,4}(\tau) \in M(P^{-1}\theta(mn)P, 0, 1)$ and $H^r_{m,n,2}(\tau) \in M(P^{-2}\theta(mn)P^2, 0, 1)$.*

Proof. As the proof of Theorem 4, from the equations $H^r_{m,n,3}(\tau) \mid P = H^r_{m,n,4}(\tau)$ and $H^r_{m,n,3}(\tau) \mid P^2 = H^r_{m,n,2}(\tau)$, the assertion follows.

Theorem 7. *Let m and n denote positive integers and suppose that r is an integer such that $r^2(n-m)(nm-1) \equiv 0 \pmod 8$. Then, $\Phi^r_{m,n,3}(\tau)$ is a modular function on $\theta(mn)$, with multiplier system $\left(\dfrac{d}{mn}\right)^r$.*

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \theta(mn)$. Then for $\tau \in \chi$,

$$\Phi_{m,n,3}(A\tau) = \frac{\vartheta(mA\tau)}{\vartheta(nA\tau)} = \frac{\vartheta(A_1 m\tau)}{\vartheta(A_2 n\tau)} = \frac{\nu(A_1)}{\nu(A_2)} \frac{\vartheta(m\tau)}{\vartheta(n\tau)}$$

where $A_1 = \begin{pmatrix} a & mb \\ c/m & d \end{pmatrix}$, $A_2 = \begin{pmatrix} a & nb \\ c/n & d \end{pmatrix}$. If A_1, $A_2 \equiv V \pmod 2$, by (4), $\nu(A) = \left(\dfrac{d}{mn}\right) e^{-\frac{\pi i c}{4}(\frac{n-m}{nm})}$. Since $r(n-m) \equiv 0 \pmod 8$, we have $\nu^r(A) = \left(\dfrac{d}{mn}\right)^r$. If A_1, $A_2 \equiv I \pmod 2$, by (4) and quadratic reciprocity law, we have $\nu(A) = \left(\dfrac{nm}{d}\right) = \left(\dfrac{d}{nm}\right)(-1)^{(\frac{nm-1}{2})(\frac{d-1}{2})}$. Since $r(nm-1) \equiv 0 \pmod 8$, $\nu^r(A) = \left(\dfrac{d}{nm}\right)^r$. Thus, in both instances, we have $\Phi^r_{m,n,3}(A\tau) = \nu^r(A)\Phi^r_{m,n,3}(\tau)$. Now we consider the Fourier epansions of $\Phi^r_{m,n,3}(\tau)$ at the cusps of $\theta(mn)$. For $\delta = m$, $\rho = n$ in the equations (7), (8), (9), (11), we obtain the Fourier expansions of $\Phi^r_{m,n,3}(\tau)$ at the parabolic points ∞, 0, $2/\Delta$, $1/\Delta$, respectively, ($\Delta \mid mn$, $\Delta > 0$). From the function $\dfrac{\vartheta(m\tau)/\vartheta(\tau)}{\vartheta(n\tau)/\vartheta(\tau)}$ and the equation (10), we note that $\Phi^r_{m,n,3}(\tau)$ has valence 0 at the parabolic points $2/\Delta$. From the same function and the equation (12), we find that $\Phi^r_{m,n,3}(\tau)$ has valence

$$\frac{1}{8}\{(mng/\Delta m_0) - (mng'/\Delta n_0)\}r$$

at the parabolic point $1/\Delta$, where $g = (m, \Delta)$, $m_0 g = m$, $g' = (n, \Delta)$, $n_0 g' = n$. Thus $\Phi^r_{m,n,3}(\tau) \in M\left(\theta(mn), 0, \left(\dfrac{d}{mn}\right)^r\right)$.

Theorem 8. *Let* m, n, r *be as in Theorem* 7. *Then* $\Phi^r_{m,n,4}(\tau) \in M(P^{-1}\theta(mn)P, 0, 1)$ *and* $\Phi^r_{m,n,2}(\tau) \in M(P^{-2}\theta(mn)P^2, 0, 1)$.

Proof. By the equations (2), since $\Phi^r_{m,n,3}(\tau) \mid P = \Phi^r_{m,n,4}(\tau)$ and $\Phi^r_{m,n,3}(\tau) \mid P^2 = \Phi^r_{m,n,2}(\tau)$ the assertion follows.

4. Conclusion

It is a simple matter, using the work already done in this paper, to formulate and prove analogous results for functions in $\Gamma_0(2mn)$. The key is the equation $\Gamma_0(2mn) = W^{-mn}\theta(mn)W^{mn}$, where m and n are odd positive integers. The functions $T_{m,n,3}(W^{mn}\tau)$, $H_{m,n,3}(W^{mn}\tau)$ and $\Phi_{m,n,3}(W^{mn}\tau)$ are modular functions on $\Gamma_0(2mn)$. The functions $\varphi_{\delta,\rho,3}(W^n\tau)$ are modular functions on $\Gamma_0(2n)$, for an odd square-free positive integer n. For example, the functions $T_{m,n,3}(W^{mn}\tau)$ are modular functions on $\Gamma_0(2mn)$ with valence 0 at the parabolic points $W^{-mn}S(i\infty)$ and valence (13) at the parabolic points $W^{-mn+\Delta}(i\infty)$, $(\Delta \mid mn, \quad \Delta > 0)$, where $S = \begin{pmatrix} 2 & b \\ \Delta & d \end{pmatrix}$ and $W = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. In this setting, the natural parabolic point in which to expand the function is $i\infty$.

References

[1] Berndt, B. C., Cheng Zhang-Liang, *Ramanujan's identities for eta functions.* Math. Ann. **292**(1992), 561-573.
[2] Kirmaci, U. S., Özdemir, M. E., *On the modular functions,* International Journal of Applied Mathematics, **2**(2000), 1385-1397.
[3] Knopp, M. I., Modular Functions in Analytic Number Theory, Markham Publishing Co, Chicago, 1970.
[4] Newman, M., *Construction and application of a class of modular functions ,* Proc. London Math. Soc. **7**(1957), 334-350.
[5] Newman, M., *Construction and application of a class of modular functions II,* Proc. London Math. Soc. **9**(1959), 373-387.
[6] Rankin, R., Modular Forms and Functions, Cambridge University Press, Cambridge, 1977.
[7] Wilson, G., *A family of modular functions arising from the theta function,* Proc. London Math. Soc. **55**(1987), 433-449.

Atatürk University, K. K. Education Faculty, Department of Mathematics, 25240, Erzurum-Turkey.