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A STUDY ON BORNOLOGICAL PROPERTIES OF THE SPACE OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Abstract. Spaces of entire functions of several complex variables occupy an important position in view of their vast applications in various branches of mathematics, for instance, the classical analysis, theory of approximation, theory of topological bases etc. With an idea of correlating entire functions with certain aspects in the theory of basis in locally convex spaces, we have investigated in this paper the bornological aspects of the space X of integral functions of several complex variables. By Y we denote the space of all power series with positive radius of convergence at the origin. We introduce bornologies on X and Y and prove that Y is a convex bornological vector space which is the completion of the convex bornological vector space X.

1. Introduction

The space of integral functions over the complex field C was introduced by Iyer [5] who defined a metric on this space by introducing a real-valued map on it. Kamthan [7] studied the properties of space of entire functions of several complex variables. In this paper we study the bornological aspect of space of entire functions of several complex variables.

A *bornology* on a set X is a family $\mathcal{B}$ of subset of X satisfying the following axioms:

(i) $\mathcal{B}$ is a covering of X, i.e. $X = \bigcup_{B\in\mathcal{B}} B$;

(ii) $\mathcal{B}$ is hereditary under inclusion, i.e. if $A \in \mathcal{B}$ and B is a subset of X contained in A, then $B \in \mathcal{B}$;

(iii) $\mathcal{B}$ is stable under finite union.

A pair $(X, \mathcal{B})$ consisting of a set X and a bornology $\mathcal{B}$ on X is called a *bornological space,* and the elements of $\mathcal{B}$ are called the *bounded subsets* of X.

A *base of a bornology* $\mathcal{B}$ *on* X *is any subfamily* $\mathcal{B}_\circ$ of $\mathcal{B}$ such that every element of $\mathcal{B}$ is contained in an element of $\mathcal{B}_\circ$. A family $\mathcal{B}_\circ$ of subsets of X is a base for a bornology on X if and only if $\mathcal{B}_\circ$ covers X and every finite union of elements of $\mathcal{B}_\circ$ is contained in a member of $\mathcal{B}_\circ$. Then the collection of those subsets of X which are contained in an element of $\mathcal{B}_\circ$ defines a bornology $\mathcal{B}$ on X having $\mathcal{B}_\circ$ as a base. A bornology is said to be a *bornology with a countable base* if it possesses a base consisting of a sequence

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of bounded sets. Such a sequence can always be assumed to be increasing. For further definitions and notations, we shall refer to [2] and [3].

2. The Space X

Let C denote the complex plane, and I be the set of non-negative integers. We write for $n \in I$, $n \geq 1$,

$$C^n = \{(z_1, z_2, \ldots, z_n);\ z_i \in C, \quad 1 \leq i \leq n\},$$
$$I^n = \{(p_1, p_2, \ldots, p_n);\ p_i \in I, \quad 1 \leq i \leq n\}.$$

C^n and I^n are respectively Banach and metric spaces under the functions:

$$\|(z_1, \ldots, z_n)\| = |z_1| + \cdots + |z_n|; \quad \|(p_1, \ldots, p_n)\| = p_1 + \cdots + p_n.$$

We are concerned here with the space of entire functions from C^n to C under the usual pointwise addition and scalar multiplicatioin. For the sake of simplicity we consider the case when $n = 2$, though our results can be easily extended to any positive integer n.

Let therefore, X be the space of all entire functions $f : C^2 \to C$, where

$$f(z_1, z_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{m,n} {z_1}^m {z_2}^n,\ a_{m,n} \in C \quad \text{for } m, n \geq 0, \tag{2.1}$$

and $\lim\limits_{\|m,n\| \to \infty} |a_{mn}|^{1/(m+n)} = 0$, where corresponding to each $f \in X$, $a_{m,n}$'s are uniquely determined coefficients in C. For each $f \in X$ there is an associated real number $\|f\|$ defined by

$$\|f\| = \sup\left\{|a_{0,0}|, |a_{m,n}|^{1/(m+n)}, \quad m, n \geq 0,\ m + n \neq 0\right\},$$

satisfying for all $f, g \in X$
(a) $\|0\| = 0$;
(b) $\|f\| \geq 0$;
(c) $\|f\| = 0 \Leftrightarrow f \equiv 0$;
(d) $\|-f\| = \|f\|$;
(e) $\|f + g\| \leq \|f\| + \|g\|$.
Then $\|\cdot\|$ is a total paranorm on X. We define a bornology on X with the help of this paranorm. We denote by B_J the set $\{f \in X : \|f\| \leq J\}$. Then the family $\mathcal{B}_\circ = \{B_J : J = 1, 2, \ldots\}$ forms a base for a bornology $\mathcal{B}$ on X.

It is known that $(X, \mathcal{B})$ is a separable convex bornological vector space with a countable base.

Definition 1. A set P is said to be *bornivorous* if for every bounded set S there exists a $\lambda \in C$, $\lambda \neq 0$ such that $\mu S \subset P$ for all $\mu \in C$ for which $|\mu| \leq |\lambda|$.
We now prove

Theorem 2.1. *$\mathcal{B}$ contains no bornivorous set.*

Proof. Suppose $\mathcal{B}$ contains a borni vorous set A. Then there exists a set $S_i \in \mathcal{B}$ such that $A \subset S_i$ and consequently S_i is also bornivorous. We now assert that if $i_1 > i$, then $\lambda S_{i_1} \not\subset S_i$ for any $\lambda \in C$ which leads to a contradiction. If $i_1 > i$, it is easy to see that $\lambda S_{i_1} \not\subset S_i$ for any $\lambda \in C$ such that $|\lambda| \geq 1$ Now we prove that $\lambda S_{i_1} \not\subset S_i$ for any $\lambda \in C$ such that $|\lambda| < 1$ also. Let thus $|\lambda| < 1$. Since $i_1/i > 1$, we can choose m, n such that $1 < 1/|\lambda| < (i_1/i)^{m+n}$. Now let $a_{m,n} \in C$ be such that $\dfrac{i^{m+n}}{|\lambda|} < |a_{m,n}| \leq i_1^{m+n}$ and let $f = a_{m,n} z_1^m z_2^n$. Then $\|f\| = |a_{m,n}|^{1/(m+n)} \leq i_1$, and hence $f \in S_{i_1}$. Now $\|\lambda f\| = \|\lambda a_{m,n} z_1^m z_2^n\| = |\lambda a_{m,n}|^{1/(m+n)} > i$ and hence $\lambda f \notin S_{i_1}$. Thus $\lambda S_{i_1} \not\subset S_i$ for any $\lambda \in C$. This proves Theorem 2.1.

Definition 2. Let E be a bornological vector space. A sequence $\{x_n\}$ in E is said to be *Mackey-convergent* to a point $x \in E$, if there exists a decreasing sequence of positive real numbers $\{\lambda_n\}$ tending to zero such that the sequence $\left\{ \dfrac{x_n - x}{\lambda_n} \right\}$ is bounded.

Definition 3. Let E be a separated convex bornological space. A sequence $\{x_n\}$ in E is said to be a *bornologcal Cauchy sequence* (*or a Mackey-Cauchy sequence*) in E *if* there exists a bounded disk $B \subset E$ such that $\{x_n\}$ is a Cauchy sequence in E_B.

Theorem 2.2. *The Mackey-convergence in a bornological vector space E is topologisable if and only if E has a bounded bornivorous set.*

For proof of Theorem 2.2, see Hogbe-Nlend [2, Proposition 1, p.12].

Corollary 2.1. *The Mackey-convergence of X is not topologisable.*

Proof. Suppose the Mackey-convergence of X is topologisable. Then by Theorem 2.2, X has a bounded bornivorous set, and this contradicts Theorem 2.1.

3. The Bornological Dual of X

The following result is due to Kamthan [7, Lemma 2.1]:

Lemma 3.1. *A necessary and sufficient condition that $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{m,n} a_{m,n}$ should be convergent for every sequence $\{a_{m,n}\}$ satisfying*

$$|a_{m,n}|^{1/(m+n)} \to 0 \text{ as } m, n \to \infty, \text{ is that}$$
$$\left\{ |c_{0,0}|; |c_{m,n}|^{1/(m+n)}, \quad m, n \geq 0, \quad m+n \neq 0 \right\} \text{ should be bounded.}$$

Kamthan [7] has also proved that every continuous linear functional ϕ on X is of the *form*

$$\phi(f) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{m,n}c_{m,m}, \quad \text{where} \quad f(z_1,z_2) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{m,n}z_1^m z_2^n, \ \text{and the} \tag{3.1}$$

sequence $\left\{|c_{0,0}|; |c_{m,n}|^{1/(m+n)},\ m,n \geq 0,\ m+n \neq 0\right\}$ *is bounded. Moreover, if any double sequence* $\{c_{m,n} : m,n \geq 0\}$ *satisfies Lemma* 3.1*, then the mapping*

$\phi : X \to C$ *whose value at any* $f \in X$ *is given by (3.1) represents a continuous linear functional on* X*.*

Lemma 3.2. *A linear functional* $\phi : X \to C$ *is bounded if and only if* ϕ *maps every Mackey-convergent sequence to a bounded sequence in C.*
For the proof of above, we refer to [2, *propositions* 1, p. 10].
We now prove

Theorem 3.1. *A linear functional* $\phi = \sum\limits_{n=0}^{\infty}\sum\limits_{m=0}^{\infty} c_{m,n}a_{m,n}$ *on* X *is bounded if and only if* $\lim\limits_{\|m,n\|\to\infty} |c_{m,n}|^{1/(m+n)} = 0$.

Proof. First suppose that $|c_{m,n}|^{1/(m+n)} \to 0$ as $m,n \to \infty$. Let $\{f_q\}$ be a sequence in X such that $f_q \xrightarrow{M} 0$. Then there exists a constant k and a decreasing sequence $\{\lambda_q\}$ of scalars converging to zero such that $\|f_q/\lambda_q\| \leq k$ i.e. $\left|a_{0,0}^{(q)}\right| \leq |\lambda_q|\, k$ and $|a_{m,n}^{(q)}| \leq |\lambda_q| k^{m+n}$, $m,n \geq 1$.
Since $|c_{m,n}|^{1/(m+n)} \to 0$, there exists N such that $|c_{m,n}|^{1/(m+n)} \leq \dfrac{1}{2k}$ for $\|(m,n)\| \geq N$.
Hence $|c_{m,n}| \leq (2k)^{-m-n}$, $\|(m,n)\| \geq N$.
Now

$$\begin{aligned}
|\phi(f_q)| &= \left|\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{m,n}a_{m,n}^{(q)}\right| \\
&\leq \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |c_{m,n}|\left|a_{m,n}^{(q)}\right| \\
&\leq \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |c_{m,n}|\,|\lambda_q|\,k^{m+n} \\
&\leq \sum_{\|(m,n)\|\leq N} |c_{m,n}|\,|\lambda_q|\,k^{m+n} + \sum_{\|(m,n)\|>N}^{\infty} |\lambda_q|\,2^{-m-n} < \infty.
\end{aligned}$$

Hence ϕ is bounded on every sequence which Mackey-converges to zero and consequently by Lemma 3.2 ϕ is bounded.

To prove the converse part of the theorem, let ϕ be as mentioned in the hypothesis such that

$$\limsup_{\|(m,n)\|\to\infty} |c_{m,n}|^{1/(m+n)} = \rho > 0.$$

Then given $\eta > 0$ such that $\eta < \rho$, there exist divergent increasing sequences $\{m_q\}$, $\{n_t\}$ of integers such that

$$|c_{m,n}| \geq \eta^{m+n} \quad \text{for all} \quad m = m_q,\ n = n_1.$$

Choose $\pi \in R$ such that $\pi > 1$ and $\pi\eta > 1$. Consider the sequence $\{f_{m,n}\}$ where $f_{m,n} = \pi^{m+n} z_1^m z_2^n \in X$ and define $\lambda_{mn} \in C$ as $\lambda_{m,n} = 1/\pi^{m+n}$. Then $\lim_{\|(m,n)\| \to \infty} \lambda_{m,n} = 0$ and $\|f_{m,n}/\lambda_{m,n}\| = \|\pi^{2(m+n)} z_1^m z_2^n\| = |\pi|^2 < \infty$. Consequently $f_{m,n} \xrightarrow{M} 0$. But $\phi(f_{m,n}) = c_{mn}\pi^{m+n}$ and $|\phi(f_{m_q,n_t})| = |c_{m_q,n_t}|\pi^{m_q+n_t} > \eta^{m_q+n_t}|\pi|^{m_q+n_t}$, which is not bounded. Hence again by Lemma 3.2, ϕ is not bounded. Thus boundedness of ϕ implies that

$$\lim_{\|(m,n)\| \to \infty} |c_{m,n}|^{1/(m+n)} = 0.$$

This complete the proof of the theorem.

4. r_1, r_2-Norms On X

Let $f : C^2 \to C$, where

$$f(z_1, z_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{m,n} z_1^m z_2^n, \quad a_{m,n} \in C, \quad (z_1, z_2) \in C \times C.$$

For every $r_1, r_2 \in R^+$ an r_1, r_2 - norm $\|\cdot : r_1, r_2\|$ can be defined on X as:

$$\|\cdot : r_1, r_2\| : X \to R \text{ such that}$$
$$\|f : r_1, r_2\| = \sum_{m}^{\infty} \sum_{n}^{\infty} |a_{m,n}| r_1^m r_2^n.$$

Clearly $\|f : r_1, r_2\|$ is a norm on X. We denote the space X endowed by this norm by $X(r_1, r_2)$. We denote by $\mathcal{B}_{r_1,r_2}$ the bornology on X consisting of the sets bounded in the sense of the norm $\|\cdot : r_1, r_2\|$. We now prove.

Theorem 4.1. $\mathcal{B} = \bigcup_{r_1, r_2 > 0} \mathcal{B}_{r_1,r_2}$.

Proof. Let $B \in \mathcal{B}$. Then there exists a constant k such that $\|f\| \leq k$ for $f \in B$. Let now

$$f = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{m,n} z_1^m z_2^n \in \text{B}.$$

Then

$$|a_{0,0}| \leq k,\ |a_{m,n}||z_1|^m |z_2|^n \leq k^{m+n} |z_1|^m |z_2|^n,\ m, n \geq 1.$$

Thus

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|a_{m,n}||z_1|^m|z_2|^n \le 1+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|z_1|^m|z_2|^n < \infty \quad \text{if} \quad |z_1|,|z_2|<1/k.$$

Hence if $0<r_1,\, r_2\le 1/k$, then $B\in\mathcal{B}_{r_1,r_2}$ and so $\mathcal{B}\subset\bigcup_{r_1,r_2>0}\mathcal{B}_{r_1,r_2}$.
For the reverse inclusion let B$\in\mathcal{B}_{r_1,r_2}$. Then there exists a constant k such that

$$\|f:r_1,r_2\|\le k \text{ for all } f=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}a_{m,n}z_1^m z_2^n\in \mathrm{B},$$

i.e. $$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|a_{mn}|r_1^m r_2^n\le k,$$

i.e. $$|a_{0,0}|\le k, |a_{m,n}|^{1/(m+n)}\le\frac{k^{1/(m+n)}}{r_1r_2},\ m,n\ge 1,$$

i.e. $$\|f\|\le\sup\left\{k,\frac{k^{1/(m+n)}}{r_1r_2},\ m,n\ge 1\right\}<\infty.$$

Thus B$\in\mathcal{B}$ and hence $\bigcup_{r_1,r_2>0}\mathcal{B}_{r_1,r_2}\subset\mathcal{B}$.
This proves Theorem 4.1.

Corollary 4.1. *$(X,\mathcal{B})$ is the bornological inductive limit of normed space $\{X(r_1,r_2)\}_{r_1,r_2\in R^+}$ where the inductive limit is defined in the usual way.*
Now we prove

Lemma 4.1. *The following are equivalent for $(X,\mathcal{B})$*

(a) $f_k\xrightarrow{M}0$,
(b) *there exists a sequence $\{\lambda_k\}$ of positive real number tending to zero such that $\{f_k/\lambda_k\}$ is bounded.*

Proof. (a)$\Rightarrow$(b) is obviously true.
To prove (b)$\Rightarrow$(a), let $\{f_k\}$ be a sequence in X for which there exists a sequence $\{\lambda_k\}$ of positive real number tending to zero and a constant w such that $\|f_k/\lambda_k\|\le w$ for all k. Now there exists a positive number M such that $\lambda_k\le M$ for all k. Further, we can choose for each $i=1,2,\ldots$ a k_i such that $\lambda_k<1/i$ for all $k\ge k_i$.
Let us define a sequence $\{\lambda'_k\}$ as

$$\begin{aligned}\lambda'_k &= M \quad \text{for all } k<k_i\\ &= 1/i \quad \text{for all } k_i\le k<k_{i+1} \text{ and } i=1,2,\ldots.\end{aligned}$$

Then $\{\lambda'_k\}$ is a decreasing sequences of positive real numbers tending to zero and further $\lambda'_k\ge\lambda_k$ for all k.

Hence

$$\begin{aligned}\|f_k/\lambda_k'\| &= \|f_k\lambda_k/\lambda_k\lambda_k'\|\\ &\le A(\lambda_k/\lambda_k')\|f_k/\lambda_k\| \text{ where } A(\lambda)=\max(1,|\lambda|),\ \lambda\in C\\ &\le w.\end{aligned}$$

Therefore $f_k \xrightarrow{M} 0$.

Hence (b)⇒(a) and proof of Lemma 4.1 is complete.

Theorem 4.2. *$f_q \xrightarrow{M} 0$ in X if and only if $f_q(z_1,z_2) \xrightarrow{M} 0$ uniformly in some finite circle.*

Proof. First, let us suppose that $f_q \xrightarrow{M} 0$ and $f_q = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{mn}^q z_1^m z_2^n$.
Then there exists a constant k and a sequences $\{\lambda_q\}$ in C, tending to zero such that $\|f_q/\lambda_q\| \le k$ for all q

i.e. $\quad \left|\dfrac{a_{0,0}^q}{\lambda_q}\right| \le k$ and $\left|\dfrac{a_{m,n}^q}{\lambda_q}\right| \le k^{m+n},\ m,n\ge 1$

i.e. $\quad |a_{0,0}^q| \le |\lambda_q|k$ and $|a_{m,n}^q| \le |\lambda_q|k^{m+n},\ m,n\ge 1.$

If $(z_1,z_2)\in C\times C$ such that $|z_1|,|z_2|<1/k$, then

$$\begin{aligned}|f_q(z_1,z_2)| &= \left|\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_{m,n}^q z_1^m z_2^n\right|\\ &\le \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |a_{mn}^q|\,|z_1|^m|z_2|^n\\ &\le \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\lambda_q|\,k^{m+n}|z_1|^m|z_2|^n + |\lambda_q|\,k\\ &= |\lambda_q|\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k^{m+n}|z_1|^m|z_2|^n + k\right)\\ &\le |\lambda_q|\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} 1/2^{m+n} + k\right)\\ &\le |\lambda_q|(1+k).\end{aligned}$$

Hence

$|f_q(z_1,z_2)| \to 0$ uniformly for all z_1, z_2 such that $|z_1|$, $|z_2| < 1/k$.

Conversely, suppose there exist r_1, $r_2>0$ such that $f_q(z_1,z_2)\to 0$ uniformly for all z_1,z_2 such that $|z_1|\le r_1$, $|z_2|\le r_2$.

i.e.

$$\sup_{|z_i|\le r_i} |f_q(z_1,z_2)| \to \infty \quad \text{as} \quad q\to\infty.$$

Now

$$|f_q(z_1,z_2)| \le \sup_{|z_i|\le r_i} |f_q(z_1,z_2)| \quad \text{for all} \quad z_1,z_2 \text{ such that } |z_i|\le r_i.$$

Hence

$$|a^q_{mn}| r_1^m r_2^n \le \sup_{|z_i|\le r_i} |f_q(z_1,z_2)|,$$

i.e.

$$\left[\frac{|a^q_{mn}|}{\sup\limits_{|z_i|\le r_i} |f_q(z_1,z_2)|}\right] \le \left(\frac{1}{r_1}\right)^m \left(\frac{1}{r_1}\right)^n.$$

Choose $\xi = \max\left(\frac{1}{r_1},\frac{1}{r_2}\right)$ then, we get

$$\left[\frac{|a^q_{mn}|}{\sup\limits_{|z_i|\le r_i} |f_q(z_1,z_2)|}\right] \le \xi^{m+n}.$$

i.e.

$$\left[\frac{|a^q_{mn}|}{\sup\limits_{|z_i|\le r_i} |f_q(z_1,z_2)|}\right]^{1/(m+n)} \le \xi.$$

Let $\lambda_q = \sup\limits_{|z_i|\le r_i} |f_q(z_1,z_2)|$.

Then

$$|a^q_{0,0}| = |f_q(0,0)| \le \lambda_q \quad \text{and} \quad \lambda_q \to 0.$$

Hence

$$\begin{aligned}\|f_q/\lambda_q\| &= \sup_{m+n\ge 1}\left\{\frac{|a^q_{0,0}|}{|\lambda_q|}, \frac{|a_{q,mn}|^{1/m+n}}{|\lambda_q|}\right\}\\ &\le \max\{1,\xi\} = A(\xi),\end{aligned}$$

and hence, in view of Lemma 4.1, $f_q \xrightarrow{M} 0$.

Theorem 4.3. *The bornological dual $X^*(r_1, r_2)$ of $X(r_1, r2)$ is the same as its topological dual $X'(r_1, r_2)$.*

Proof. The proof follows immediately from the fact that a linear functional on a normed linear space is continuous if and only if it is bounded.

On $X'(r_1, r_2)$ we now define a map:

$|\cdot : 1/2(r_1 r_2)| : X'(r_1, r_2) \to R,$ where for $f = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{m,n} z_1^m z_2^n$

$|f : 1/2(r_1 r_2)| = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |a_{mn}|/2^{m+n} r_1^m r_2^n.$

It is well-known that

$$f = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} z_1^m z_2^n \in X'(r_1, r_2) \text{ if and only if } \{|c_{m,n}|/r_1^m r_2^n\} \; is$$

bounded.
Consequently the function $\|\cdot : 1/2(r_1, r_2)\|$ is well-defined and $X'(r_1, r_2)$ becomes a normed linear space relative to $\|\cdot : 1/2(r_1 r_2)\|$.
Denote by $\bar{\mathcal{B}}_{1/2(r_1 r_2)}$ the canonical bornology of $X'(r_1, r_2)$ with this norm which we call the $(1/2(r_1 r_2))$-norm.

5. The Space *Y*

Suppose that

$$Y = \left\{ g = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_{m,n} z_1^m z_2^n : b_{m,n} \in C \times C \quad \text{and} \quad \left\{ |b_{m,n}|^{1/(m+n)} \right\} \text{ is bounded} \right\}.$$

A convex bornology $\bar{\mathcal{B}}$ can be defined on Y with the help of a function $\|\cdot\| : Y \to R$ defined in a similar fashion as that on X. We note that $\bar{\mathcal{B}}$ when restricted to X gives $\mathcal{B}$. Moreover, $x = \bigcup_{r_1, r_2 > 0} X_{r_1, r_2}$, and as in the proof of Theorem 4.1 we have that $\bar{\mathcal{B}} = \bigcup_{r_1, r_2 > 0} \bar{\mathcal{B}}_{r_1, r_2}$. Consequently Y is the bornological inductive limit of the normed space $Y(r_1, r_2)$.

Theorem 5.1. *$(Y, \bar{\mathcal{B}})$ is Mackey-complete.*

Proof. We first observe that Lemma 4.1 holds for Y also. Let thus $\{f_k\}$ be a double Mackey-sequence Y. Then there exists a sequence $\{\mu_{kp}\}$ of scalars, tending to zero, such that $\left\| \dfrac{f_k - f_p}{\mu_{kp}} \right\| \leq w$, where w is some fixed real positive number.

Now we choose a sequence $\{\lambda_{kp}\}$ of scalars such that $\lambda_{kp} \geq \mu_{kp}$ for all k, p and further such that $\lambda_{k_1p_1} \leq \lambda_{k_2p_2}$ whenever $k_1 \geq k_2$ and $p_1 \geq p_2$. For this, since $\mu_{kp} \to 0$, without loss of generality we can assume that $\mu_{kp} < 1$ for all k, p. Now we set $k_1 = 1$, $p_1 = 1$ and choose (k_i, p_i) inductively such that $k_i > k_{i-1}$, $p_i > p_{i-1}$ and $\mu_{kp} < 1/i$ for $k \geq k_i$, $p \geq p_i$. Define $\{\lambda_{kp}\}$ as

$$\lambda_{kp} = \frac{1}{\min(i,j)} \quad \text{if} \quad k_i \leq k < k_{i+1} \quad \text{and} \quad p_j \leq p < p_{j+1}.$$

It is easily seen that $\{\lambda_{kp}\}$ is the required sequence. Moreover

$$\lambda_{kp} \to 0 \quad \text{and} \quad \left\| \frac{f_k - f_p}{\lambda_{kp}} \right\| \leq \left\| \frac{f_k - f_p}{\mu_{kp}} \right\| \leq w$$

i.e.

$$\frac{\left|a_{00}^k - a_{00}^p\right|}{|\lambda_{kp}|} \leq w \quad \text{and} \quad \frac{\left|a_{mn}^k - a_{mn}^p\right|}{|\lambda_{kp}|} \leq w \quad \text{for all} \quad m, n \geq 1.$$

$\{a_{00}^k\}$ and $\{a_{mn}^k\}$, $m, n \geq 1$, are Cauchy sequences and hence there exists a_{00}, a_{mn}, $m, n \geq 1$ in $C \times C$ such that $a_{00}^k \to a_{00}$ and $a_{mn}^k \to a_{mn}$ for all $m, n \geq 1$.
Now

$$\frac{\left|a_{00}^k - a_{00}^d\right|}{|\lambda_{k,k+1}|} < \frac{\left|a_{00}^k - a_{00}^d\right|}{|\lambda_{k,d}|} \quad \text{for all} \quad d \geq k+1,$$

and hence

$$\frac{\left|a_{00}^k - a_{00}^d\right|}{|\lambda_{k,k+1}|} \leq w \quad \text{for all} \quad d \geq k+1.$$

Hence as $d \to \infty$, we get $\dfrac{|a_{00}^k - a_{00}|}{|\lambda_{k,k+1}|} \leq w$.
Similarly

$$\frac{|a_{mn}^k - a_{mn}|^{1/(m+n)}}{|\lambda_{k,k+1}|} \leq w \quad \text{for all} \quad m, n \geq 1.$$

i.e.

$$\left\| \frac{f_k - f}{\lambda_{k,k+1}} \right\| \leq w \quad \text{where } f = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} z_1^m z_2^n \quad \text{and} \quad \lambda_{k,k+1} \to 0.$$

Hence $f_k \xrightarrow{M} f$.

$$\begin{aligned}
\text{Now} \quad |a_{mn}|^{1/(m+n)} &= |a_{mn}^k - a_{mn} - a_{mn}^k|^{1/(m+n)} \\
&\leq |a_{mn}^k - a_{mn}|^{1/(m+n)} + \left|a_{mn}^k\right|^{1/(m+n)} \\
&\leq |\lambda_{k,k+1}| \cdot w + |a_{mn}^k|^{1/(m+n)}.
\end{aligned}$$

Hence

$$\limsup_{m,n\to\infty} |a_{mn}|^{1/(m+n)} \le \limsup_{m,n\to\infty} |\lambda_{k,k+1}| \cdot w + \limsup_{m,n\to\infty} |a^k_{mn}|^{1/(m+n)}$$
$$\le M\ w + r_k$$
$$< \infty.$$

where $M = \sup_k |\lambda_{k,k+1}| < \infty$ and $r_k = \limsup_{m,n\to\infty} |a^k_{mn}|^{1/(m+n)} < \infty$.
Hence $f \in Y$ and therefore Y is Mackey-complete.

Corollary 5.1. *Y is complete*

Proof. In view of Theorem [2, p.33] it is enough to show that $\bar{\mathcal{B}}$ is l^1-disced. For this we show that each $B_{r_1,r_2} \in \bar{\mathcal{B}}$ is l^1-disced. Let thus $\{\lambda_i\}$ be a sequence of scalars such that
$\sum_{i=1}^{\infty} |\lambda_i| \le 1$, and $\{f_i\}$ be sequence in B_{r_1,r_2}.
Then

$$\left\| f = \sum_{i=1}^{\infty} \lambda_i f_i \right\| = \sup\left\{ \left| \sum_{i=1}^{\infty} \lambda_i a^i_{00} \right|, \left| \sum_{1}^{\infty} \lambda_i a^i_{mn} \right|^{1/(m+n)},\ m,n \ge 0,\ m+n \ne 0 \right\}$$
$$\le \sup\left\{ \sum |\lambda_i| \left|a^i_{00}\right|, \left(\sum |\lambda_i| \left|a^i_{mn}\right| \right)^{1/(m+n)},\ m,n \ge 0,\ m+n \ne 0 \right\}$$
$$\le \sup\left\{ r_i \sum |\lambda_i|, r_i \left(\sum |\lambda_i| \right)^{1/(m+n)};\ m,n \ge 0,\ m+n \ne 0 \right\} \le r_i,$$

where r_i is defined as in the previous theorem. Hence S_{r_1,r_2} is l^1-disced and the assertion follows.

Theorem 5.2. *$(X, \mathcal{B})$ is not complete.*

Proof. It is enough to show that $(X, \mathcal{B})$ is not Mackey-complete (see [2, p.33]). Thus we consider the sequence

$$f_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\frac{1}{2}\right)^{i+j} z_1^i z_2^j \qquad m,n \ge 1.$$

Then

$$\left\{ 2^{-xy} \left(f_{m,n} - f_{x,y} \right), \quad m \ge n, x \ge y \right\} \text{ is bounded in } X.$$

In other words $\{f_{m,n}\}$ is a Mackey-Cauchy sequence in X and hence in Y.
As $(Y, \bar{\mathcal{B}})$ is Mackey-complete, the Mackey-limit of $\{f_{m,n}\}$ exists in Y. In fact the Mackey-limit of $\{f_{m,n}\}$ in Y is $f_{m,n} = \sum_{i=1}^{m} \sum_{j=1}^{n} \left(\frac{1}{2}\right)^{i+j} z_1^i z_2^j$ as $\{(f_{m,n} - f)/2^{-m-n}\}$ is bounded in Y, and $f \notin X$. We now claim that the Mackey-limit of $\{f_{m,n}\}$ does not exist in X.

For otherwise, let $f_{m,n} \xrightarrow{M} g \in X$. Then $f_{m,n} \xrightarrow{M} g$ in Y. Hence $g = f$ as Y is a separated bornological vector space. This contradicts the fact that $f \notin X$. Hence $(Y, \bar{\mathcal{B}})$ is not Mackey-complete. This proves theorem 5.2. Lastly we have.

Theorem 5.3. *$(Y, \bar{\mathcal{B}})$ is the Mackey-completion of $(X, \mathcal{B})$.*

Proof. Let $f = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{m,n} z_1^m z_2^n \in Y$. Then there exists a number h such that $|c_{mn}|^{1/(m+n)} < h$ for all $m, n \geq 1$. Now consider the sequence

$$\left\{ f_{qt} = \sum_{n=0}^{q} \sum_{m=0}^{t} c_{mn} z_1^m z_2^n \right\}, \quad q, t = 1, 2, \ldots \text{ in } X.$$

Then

$$\begin{aligned}
\left\| \frac{f - f_{qt}}{\left(\frac{1}{2}\right)^{q+t}} \right\| &= \left\| \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn} z_1^m z_2^n - \sum_{n=0}^{q} \sum_{m=0}^{t} c_{mn} z_1^m z_2^n \right) \cdot 2^{q+t} \right\| \\
&= \left\| \frac{\left\{ \sum_{n=0}^{q} \sum_{m=0}^{\infty} c_{mn} z_1^m z_2^n + \sum_{n=q+1}^{\infty} \sum_{m=0}^{\infty} c_{mn} z_1^m z_2^n - \sum_{n=0}^{q} \sum_{m=0}^{t} c_{mn} z_1^m z_2^n \right\}}{\left(\frac{1}{2}\right)^{q+t}} \right\| \\
&= \left\| \sum_{n=0}^{q} \sum_{m=t+1}^{\infty} \frac{c_{mn} z_1^m z_2^n}{\left(\frac{1}{2}\right)^{q+t}} + \sum_{n=q+1}^{\infty} \sum_{m=0}^{t} \frac{c_{mn} z_1^m z_2^n}{\left(\frac{1}{2}\right)^{q+t}} + \sum_{n=q+1}^{\infty} \sum_{m=t+1}^{\infty} \frac{c_{mn} z_1^m z_2^n}{\left(\frac{1}{2}\right)^{q+t}} \right\| \\
&< 4(h + h + h) = 12h < \infty,
\end{aligned}$$

i.e. $f_{q,t} \xrightarrow{M} f$ in X. Thus every $f \in Y$ can be written as the Mackey-limit of a sequence $\{f_{m,n}\}$ in X.

Corollary 5.2. *$(Y, \bar{\mathcal{B}})$ is the completion of $(X, \mathcal{B})$.*
The proof follows immediately from Theorem 5.1 and 5.3.

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