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POPOVICIU'S AND BELLMAN'S INEQUALITIES IN p-SEMI-INNER PRODUCT SPACES

Dedicated to the Memory of Dr. I. Franić

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Abstract. In this paper, we study Popoviciu's and Bellman's inequalities in p-semi-inner product spaces and give some related results.

1. Introduction and Preliminaries

The following generalization of the well-known Aczél's inequality (see, for example, [4, p.117]) is refered in the literature as Popoviciu's inequality:

Theorem 1.1. *Let a_0, b_0 and a_i, b_i, p_i $(i = 1, 2, \dots, n)$ be nonnegative real numbers such that*

$$\sum_{i=1}^{n} p_i a_i^p \le a_0^p, \qquad \sum_{i=1}^{n} p_i b_i^q \le b_0^q, \tag{1.1}$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then the following inequality holds

$$\left(a_0^p - \sum_{i=1}^{n} p_i a_i^p\right)^{\frac{1}{p}} \left(b_0^q - \sum_{i=1}^{n} p_i b_i^q\right)^{\frac{1}{q}} \le a_0 b_0 - \sum_{i=1}^{n} p_i a_i b_i. \tag{1.2}$$

A result closely related to the one stated above is given in the following theorem and is refered in the literature as Bellman's inequality:

Theorem 1.2. *Let a_0, b_0 and a_i, b_i, p_i $(i = 1, 2, \dots, n)$ be nonnegative real numbers such that*

$$\sum_{i=1}^{n} p_i a_i^p \le a_0^p, \qquad \sum_{i=1}^{n} p_i b_i^p \le b_0^p, \tag{1.3}$$

where $p > 1$. Then the following inequality holds

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$$\left(a_0^p - \sum_{i=1}^n p_i a_i^p\right)^{\frac{1}{p}} + \left(b_0^p - \sum_{i=1}^n p_i b_i^p\right)^{\frac{1}{p}} \leq \left[(a_0+b_0)^p - \sum_{i=1}^n p_i(a_i+b_i)^p\right]^{\frac{1}{p}}. \tag{1.4}$$

Remark. In fact, T. Popoviciu and R. Bellman proved Theorem 1.1 and Theorem 1.2, respectively, for the case $p_i = 1$ $(i = 1, 2, \ldots, n)$ (see [4], pp.118-119). The result stated in Theorem 1.1 follows immediately from the original Popoviciu's result by replacing a_i with $p_i^{1/p} a_i$ and b_i with $p_i^{1/q} b_i$ $(i = 1, 2, \ldots, n)$. Similarly, the result stated in Theorem 1.2 is easily obtained from the original Bellman's result by replacing a_i with $p_i^{1/p} a_i$ and b_i with $p_i^{1/p} b_i$ $(i = 1, 2, \ldots, n)$.

Recently, M. Matić and J. Pečarić [3] proved the following refinements of the above results:

Theorem 1.3. *Let a_0, b_0 and a_i, b_i, p_i, q_i $(i = 1, 2, \ldots, n)$ be nonnegative real numbers such that the condition* (1.1) *is valid and*

$$0 \leq q_i \leq p_i \quad (i = 1, 2, \ldots, n).$$

Then the following inequalities hold

$$\begin{aligned} 0 &\leq \left(\sum_{i=1}^n q_i a_i^p\right)^{1/p} \left(\sum_{i=1}^n q_i b_i^q\right)^{1/q} - \sum_{i=1}^n q_i a_i b_i \\ &\leq a_0 b_0 - \sum_{i=1}^n p_i a_i b_i - \left(a_0^p - \sum_{i=1}^n p_i a_i^p\right)^{\frac{1}{p}} \left(b_0^q - \sum_{i=1}^n p_i b_i^q\right)^{\frac{1}{q}}. \end{aligned} \tag{1.5}$$

Theorem 1.4. *Let a_0, b_0 and a_i, b_i, p_i, q_i $(i = 1, 2, \ldots, n)$ be nonnegative real numbers such that the condition* (1.3) *is valid and*

$$0 \leq q_i \leq p_i \quad (i = 1, 2, \ldots, n).$$

Then the following inequalities hold

$$\begin{aligned} 0 &\leq \left[\left(\sum_{i=1}^n q_i a_i^p\right)^{1/p} + \left(\sum_{i=1}^n q_i b_i^p\right)^{1/p}\right]^p - \sum_{i=1}^n q_i (a_i+b_i)^p \\ &\leq (a_0+b_0)^p - \sum_{i=1}^n p_i (a_i+b_i)^p \\ &\quad - \left[\left(a_0^p - \sum_{i=1}^n p_i a_i^p\right)^{1/p} + \left(b_0^p - \sum_{i=1}^n p_i b_i^p\right)^{1/p}\right]^p. \end{aligned} \tag{1.6}$$

In this paper, we give results related to Popoviciu's and Bellman's inequalities which hold in p-semi-inner product spaces and which can be regarded as a generalizations of the results stated above. The notation of a 2-semi-inner product space was introduced by A. H. Siddiqui and S. M. Rizvi [5]. This was generalized to the concept of p-semi-inner-product space by I. Franić [1]. To avoid the problem in the proof of the main result from [1] and [5], Y. Ho and A. White [2] modified one part in the definition of p-semi-inner product, the condition (PS-1) on positive definiteness, and then stated and proved the main result from [1] and [5] with that modified definition. Here we use the definition of p-semi-inner product space given in [2]:

Definition. Let X be a real linear space of dimension greater than one and let $[\cdot \cdot|\cdot]$ be a real-valued function defined on $X \times X \times X$ such that for some $p \in (1, \infty)$ we have:

(PS-1) (i) $[x, x|z] \geq 0$,
(ii) $[x, x|z] = 0$ if and only if x and z are linearly dependent,
(PS-2) $[\alpha x, y|z]{=}\alpha[x.y|z]$, where α is a real number,
(PS-3) $[x + x', y|z] = [x, y|z] + [x', y|z]$,
(PS-4) $|[x, y|z]| \leq [x, x|z]^{1p}[y, y|z]^{\frac{p-1}{p}}$.

Then $[\cdot, \cdot|\cdot]$ is called a p-semi-inner product on X and $(X, [\cdot, \cdot|\cdot])$ is called a p-semi-inner product space (in short, a p-SIP space).

We are interested in the following two simple properties which are valid in any p-SIP space X:

$$[\alpha x, \alpha x|y] = |\alpha|^p[x, x|y] \tag{1.7}$$

and

$$[x + y, x + y|z]^{\frac{1}{p}} \leq [x, x|z]^{\frac{1}{p}} + [y, y|z]^{\frac{1}{p}}. \tag{1.8}$$

Indeed, when $\alpha = 0$ or x and y are linearly dependent, (1.7) is trivially fulfilled. If $\alpha \neq 0$ and x and y are linearly independent, then, by (i) of (PS-1) and (PS-2), we have

$$[\alpha x, \alpha x|y] = |[\alpha x, \alpha x|y]| = |\alpha||[x, \alpha x|y]|. \tag{1.9}$$

By (PS-4), we have

$$|[x, \alpha x|y]| \leq [x, x|y]^{\frac{1}{p}}[\alpha x, \alpha x|y]^{\frac{p-1}{p}}$$

and so, by (1.9),

$$[\alpha x, \alpha x|y] \leq |\alpha|[x, x|y]^{\frac{1}{p}}[\alpha x, \alpha x|y]^{\frac{p-1}{p}}. \tag{1.10}$$

Thus, dividing (1.10) by $[\alpha x, \alpha x|y]^{\frac{p-1}{p}}$, we get

$$[\alpha x, \alpha x|y]^{\frac{1}{p}} \leq |\alpha|[x, x|y]^{\frac{1}{p}}. \tag{1.11}$$

On the other hand, from (1.11) it follows

$$[x, x|y]^{\frac{1}{p}} = \left[\frac{1}{\alpha}\alpha x, \frac{1}{\alpha}\alpha x|y\right]^{\frac{1}{p}} \leq \left|\frac{1}{\alpha}\right|[\alpha x, \alpha x|y]^{\frac{1}{p}}$$

or, equivalently,

$$|\alpha|[x,x|y]^{\frac{1}{p}} \le [\alpha x, \alpha x|y]^{\frac{1}{p}}. \tag{1.12}$$

Now (1.7) follows from (1.11) and (1.12). To prove (1.8), first suppose $[x+y, x+y|z] \ne 0$. Then we have

$$\begin{aligned}[x+y\ ,\ x+y|z] \\ &= [x, x+y|z] + [y, x+y|z] \\ &\le [x,x|z]^{\frac{1}{p}}[x+y, x+y|z]^{\frac{p-1}{p}} + [y,y|z]^{\frac{1}{p}}[x+y, x+y|z]^{\frac{p-1}{p}} \\ &= ([x,x|z]^{\frac{1}{p}} + [y,y|z]^{\frac{1}{p}})[x+y, x+y|z]^{\frac{p-1}{p}}. \end{aligned} \tag{1.13}$$

Now, dividing (1.13) by $[x+y, x+y|z]^{\frac{p-1}{p}}$, we get (1.8). In the case when $[x+y, x+y|z] = 0$, that is, when $x+y$ and z are linearly dependent, the inequality (1.8) obviously holds.

2. Popoviciu's Inequality

In this section, we study an analogue of Popoviciu's inequality in p-semi- inner product spaces. An expected generalization of Popoviciu's inequality for p-SIP space is as follows:

Theorem 2.1. *Let $(X, [\cdot, \cdot|\cdot])$ be a p-SIP space. Let a_0, b_0 be nonnegative numbers and $x, y, z \in X$ vectors such that*

$$[x,x|z] \le a_0^p, \qquad [y,y|z] \le b_0^q, \tag{2.1}$$

where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then we have

$$(a_0^p - [x,x|z])^{\frac{1}{p}}(b_0^q - [y,y|z])^{\frac{1}{q}} \le a_0 b_0 - |[x,y|z]|. \tag{2.2}$$

Instead of proving this result here, we state and prove the followig theorem which is more general:

Theorem 2.2. *Let $(X, [\cdot, \cdot|\cdot])$ be a p-SIP space and $x_i, y_i, z_i \in X$ $(i = 1, 2, \cdots, n)$ be given vectors. If a_0, b_0 and p_i $(i = 1, 2, \cdots, n)$ are nonnegative numbers such that*

$$\sum_{i=1}^n p_i[x_i, x_i|z_i] \le a_0^p, \qquad \sum_{i=1}^n p_i[y_i, y_i|z_i] \le b_0^q, \tag{2.3}$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have

$$\begin{aligned}\left(a_0^p - \sum_{i=1}^n p_i[x_i, x_i|z_i]\right)^{\frac{1}{p}} \left(b_0^q - \sum_{i=1}^n p_i[y_i, y_i|z_i]\right)^{\frac{1}{q}} \\ \le a_0 b_0 - \sum_{i=1}^n p_i|[x_i, y_i|z_i]|. \end{aligned} \tag{2.4}$$

Proof. Set $a_i = [x_i, x_i|z_i]^{\frac{1}{p}}$ and $b_i = [y_i, y_i|z_i]^{\frac{1}{q}}$ $(i = 1, 2, \cdots, n)$ and apply Theorem 1.1 to obtain the inequality

$$\left(a_0^p - \sum_{i=1}^{n} p_i[x_i, x_i|z_i]\right)^{\frac{1}{p}} \left(b_0^q - \sum_{i=1}^{n} p_i[y_i, y_i|z_i]\right)^{\frac{1}{q}} \leq a_0 b_0 - \sum_{i=1}^{n} p_i[x_i, x_i|z_i]^{\frac{1}{p}}[y_i, y_i|z_i])^{\frac{1}{q}}. \tag{2.5}$$

Now, from $\frac{1}{p} + \frac{1}{q} = 1$, we get $\frac{p-1}{p} = \frac{1}{q}$ so that, by (PS-4), we have

$$|[x_i, y_i|z_i]| \leq [x_i, x_i|z_i]^{\frac{1}{p}}[y_i, y_i|z_i]^{\frac{1}{q}} \ (i = 1, 2, \cdots, n).$$

Using these inequalities and (2.5), we get (2.4). This completes the proof.

Remark. In the case when $n = 1$, $p_1 = 1$, $x_1 = x$, $y_1 = y$ and $z_1 = z$, Theorem 2.2 reduces to Theorem 2.1.

Next, we give a result for p-SIP spaces anologous to the result stated in Theorem 1.3:

Theorem 2.3. *Let $(X, [\cdot, \cdot|\cdot])$ be a p-SIP space and $x_i, y_i, z_i \in X$ $(i = 1, 2, \cdots, n)$ be given vectors. If a_0, b_0, p_i and q_i $(i = 1, 2, \cdots, n)$ are nonnegative numbers such that*

$$0 \leq q_i \leq p_i \qquad (i = 1, 2, \cdots, n)$$

and

$$\sum_{i=1}^{n} p_i[x_i, x_i|z_i] \leq a_0^p, \qquad \sum_{i=1}^{n} p_i[y_i, y_i|z_i] \leq b_0^q, \tag{2.6}$$

where $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then we have

$$\begin{aligned} 0 &\leq \left(\sum_{i=1}^{n} q_i[x_i, x_i|z_i]\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} q_i[y_i, y_i|z_i]\right)^{\frac{1}{q}} - \sum_{i=1}^{n} q_i|[x_i, y_i|z_i]| \\ &\leq a_0 b_0 - \sum_{i=1}^{n} p_i|[x_i, y_i|z_i]| - \left(a_0^p - \sum_{i=1}^{n} p_i[x_i, x_i|z_i]\right)^{\frac{1}{p}} \left(b_0^q - \sum_{i=1}^{n} p_i[y_i, y_i|z_i]\right)^{\frac{1}{q}}. \end{aligned} \tag{2.7}$$

Proof. Since $0 \leq p_i - q_i \leq p_i$ $(i = 1, 2, \cdots, n)$, from (2.6) it follows that

$$a_0^p - \sum_{i=1}^{n} (p_i - q_i)[x_i, x_i|z_i] \geq a_0^p - \sum_{i=1}^{n} p_i[x_i, x_i|z_i] \geq 0$$

and

$$b_0^q - \sum_{i=1}^{n} (p_i - q_i)[y_i, y_i|z_i] \geq b_0^q - \sum_{i=1}^{n} p_i[y_i, y_i|z_i] \geq 0.$$

Therefore, we can apply (2.4) with $p_i - q_i$ in place of p_i $(i = 1, 2, \cdots, n)$ to obtain the inequality

$$\begin{aligned}&\left(a_0^p - \sum_{i=1}^n p_i[x_i, x_i|z_i] + \sum_{i=1}^n q_i[x_i, x_i|z_i]\right)^{\frac{1}{p}}\\ &\qquad\times\left(b_0^p - \sum_{i=1}^n p_i[y_i, y_i|z_i] + \sum_{i=1}^n q_i[y_i, y_i|z_i]\right)^{\frac{1}{q}}\\ &\le a_0 b_0 - \sum_{i=1}^n p_i|[x_i, y_i|z_i]| + \sum_{i=1}^n q_i|[x_i, y_i|z_i]|.\end{aligned} \tag{2.8}$$

On the other hand, applying the well-known discrete Hölder inequality, we get

$$\begin{aligned}&\left(a_0^p - \sum_{i=1}^n p_i[x_i, x_i|z_i]\right)^{\frac{1}{p}}\left(b_0^p - \sum_{i=1}^n p_i[y_i, y_i|z_i]\right)^{\frac{1}{q}}\\ &\qquad+\left(\sum_{i=1}^n q_i[x_i, x_i|z_i]\right)^{\frac{1}{p}}\left(\sum_{i=1}^n q_i[y_i, y_i|z_i]\right)^{\frac{1}{q}}\\ &\le \left(a_0^p - \sum_{i=1}^n p_i[x_i, x_i|z_i] + \sum_{i=1}^n q_i[x_i, x_i|z_i]\right)^{\frac{1}{p}}\\ &\qquad\times\left(b_0^q - \sum_{i=1}^n p_i[y_i, y_i|z_i] + \sum_{i=1}^n q_i[y_i, y_i|z_i]\right)^{\frac{1}{q}}.\end{aligned} \tag{2.9}$$

Thus, from (2.8) and (2.9) it follows that

$$\begin{aligned}&\left(a_0^p - \sum_{i=1}^n p_i[x_i, x_i|z_i]\right)^{\frac{1}{p}}\left(b_0^p - \sum_{i=1}^n p_i[y_i, y_i|z_i]\right)^{\frac{1}{q}} + \left(\sum_{i=1}^n q_i[x_i, x_i|z_i]\right)^{\frac{1}{p}}\left(\sum_{i=1}^n q_i[y_i, y_i|z_i]\right)^{\frac{1}{q}}\\ &\le a_0 b_0 - \sum_{i=1}^n p_i|[x_i, y_i|z_i]| + \sum_{i=1}^n q_i|[x_i, y_i|z_i]|,\end{aligned}$$

which is equivalent to the second inequality in (2.7). The first inequality in (2.7) is a simple consequence of the weighted Hölder discrete inequality and (PS-4).
Namely, we have

$$\left(\sum_{i=1}^n q_i[x_i, x_i|z_i]\right)^{\frac{1}{p}}\left(\sum_{i=1}^n q_i[y_i, y_i|z_i]\right)^{\frac{1}{q}} \ge \sum_{i=1}^n q_i[x_i, x_i|z_i]^{\frac{1}{p}}[y_i, y_i|z_i]^{\frac{1}{q}} \ge \sum_{i=1}^n q_i|[x_i, y_i|z_i]|.$$

This completes the proof.

3. Bellman's Inequality

In this section, we give analogues of the results related to Bellman's inequality in p-semi-inner product spaces. The simplest generalization of Bellman's inequality for p-SIP space is given in the next theorem:

Theorem 3.1. *Let $(X,[\cdot,\cdot|\cdot])$ be a p-SIP space and $x,y,z\in X$ be given vectors. If a_0 and b_0 are nonnegative numbers such that*

$$[x,x|z]\le a_0^p,\qquad [y,y|z]\le b_0^p, \tag{3.1}$$

then we have

$$(a_0^p-[x,x|z])^{\frac{1}{p}}+(b_0^p-[y,y|z])^{\frac{1}{p}}\le((a_0+b_0)^p-[x+y,x+y|z])^{\frac{1}{p}}. \tag{3.2}$$

We give the proof for thr following more general result:

Theorem 3.2. *Let $(X,[\cdot,\cdot|\cdot])$ be a p-SIP space and let $x_i,y_i,z_i\in X$ $(i=1,2,\cdots,n)$ be given vectors. If a_0,b_0 and p_i $(i=1,2,\cdots,n)$ are nonnegative numbers such that*

$$\sum_{i=1}^n p_i[x_i,x_i|z_i]\le a_0^p,\qquad \sum_{i=1}^n p_i[y_i,y_i|z_i]\le b_0^p, \tag{3.3}$$

then we have

$$\left(a_0^p-\sum_{i=1}^n p_i[x_i,x_i|z_i]\right)^{\frac{1}{p}}+\left(b_0^p-\sum_{i=1}^n p_i[y_i,y_i|z_i]\right)^{\frac{1}{p}}$$
$$\le\left[(a_0+b_0)^p-\sum_{i=1}^n p_i[x_i+y_i,x_i+y_i|z_i]\right]^{\frac{1}{p}}. \tag{3.4}$$

Proof. Setting $a_i=[x_i,x_i|z_i]^{\frac{1}{p}}$ and $b_i=[y_i,y_i|z_i]^{\frac{1}{p}}$ and then applying Theorem 1.2, we get

$$\left(a_0^p-\sum_{i=1}^n p_i[x_i.x_i|z_i]\right)^{\frac{1}{p}}+\left(b_0^p-\sum_{i=1}^n p_i[y_i,y_i|z_i]\right)^{\frac{1}{p}}$$
$$\le\left[(a_0+b_0)^p-\sum_{i=1}^n p_i\left([x_i,x_i|z_i]^{\frac{1}{p}}+[y_i,y_i|z_i]^{\frac{1}{p}}\right)^p\right]^{\frac{1}{p}}. \tag{3.5}$$

On the other side, by (1.8), we have

$$[x_i,x_i|z_i]^{\frac{1}{p}}+[y_i,y_i|z_i]^{\frac{1}{p}}\ge[x_i+y_i,x_i+y_i|z_i]^{\frac{1}{p}}\qquad (i=1,2,\cdots,n).$$

Using these inequalities, we get

$$\sum_{i=1}^{n} p_i[x_i+y_i, x_i+y_i|z_i] \le \sum_{i=1}^{n} p_i\left([x_i,x_i|z_i]^{\frac{1}{p}} + [y_i,y_i|z_i]^{\frac{1}{p}}\right)^p$$

$$\le \left[\left(\sum_{i=1}^{n} p_i[x_i,x_i|z_i]\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n}[y_i,y_i|z_i]\right)^{\frac{1}{p}}\right]^p. \qquad (3.6)$$

Now it is easy to obtain (3.4) from (3.5) and (3.6). This completes the proof.

Remark. In the case when $n = 1$, $p_1 = 1$, $x_1 = x$, $y_1 = y$ and $z_1 = z$, Theorem 3.2 reduces to Theorem 3.1.

As we may expect, am analogue of the result stated in Theorem 1.4 for p-SIP can be also proved:

Theorem 3.3. *Let $(X, [\cdot,\cdot|\cdot])$ be a p-SIP space and let $x_i, y_i, z_i \in X$ $(i = 1, 2, \cdots, n)$ be given vectors. If a_0, b_0 and p_i, q_i $(i = 1, 2, \cdots, n)$ are nonnegative numbers such that the conditions* (3.3) *are satisfied and*

$$0 \le q_i \le p_i \qquad (i = 1, 2, \cdots, n),$$

then we have

$$0 \le \left[\left(\sum_{i=1}^{n} q_i[x_i,x_i|z_i]\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} q_i[y_i,y_i|z_i]\right)^{\frac{1}{p}}\right]^p - \sum_{i=1}^{n} q_i[x_i+y_i, x_i+y_i|z_i]$$

$$\le (a_0+b_0)^p - \sum_{i=1}^{n} p_i[x_i+y_i, x_i+y_i|z_i]$$

$$-\left[\left(a_0^p - \sum_{i=1}^{n} p_i[x_i,x_i|z_i]\right)^{\frac{1}{p}} + \left(b_0^p - \sum_{i=1}^{n} p_i[y_i,y_i|z_i]\right)^{\frac{1}{p}}\right]^p. \qquad (3.7)$$

Proof. Since $0 \le p_i - q_i \le p_i$ $(i = 1, 2, \cdots, n)$, from (3.3) it follows that

$$a_0^p - \sum_{i=1}^{n}(p_i - q_i)[x_i,x_i|z_i] \ge a_0^p - \sum_{i=1}^{n} p_i[x_i,x_i|z_i] \ge 0$$

and

$$b_0^p - \sum_{i=1}^{n}(p_i - q_i)[y_i,y_i|z_i] \ge b_0^p - \sum_{i=1}^{n} p_i[y_i,y_i|z_i] \ge 0.$$

Therefore, we can replace p_i in (3.4) with $p_i - q_i$ $(i = 1, 2, \cdots, n)$ to obtain the inequality

$$\left(a_0^p - \sum_{i=1}^n p_i[x_i, x_i|z_i] + \sum_{i=1}^n q_i[x_i, x_i|z_i] \right)^{\frac{1}{p}} + \left(b_0^p - \sum_{i=1}^n p_i[y_i, y_i|z_i] + \sum_{i=1}^n q_i[y_i, y_i|z_i] \right)^{\frac{1}{p}}$$
$$\leq \left[(a_0 + b_0)^p - \sum_{i=1}^n p_i[x_i + y_i, x_i + y_i|z_i] + \sum_{i=1}^n q_i[x_i + y_i, x_i + y_i|z_i] \right]^{\frac{1}{p}}. \tag{3.8}$$

Also, by the Minkowski's inequality, we have

$$\Bigg[\left(a_0^p - \sum_{i=1}^n p_i[x_i, x_i|z_i] + \sum_{i=1}^n q_i[x_i, x_i|z_i] \right)^{\frac{1}{p}}$$
$$+ \left(b_0^p - \sum_{i=1}^n p_i[y_i, y_i|z_i] + \sum_{i=1}^n q_i[y_i, y_i|z_i] \right)^{\frac{1}{p}} \Bigg]^p$$
$$\geq \left[\left(a_0^p - \sum_{i=1}^n p_i[x_i, x_i|z_i] \right)^{\frac{1}{p}} + \left(b_0^p - \sum_{i=1}^n p_i[y_i, y_i|z_i] \right)^{\frac{1}{p}} \right]^p$$
$$+ \left[\left(\sum_{i=1}^n q_i[x_i, x_i|z_i] \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n q_i[y_i, y_i|z_i] \right)^{\frac{1}{p}} \right]^p. \tag{3.9}$$

Now, from the inequalities (3.8) and (3.9), we get the inequality

$$(a_0 + b_0)^p - \sum_{i=1}^n p_i[x_i + y_i, x_i + y_i|z_i] + \sum_{i=1}^n q_i[x_i + y_i, x_i + y_i|z_i]$$
$$\geq \left[\left(a_0^p - \sum_{i=1}^n p_i[x_i, x_i|z_i] \right)^{\frac{1}{p}} + \left(b_0^p - \sum_{i=1}^n p_i[y_i, y_i|z_i] \right)^{\frac{1}{p}} \right]^p$$
$$+ \left[\left(\sum_{i=1}^n q_i[x_i, x_i|z_i] \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n q_i[y_i, y_i|z_i] \right)^{\frac{1}{p}} \right]^p,$$

which is equivalent to the second inequality in (3.7). The first inequality in (3.7) is a simple consequence of (1.8) and weighted discrete Minkowski's inequality since we have

$$\left[\left(\sum_{i=1}^n q_i[x_i, x_i|z_i] \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n q_i[y_i, y_i|z_i] \right)^{\frac{1}{p}} \right]^p \geq \sum_{i=1}^n q_i \left([x_i, x_i|z_i]^{\frac{1}{p}} + [y_i, y_i|z_i]^{\frac{1}{p}} \right)^p$$
$$\geq \sum_{i=1}^n q_i[x_i + y_i, x_i + y_i|z_i].$$

This completes the proof.

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