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CHEBYSHEV TYPE INEQUALITIES INVOLVING GENERALIZED KATUGAMPOLA FRACTIONAL INTEGRAL OPERATORS

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Abstract. A number of Chebyshev type inequalities involving various fractional integral operators have, recently, been presented. Here, motivated essentially by the earlier works and their applications in diverse research subjects, we aim to establish several Chebyshev type inequalities involving generalized Katugampola fractional integral operator. Relevant connections of the results presented here with those (known and new) involving relatively simpler and familiar fractional integral operators are also pointed out.

1. Introduction and preliminaries

Many integral inequalities of various types have been presented in the literature. Among them, we choose to recall the following Chebyshev inequality (see [3]):

$$\frac{1}{b-a}\int_a^b f(x)g(x)\,dx \geq \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right), \tag{1.1}$$

where f and g are two integrable and synchronous functions on $[a,b]$. Here, two functions f and g are called *synchronous* on $[a,b]$ if

$$\big(f(x)-f(y)\big)\big(g(x)-g(y)\big) \geq 0 \quad (x,\, y \in [a,b]).$$

The inequality (1.1) has many applications in diverse research subjects such as numerical quadrature, transform theory, probability, existence of solutions of differential equations and statistical problems. My authors have investigated generalizations of the Chebyshev inequality (1.1) which are called Chebyshev type inequalities. Take some examples. Niculescu and Roventa [8] proved that for two functions $f, g \in L^{\infty}([a,b])$, the Chebyshev inequality holds under the following assumption:

$$\left(f(x)-\frac{1}{x-a}\int_a^b f(x)dx\right)\left(g(x)-\frac{1}{x-a}\int_a^b g(x)dx\right) \geq 0.$$

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Dahmani et al. [5] established the Chebyshev inequality without synchronous function condition. Also, recently, many authors have presented Chebyshev type inequalities involving various fractional integral operators (see, e.g., [10] and the references therein). Sousa et al. [13] established two Grüss-type inequalities involving generalized Katugampola fractional integral operator (1.8).

Motivated by the above-cited works, in this paper, we aim to establish Chebyshev type inequalities with two synchronous functions involving generalized Katugampola fractional integral operator (1.8). Also, we we present a Chebyshev type inequality without two synchronous function assumption by substituting another condition. Further, the results presented here, being very general, are pointed out to include various known and new inequalities involving some relatively simpler and familiar fractional integral operators as their special cases.

For our purpose, we recall some definitions. The beta function $B(\alpha,\beta)$ is defined by (see, e.g., [14, Section 1.1])

$$B(\alpha,\beta)=\begin{cases}\displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\Re(\alpha)>0;\ \Re(\beta)>0)\\ \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & \left(\alpha,\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-\right),\end{cases} \tag{1.2}$$

where Γ is the familiar Gamma function. Here and in the following, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$, and $\mathbb{Z}_0^-$ be the sets of complex numbers, real numbers, positive real numbers, positive integers, and non-positive integers, respectively, and let $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$.

Let $[a,b]$ $(-\infty<a<b<\infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann-Liouville fractional integrals (left-sided) of a function f of order $\alpha\in\mathbb{C}$ with $\Re(\alpha)>0$ are defined by (see, e.g., [2, 7, 9, 11])

$$\left(J_{a+}^{\alpha}f\right)(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{f(t)}{(x-t)^{1-\alpha}}\,dt \quad (x>a). \tag{1.3}$$

The Liouville fractional integrals (left-sided) of a real function f of order $\alpha\in\mathbb{C}$ with $\Re(\alpha)>0$ are defined by (see, e.g., [2, 7, 9, 11])

$$\left(I_{0+}^{\alpha}f\right)(x):=\frac{1}{\Gamma(\alpha)}\int_0^x\frac{f(t)}{(x-t)^{1-\alpha}}\,dt \quad \left(x\in\mathbb{R}^+\right). \tag{1.4}$$

Let (a,b) $(-\infty<a<b<\infty)$ be a finite or an infinite interval on the half-axis $\mathbb{R}^+$. The Hadamard fractional integrals (left-sided) of a real function $f\in L(a,b)$ of order $\alpha\in\mathbb{C}$ with $\Re(\alpha)>0$ are defined by (see, e.g., [2, 7, 9, 11])

$$\left(H_{a+}^{\alpha}f\right)(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\left(\log\frac{x}{t}\right)^{\alpha-1}\frac{f(t)}{t}\,dt \quad (a<x<b). \tag{1.5}$$

Let (a,b) $(-\infty \le a < b \le \infty)$ be a finite or an infinite interval on the half-axis $\mathbb{R}^+$. Also, let $\Re(\alpha) > 0$, $\sigma \in \mathbb{R}^+$, and $\eta \in \mathbb{C}$. The Erdélyi-Kober fractional integrals (left-sided) of a real function $f \in L(a,b)$ of order $\alpha \in \mathbb{C}$ are defined by (see, e.g., [7, 9])

$$\left(I^{\alpha}_{a+,\sigma,\eta} f\right)(x) := \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \frac{t^{\sigma(\eta+1)-1}}{(x^\alpha - t^\alpha)^{1-\alpha}} f(t)\,dt \quad (0 \le a < x < b \le \infty). \tag{1.6}$$

Let $[a,b] \subset \mathbb{R}$ be a finite interval. The Katugampola fractional integrals (left-sided) of a real function $f \in X^p_c(a,b)$ of order $\alpha \in \mathbb{C}$ with $\Re(\alpha > 0)$ and $\rho \in \mathbb{R}^+$ are defined by (see [6])

$$\left({}^\rho I^{\alpha}_{a+} f\right)(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t)\,dt \quad (x > a). \tag{1.7}$$

Here, the space $X^p_c(a,b)$ $(c \in \mathbb{R},\ 1 \le p \le \infty)$ consist of those complex-valued Lebesque measurable functions φ on (a,b) for which $\|\varphi\|_{X^p_c} < \infty$, with

$$\|\varphi\|_{X^p_c} = \left(\int_a^b |x^c \varphi(x)|^p \frac{dx}{x}\right)^{1/p} \qquad (1 \le p < \infty)$$

and

$$\|\varphi\|_{X^p_c} = \operatorname{esssup}_{x \in (a,b)} [x^c |\varphi(x)|].$$

Let $0 \le a < x < b \le \infty$. Also, let $\varphi \in X^p_c(a,b)$, $\alpha \in \mathbb{R}^+$, and $\beta, \rho, \eta, \kappa \in \mathbb{R}$. Then the fractional integrals (left-sided and right-sided) of a function φ are defined, respectively, by (see [12])

$$\left({}^\rho I^{\alpha,\beta}_{a+,\eta,\kappa} \varphi\right)(x) := \frac{\rho^{1-\beta} x^\kappa}{\Gamma(\alpha)} \int_a^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} \varphi(\tau)\,d\tau \tag{1.8}$$

and

$$\left({}^\rho I^{\alpha,\beta}_{b-,\eta,\kappa} \varphi\right)(x) := \frac{\rho^{1-\beta} x^{\rho\eta}}{\Gamma(\alpha)} \int_x^b \frac{\tau^{\kappa+\rho-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} \varphi(\tau)\,d\tau. \tag{1.9}$$

Remark 1.1. The fractional integral (1.8) contains five well-known fractional integrals as its particular cases (see also [13]):

(i) Setting $\kappa = 0$, $\eta = 0$ and taking the limit $\rho \to 1$ in (1.8), the integral operator (1.8) reduces to the Riemann-Liouville fractional integral (1.3) (see also [7, p. 69]).

(ii) Setting $\kappa = 0$, $\eta = 0$, $a = 0$ and taking the limit $\rho \to 1$ in (1.8), the integral operator (1.8) reduces to the Liouville fractional integral (1.4) (see also [7, p.79]).

(iii) Setting $\beta = \alpha$, $\kappa = 0$, $\eta = 0$, and taking the limit $\rho \to 0^+$ with L'Hôspital's rule in (1.8), the integral operator (1.8) reduces to the Hadamard fractional integral (1.5) (see also [7, p. 110]).

(iv) Setting $\beta = 0$ and $\kappa = -\rho(\alpha+\eta)$ in (1.8), the integral operator (1.8) reduces to the Erdélyi-Kober fractional integral (1.6) (see also [7, p.105]).

(v) Setting $\beta=\alpha$, $\kappa=0$ and $\eta=0$ in (1.8), the integral operator (1.8) reduces to the Katugampola fractional integral (1.7) (see also [6]).

2. Chebyshev type inequalities with synchronous functions

Here, we establish Chebyshev type inequalities with synchronous functions involving the Katugampola fractional integrals (1.7). By using (1.2), we obtain (see [13, Eq. (3.1)])

$$\frac{\rho^{1-\beta}x^{\kappa}}{\Gamma(\alpha)}\int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}\,d\tau=\frac{x^{\kappa+\rho(\eta+\alpha)}\,\Gamma(\eta+1)}{\rho^{\beta}\,\Gamma(\alpha+\eta+1)}:=\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta) \tag{2.1}$$

$$\left(\alpha,\,x\in\mathbb{R}^+;\ \beta,\,\rho,\,\eta,\,\kappa\in\mathbb{R}\right).$$

We also let

$$\left({}^{\rho}I_{0+,\eta,\kappa}^{\alpha,\beta}\varphi\right)(x):=\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}\varphi\right)(x). \tag{2.2}$$

Theorem 2.1. *Let $\beta,\kappa\in\mathbb{R}$, $x,\,\alpha,\,\rho\in\mathbb{R}^+$, and $\eta\in\mathbb{R}_0^+$. Also, let f and g be two integrable functions which are synchronous on $[0,\infty)$. Then*

$$\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}fg\right)(x)\geq\frac{1}{\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)}\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f\right)(x)\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g\right)(x). \tag{2.3}$$

Proof. Since f and g are synchronous on $[0,\infty)$, we have

$$(f(\tau)-f(\xi))(g(\tau)-g(\xi))\geq 0\quad\left(\tau,\,\xi\in\mathbb{R}_0^+\right),$$

or, equivalently,

$$f(\tau)g(\tau)+f(\xi)g(\xi)\geq f(\tau)g(\xi)+f(\xi)g(\tau)\quad\left(\tau,\,\xi\in\mathbb{R}_0^+\right). \tag{2.4}$$

Multiplying both sides of (2.4) by

$$\frac{\rho^{1-\beta}x^{\kappa}}{\Gamma(\alpha)}\,\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}\quad\left(x\in\mathbb{R}^+,\ 0<\tau<x\right)$$

and integrating both sides of the resulting inequality with respect to the variable τ from 0 and x, we get

$$\begin{aligned}&\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}fg\right)(x)+f(\xi)g(\xi)\frac{\rho^{1-\beta}x^{\kappa}}{\Gamma(\alpha)}\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}\,d\tau\\&\geq g(\xi)\frac{\rho^{1-\beta}x^{\kappa}}{\Gamma(\alpha)}\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}f(\tau)\,d\tau+f(\xi)\frac{\rho^{1-\beta}x^{\kappa}}{\Gamma(\alpha)}\int_0^x\frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}}g(\tau)\,d\tau.\end{aligned}$$

We find from (1.8), (2.1) and (2.2) that

$$\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}fg\right)(x)+f(\xi)g(\xi)\,\Lambda_{x,\kappa}^{\rho,\beta}(\alpha,\eta)\geq g(\xi)\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}f\right)(x)+f(\xi)\left({}^{\rho}I_{\eta,\kappa}^{\alpha,\beta}g\right)(x). \tag{2.5}$$

Multiplying both sides of (2.5) by

$$\frac{\rho^{1-\beta}x^{\kappa}}{\Gamma(\alpha)}\frac{\xi^{\rho(\eta+1)-1}}{(x^{\rho}-\xi^{\rho})^{1-\alpha}} \quad \left(x\in\mathbb{R}^{+},\, 0<\tau<x\right)$$

and integrating both sides of the resulting inequality with respect to the variable ξ from 0 and x, similarly as above, we obtain

$$\begin{aligned}&\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}fg\right)(x)\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)+\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}fg\right)(x)\\ &\geq\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}f\right)(x)\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}g\right)(x)+\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}g\right)(x)\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}f\right)(x),\end{aligned}$$

which, upon simplifying, leads to (2.3). □

Remark 2.1. We consider some particular cases of the result in Theorem 2.1.

(i) Setting $\kappa=0$, $\eta=0$ and taking the limit $\rho\to 1$ in the result in Theorem 2.1 yields the inequality in [1, Theorem 3.1].

(ii) Setting $\beta=\alpha$, $\kappa=0$, $\eta=0$ and taking the limit $\rho\to 0^{+}$ in the result in Theorem 2.1 gives the inequality in [4, Theorem 3.1].

(iii) Setting $\beta=0$ and $\kappa=-\rho(\alpha+\eta)$ in the result in Theorem 2.1 yields the inequality in [10, Theorem 1].

(iv) Setting $\beta=\alpha$, $\kappa=0$ and $\eta=0$ in the result in Theorem 2.1, under the corresponding reduced assumption, we obtain

$$\left({}^{\rho}I^{\alpha}_{0+}fg\right)(x)\geq\frac{\rho^{\alpha}\Gamma(\alpha+1)}{x^{\rho\alpha}}\left({}^{\rho}I^{\alpha}_{0+}f\right)(x)\left({}^{\rho}I^{\alpha}_{0+}g\right)(x). \tag{2.6}$$

Theorem 2.2. *Let $\beta,\kappa\in\mathbb{R}$, x, α, ρ, $\sigma\in\mathbb{R}^{+}$, and $\eta\in\mathbb{R}^{+}_{0}$. Also, let f and g be two integrable functions which are synchronous on* $[0,\infty)$*. Then*

$$\begin{aligned}&\Lambda^{\rho,\beta}_{x,\kappa}(\sigma,\eta)\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}fg\right)(x)+\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\left({}^{\rho}I^{\sigma,\beta}_{\eta,\kappa}fg\right)(x)\\ &\geq\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}f\right)(x)\left({}^{\rho}I^{\sigma,\beta}_{\eta,\kappa}g\right)(x)+\left({}^{\rho}I^{\sigma,\beta}_{\eta,\kappa}f\right)(x)\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa}g\right)(x).\end{aligned} \tag{2.7}$$

Proof. Multiplying both sides of (2.5) by

$$\frac{\rho^{1-\beta}x^{\kappa}}{\Gamma(\sigma)}\frac{\xi^{\rho(n+1)-1}}{(x^{\rho}-\xi^{\rho})^{1-\sigma}} \quad \left(x\in\mathbb{R}^{+},0<\xi<x\right)$$

and integrating the resulting inequality with respect to the variable ξ, from 0 and x, similarly as in the proof of Theorem 2.1, we get the desired result. □

Remark 2.2. We consider some particular cases of the result in Theorem 2.2.

(i) Setting $\kappa = 0$, $\eta = 0$ and taking the limit $\rho \to 1$ in the result in Theorem 2.2 yields the inequality in [1, Theorem 3.2].

(ii) Setting $\beta = \alpha$, $\kappa = 0$, $\eta = 0$ and taking the limit $\rho \to 0^+$ in the result in Theorem 2.2 gives the inequality in [4, Theorem 3.2].

(iii) Setting $\beta = 0$ and $\kappa = -\rho(\alpha+\eta)$ in the result in Theorem 2.2 yields the inequality in [10, Theorem 2].

(iv) Setting $\beta = \alpha$, $\kappa = 0$ and $\eta = 0$ in the result in Theorem 2.2, under the corresponding reduced assumption, we obtain

$$\begin{aligned} &\frac{\rho^{\alpha}\Gamma(\sigma+1)}{x^{\rho\sigma}}\left({}^{\rho}I^{\alpha}_{0+} fg\right)(x) + \frac{\rho^{\alpha}\Gamma(\alpha+1)}{x^{\rho\alpha}}\left({}^{\rho}I^{\sigma}_{0+} fg\right)(x) \\ &\geq \left({}^{\rho}I^{\alpha}_{0+} f\right)(x)\left({}^{\rho}I^{\sigma}_{0+} g\right)(x) + \left({}^{\rho}I^{\sigma}_{0+} f\right)(x)\left({}^{\rho}I^{\alpha}_{0+} g\right)(x). \end{aligned} \tag{2.8}$$

Theorem 2.3. *Let $\beta, \kappa \in \mathbb{R}$, x, α, $\rho \in \mathbb{R}^+$, and $\eta \in \mathbb{R}^+_0$. Also, let*

$$f_j : \mathbb{R}^+_0 \to \mathbb{R} \quad (j = 1, \ldots, n;\ n \in \mathbb{N})$$

be increasing functions. Then

$$\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa} \prod_{j=1}^{n} f_j\right)(x) \geq \frac{1}{\left\{\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\right\}^{n-1}} \prod_{j=1}^{n}\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa} f_j\right)(x) \quad (n \in \mathbb{N}). \tag{2.9}$$

Proof. We prove this theorem by induction on $n \in \mathbb{N}$. Obviously, the case $n = 1$ of (2.9) holds.

For $n = 2$, since f_1 and f_2 are increasing on $\mathbb{R}^+_0$, we find

$$\left(f_1(\tau) - f_2(\xi)\right)\left(f_2(\tau) - f_2(\xi)\right) \geq 0 \quad \left(\tau, \xi \in \mathbb{R}^+_0\right).$$

Now, the proof of (2.9) for $n = 2$ would run parallel to that of Theorem 2.1.

Assume that the inequality (2.9) is true for some $n \in \mathbb{N}$. We observe that $f := \prod_{j=1}^{n} f_j$ is increasing on $\mathbb{R}^+_0$ for each f_j increasing. Let $g := f_{n+1}$. Then, applying the case $n = 2$ to the functions f and g, we have

$$\begin{aligned} \left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa} \prod_{j=1}^{n} f_j \cdot f_{n+1}\right)(x) &\geq \frac{1}{\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)}\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa} \prod_{j=1}^{n} f_j\right)(x) \cdot \left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa} f_{n+1}\right)(x) \\ &\geq \frac{1}{\left\{\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)\right\}^{n}} \prod_{j=1}^{n+1}\left({}^{\rho}I^{\alpha,\beta}_{\eta,\kappa} f_j\right)(x), \end{aligned}$$

where the induction hypothesis for n is used for the second inequality. This completes the proof. □

Remark 2.3. We consider some particular cases of the result in Theorem 2.3.

(i) Setting $\kappa = 0$, $\eta = 0$ and taking the limit $\rho \to 1$ in the result in Theorem 2.3 yields the inequality in [1, Theorem 3.3].

(ii) Setting $\beta = \alpha$, $\kappa = 0$, $\eta = 0$ and taking the limit $\rho \to 0^+$ in the result in Theorem 2.3 gives the inequality in [4, Theorem 3.3].

(iii) Setting $\beta = 0$ and $\kappa = -\rho(\alpha+\eta)$ in the result in Theorem 2.3 yields the inequality in [10, Theorem 3].

(iv) Setting $\beta = \alpha$, $\kappa = 0$ and $\eta = 0$ in the result in Theorem 2.3, under the corresponding reduced assumption, we obtain

$$\left({}^\rho I^\alpha_{a+} \prod_{j=1}^n f_j\right)(x) \geq \left(\frac{\rho^\alpha \Gamma(\alpha+1)}{x^{\rho\alpha}}\right)^{n-1} \prod_{j=1}^n \left({}^\rho I^\alpha_{a+} f_j\right)(x) \quad (n \in \mathbb{N}). \tag{2.10}$$

Theorem 2.4. *Let $\beta, \kappa \in \mathbb{R}$, x, α, $\rho \in \mathbb{R}^+$, and $\eta \in \mathbb{R}^+_0$. Also, let two functions $f, g : \mathbb{R}^+_0 \to \mathbb{R}$ such that f is increasing and g is differentiable with g' bounded below, and let $m := \inf_{t\in\mathbb{R}^+_0} g'(t)$. Then*

$$\begin{aligned}\left({}^\rho I^{\alpha,\beta}_{\eta,\kappa} f g\right)(x) \geq\ & \frac{1}{\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)} \left({}^\rho I^{\alpha,\beta}_{\eta,\kappa} f\right)(x) \left({}^\rho I^{\alpha,\beta}_{\eta,\kappa} g\right)(x) \\ & - \frac{m x \Gamma(\eta+1+\frac{1}{\rho})\Gamma(\eta+\alpha+1)}{\Gamma(\eta+\alpha+1+\frac{1}{\rho})\Gamma(\eta+1)} \left({}^\rho I^{\alpha,\beta}_{\eta,\kappa} f\right)(x) + m \left({}^\rho I^{\alpha,\beta}_{\eta,\kappa} i \cdot f\right)(x),\end{aligned} \tag{2.11}$$

where $i(x) = x$ is the identity function.

Proof. Let $h(x) := g(x) - mx$ $\left(x \in \mathbb{R}^+_0\right)$. We find that h is differentiable and increasing on $\mathbb{R}^+_0$. As in the process of Theorem 2.3, for clarity, let $p(x) := mx$, we obtain

$$\begin{aligned}&\left({}^\rho I^{\alpha,\beta}_{\eta,\kappa} f(g-p)\right)(x) \\ &\quad \geq \frac{1}{\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)} \left({}^\rho I^{\alpha,\beta}_{\eta,\kappa} f\right)(x) \left({}^\rho I^{\alpha,\beta}_{\eta,\kappa} (g-p)\right)(x) \\ &\quad = \frac{1}{\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)} \left({}^\rho I^{\alpha,\beta}_{\eta,\kappa} f\right)(x) \left({}^\rho I^{\alpha,\beta}_{\eta,\kappa} g\right)(x) - \frac{1}{\Lambda^{\rho,\beta}_{x,\kappa}(\alpha,\eta)} \left({}^\rho I^{\alpha,\beta}_{\eta,\kappa} f\right)(x) \left({}^\rho I^{\alpha,\beta}_{\eta,\kappa} p\right)(x).\end{aligned} \tag{2.12}$$

By using (1.8) with the aid of (1.2), we get

$$\left({}^\rho I^{\alpha,\beta}_{\eta,\kappa} p\right)(x) = \frac{m x^{\kappa+\rho(\alpha+\eta)+1} \Gamma\left(\eta+\frac{1}{\rho}+1\right)}{\rho^\beta \Gamma\left(\alpha+\eta+\frac{1}{\rho}+1\right)}. \tag{2.13}$$

We have

$$\left({}^\rho I^{\alpha,\beta}_{\eta,\kappa} f(g-p)\right)(x) = \left({}^\rho I^{\alpha,\beta}_{\eta,\kappa} f g\right)(x) - m \left({}^\rho I^{\alpha,\beta}_{\eta,\kappa} i \cdot f\right)(x). \tag{2.14}$$

Finally, using (2.13) and (2.14) in (2.12), we obtain the result (2.11). □

Remark 2.4. We consider some particular cases of the result in Theorem 2.4.

(i) Setting $\kappa = 0$, $\eta = 0$ and taking the limit $\rho \to 1$ in the result in Theorem 2.4 yields the inequality in [1, Theorem 3.4].

(ii) Setting $\beta = \alpha$, $\kappa = 0$ and $\eta = 0$ in the result in Theorem 2.4, under the corresponding reduced assumption, we obtain

$$\begin{aligned}\left({}^{\rho} I^{\alpha}_{a+} f g\right)(x) \geq & \frac{\rho^{\alpha}\Gamma(\alpha+1)}{x^{\rho\alpha}} \left({}^{\rho} I^{\alpha}_{a+} f\right)(x) \left({}^{\rho} I^{\alpha}_{a+} g\right)(x) \\ & - \frac{m x \Gamma(1+\frac{1}{\rho})\Gamma(\alpha+1)}{\Gamma(\alpha+1+\frac{1}{\rho})} \left({}^{\rho} I^{\alpha}_{a+} f\right)(x) + m \left({}^{\rho} I^{\alpha}_{a+} i \cdot f\right)(x). \end{aligned} \tag{2.15}$$

3. Chebyshev type inequality without synchronous functions

Here, we establish a Chebyshev type inequality involving the fractional integral operator (1.8) without synchronous functions. To do this, we begin with the following lemma.

Lemma 3.1. *Let $\beta, \kappa \in \mathbb{R}$, $\alpha \geq 1$, $\rho \in \mathbb{R}^{+}$, and $\eta \in \mathbb{R}^{+}_{0}$. Also, let $f, g : [a,b] \to \mathbb{R}$ be functions such that f is differentiable and g is integrable on $[a,b]$, and $x \in (a,b]$. Then*

$$\left({}^{\rho} I^{\alpha,\beta}_{\eta,\kappa} f g\right)(x) = \frac{1}{x-a}\int_a^x f(s)\,ds \cdot \left({}^{\rho} I^{\alpha,\beta}_{\eta,\kappa} g\right)(x) + \frac{\rho^{1-\beta} x^{\kappa}}{\Gamma(\alpha)} \int_a^x \mathscr{H}^{a,x}_{\alpha,\eta,\rho}(\tau)\,d\tau, \tag{3.1}$$

where

$$\begin{aligned}\mathscr{H}^{a,x}_{\alpha,\eta,\rho}(\tau) := & \left\{ f(\tau) - \frac{1}{\tau-a}\int_a^{\tau} f(s)\,ds \right\} \\ & \times \left\{ \frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}} g(\tau) - \frac{1}{\tau-a}\int_a^{\tau} \frac{s^{\rho(\eta+1)-1}}{(x^{\rho}-s^{\rho})^{1-\alpha}} g(s)\,ds \right\}. \end{aligned} \tag{3.2}$$

Proof. Integrating by parts, we have

$$\begin{aligned} & \int_a^x f(\tau) \frac{\tau^{\rho(\eta+1)-1}}{(x^{\rho}-\tau^{\rho})^{1-\alpha}} g(\tau)\,d\tau \\ & = \left(f(\tau) \int_a^{\tau} \frac{s^{\rho(\eta+1)-1}}{(x^{\rho}-s^{\rho})^{1-\alpha}} g(s)\,ds \right)\Bigg|_a^x - \int_a^x \left(f^{'}(\tau) \int_a^{\tau} \frac{s^{\rho(\eta+1)-1}}{(x^{\rho}-s^{\rho})^{1-\alpha}} g(s)\,ds \right) d\tau \\ & = f(x) \int_a^x \frac{s^{\rho(\eta+1)-1}}{(x^{\rho}-s^{\rho})^{1-\alpha}} g(s)\,ds - \int_a^x u(\tau)\,v'(\tau)\,d\tau, \end{aligned} \tag{3.3}$$

where

$$u(\tau) := \frac{1}{\tau-a}\int_a^{\tau} \frac{s^{\rho(\eta+1)-1}}{(x^{\rho}-s^{\rho})^{1-\alpha}} g(s)\,ds \quad \text{and} \quad v'(\tau) := (\tau-a) f^{'}(\tau). \tag{3.4}$$

We find

$$u^{'}(\tau) = -\frac{1}{(\tau-a)^2}\int_a^{\tau} \frac{s^{\rho(\eta+1)-1}}{(x^{\rho}-s^{\rho})^{1-\alpha}} g(s)\,ds + \frac{\tau^{\rho(\eta+1)-1}}{(\tau-a)(x^{\rho}-\tau^{\rho})^{1-\alpha}} g(\tau) \tag{3.5}$$

and

$$v(\tau)=\int_a^\tau (s-a)\,f^{'}(s)\,ds=(\tau-a)\,f(\tau)-\int_a^\tau f(s)\,ds. \tag{3.6}$$

We have

$$\int_a^x u(\tau)\,v'(\tau)\,d\tau=u(\tau)\,v(\tau)\Big|_a^x-\int_a^x u'(\tau)\,v(\tau)\,d\tau. \tag{3.7}$$

Setting (3.4), (3.5), and (3.6) in (3.7), we get

$$\begin{aligned}\int_a^x u(\tau)\,v'(\tau)\,d\tau &= \int_a^x \frac{s^{\rho(\eta+1)-1}}{(x^\rho-s^\rho)^{1-\alpha}}g(s)\,ds\left\{f(x)-\frac{1}{x-a}\int_a^x f(s)\,ds\right\}\\ &\quad+\int_a^x\left\{\frac{1}{\tau-a}\int_a^\tau \frac{s^{\rho(\eta+1)-1}}{(x^\rho-s^\rho)^{1-\alpha}}g(s)\,ds-\frac{\tau^{\rho(\eta+1)-1}}{(x^\rho-\tau^\rho)^{1-\alpha}}\,g(\tau)\right\}\\ &\quad\times\left\{f(\tau)-\frac{1}{\tau-a}\int_a^\tau f(s)\,ds\right\}d\tau. \end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.3) and multiplying both sides of the resulting identity by $\frac{\rho^{1-\beta}x^\kappa}{\Gamma(\alpha)}$, and using (1.8), we obtain the desired result (3.1). □

Theorem 3.1. *Let $\beta,\kappa\in\mathbb{R}$, $\alpha\geq 1$, $\rho\in\mathbb{R}^+$, and $\eta\in\mathbb{R}_0^+$. Also, let $f,\,g:[a,b]\to\mathbb{R}$ be functions such that f is differentiable and g is integrable on $[a,b]$. Further, let $\mathscr{H}^{\alpha,x}_{\alpha,\eta,\rho}(\tau)\geq 0$ $(\tau\in(a,b])$ and $x\in(a,b]$. Then*

$$\frac{1}{x-a}\left({}^\rho I^{\alpha,\beta}_{\eta,\kappa}fg\right)(x)\geq\left\{\frac{1}{x-a}\int_a^x f(s)\,ds\right\}\left\{\frac{1}{x-a}\left({}^\rho I^{\alpha,\beta}_{\eta,\kappa}g\right)(x)\right\}. \tag{3.9}$$

Proof. Obviously, the result here follows from Lemma 3.1. □

Remark 3.1. We consider some particular cases of the result in Theorem 3.1.

(i) Setting $\kappa=0$, $\eta=0$ and taking the limit $\rho\to 1$ in the result in Theorem 3.1 yields the inequality in [5, Theorem 3.1].

(ii) Setting $\beta=0$ and $\kappa=-\rho(\alpha+\eta)$ in the result in Theorem 3.1, under the corresponding reduced assumption, we obtain

$$\frac{1}{x-a}\left(I^\alpha_{a+,\sigma,\eta}fg\right)(x)\geq\left\{\frac{1}{x-a}\int_a^x f(s)\,ds\right\}\left\{\frac{1}{x-a}\left(I^\alpha_{a+\sigma,\eta}g\right)(x)\right\}. \tag{3.10}$$

(iii) Setting $\beta=\alpha$, $\kappa=0$, $\eta=0$ in the result in Theorem 3.1 and taking the limit $\rho\to 0+$ with the aid of L'Hôpital's rule, we get

$$\frac{1}{x-a}\left(H^\alpha_{a+}fg\right)(x)\geq\left\{\frac{1}{x-a}\int_a^x f(s)\,ds\right\}\left\{\frac{1}{x-a}\left(H^\alpha_{a+}g\right)(x)\right\}. \tag{3.11}$$

(iv) Setting $\beta=\alpha$, $\kappa=0$ and $\eta=0$ in the result in Theorem 3.1, under the corresponding reduced assumption, we obtain

$$\frac{1}{x-a}\left({}^\rho I^\alpha_{a+}fg\right)(x)\geq\left\{\frac{1}{x-a}\int_a^x f(s)\,ds\right\}\left\{\frac{1}{x-a}\left({}^\rho I^\alpha_{a+}g\right)(x)\right\}. \tag{3.12}$$

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