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A LINK BETWEEN HARMONICITY OF 2-DISTANCE FUNCTIONS AND INCOMPRESSIBILITY OF CANONICAL VECTOR FIELDS

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Abstract. Let M be a Riemannian submanifold of a Riemannian manifold $\tilde{M}$ equipped with a concurrent vector field $\tilde{Z}$. Let Z denote the restriction of $\tilde{Z}$ along M and let Z^T be the tangential component of Z on M, called the canonical vector field of M. The 2-distance function δ_Z^2 of M (associated with Z) is defined by $\delta_Z^2 = \langle Z, Z\rangle$.

In this article, we initiate the study of submanifolds M of $\tilde{M}$ with incompressible canonical vector field Z^T arisen from a concurrent vector field $\tilde{Z}$ on the ambient space $\tilde{M}$. First, we derive some necessary and sufficient conditions for such canonical vector fields to be incompressible. In particular, we prove that the 2-distance function δ_Z^2 is harmonic if and only if the canonical vector field Z^T on M is an incompressible vector field. Then we provide some applications of our main results.

1. Incompressible vector fields

In *fluid mechanics*, many liquids are hard to compress (i.e., their volumes or densities don't change much when you squeeze them), so that the density ρ is essentially a constant. For such an incompressible fluid the equation of continuity simplifies to the divergence of the flow velocity v is zero, i.e.,

$$\mathrm{div}(v) = 0 \quad \text{(incompressible)}, \tag{1.1}$$

so that the velocity field[1] v is an incompressible vector field (also known as a solenoidal vector field or a divergence-free vector field). This condition is analogous to the condition $\mathrm{div}(B) = 0$ *in electromagnetism* that the magnetic field B has zero divergence.

It is well-known that incompressible vector fields are important in magnetohydrodynamics. Moreover, magnetic fields are widely used throughout modern technology, particularly in electrical engineering and electromechanics (cf. e.g., [1, 15, 16]).

Based on the reasons mentioned above, one has the following.

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[1]All vector fields, functions, immersions and manifolds are assumed to be smooth.

Definition 1.1. A vector field X on a Riemannian manifold M is called *incompressible* if the divergence of X is zero, i.e., $\operatorname{div}(X) = 0$.

Let $\phi : M \to \tilde{M}$ be an isometric immersion of a Riemannian manifold M into another Riemannian manifold $\tilde{M}$. Denote by $\langle\ ,\ \rangle$ the inner product of M as well as of $\tilde{M}$. Assume that $\tilde{Y}$ is a vector field of $\tilde{M}$. Denote by Y the restriction of $\tilde{Y}$ along M. Then Y admits an orthogonal decomposition:

$$Y = Y^T + Y^\perp, \tag{1.2}$$

where Y^T and $Y^\perp$ are the tangential and the normal components of Y, respectively. The tangent vector field Y^T of M is called the *canonical vector field* of M associated with Y.

For a submanifold M of a Euclidean space $\mathbb{E}^m$, the most elementary and natural vector field on M is the position vector field $\mathbf{x}$. The tangential component $\mathbf{x}^T$ of $\mathbf{x}$ is simply called the *canonical vector field* of M [11, 12]. It is well-known that the position vector field of $\mathbb{E}^m$ is a concurrent vector field (see Definition 2.2 and Example 2.1).

In earlier articles, we have investigated Euclidean submanifolds whose canonical vector fields are concurrent [6, 8], concircular [14], torse-forming [13], conformal [12], or incompressible [11]. (See also recent surveys [9, 10] for several topics on position vector fields on Euclidean submanifolds.)

In this article, we initiate the investigation of submanifolds M of $\tilde{M}$ with incompressible canonical vector field Z^T arisen from a concurrent vector field $\tilde{Z}$ on the ambient space $\tilde{M}$. First, we derive some necessary and sufficient conditions for such canonical vector fields to be incompressible. In particular, we prove that the 2-distance function δ_Z^2 is harmonic if and only if the canonical vector field Z^T on M is an incompressible vector field. Then we provide some applications of our main results.

2. Preliminaries

Let $\phi : M \to \tilde{M}$ be an isometric immersion of a connected Riemannian n-manifold M into a Riemannian m-manifold $\tilde{M}$. For each point $p \in M$, we denote by T_pM and $T_p^\perp M$ the tangent space and the normal space of M at p, respectively. Let ∇ and $\tilde{\nabla}$ denote the Levi–Civita connections of M and $\mathbb{E}^m$, respectively.

The formula of Gauss and the formula of Weingarten are then given respectively by (cf. [3, 4, 7])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.2}$$

for vector fields X, Y tangent to M and ξ normal to M, where h denotes the second fundamental form, D is the normal connection and A is the shape operator of M.

For each normal vector ξ at p, the shape operator A_ξ is a self-adjoint endomorphism of T_pM. The second fundamental form h and the shape operator A are related by

$$\langle A_\xi X, Y\rangle = \langle h(X,Y),\xi\rangle. \tag{2.3}$$

The mean curvature vector field H of an n-dimensional submanifold M is defined by

$$H = \left(\frac{1}{n}\right)\text{trace}\, h. \tag{2.4}$$

Let $\{e_1,\dots,e_n\}$ be an orthonormal frame on M, then the *divergence* of a vector field X on M, denoted by $\text{div}(X)$, is defined by

$$\text{div}(X) = \sum_{j=1}^{n} \langle \nabla_{e_i} X, e_i\rangle. \tag{2.5}$$

The *gradient* ∇f of a function f on M is defined by

$$\langle \nabla f, Y\rangle = Yf$$

for any vector Y tangent to M. Hence, in terms of an orthonormal frame $\{e_1,\dots,e_n\}$ on M, we have

$$\nabla f = \sum_{i=1}^{n} (e_i f) e_i. \tag{2.6}$$

And the *Laplacian* Δ of M acting on a function f on M is given by

$$\Delta f = -\sum_{i=1}^{n} \left\{ e_i e_i(f) - \nabla_{e_i} e_i(f)\right\}. \tag{2.7}$$

Now, we present some basic definitions for later use.

Definition 2.2. A vector field $\tilde{Z}$ on a Riemannian manifold $\tilde{M}$ is called a *concurrent vector field* if it satisfies

$$\tilde{\nabla}_X \tilde{Z} = X \tag{2.8}$$

for all vectors X tangent to $\tilde{M}$, where $\tilde{\nabla}$ denotes the Levi-Civita connection of $\tilde{M}$ (cf. [19, 20])

Concurrent vector fields play some important roles in differential geometry and mathematical physics. For instance, it was proved in [19] that if the holonomy group of a Riemannian manifold $\tilde{M}$ leaves a point invariant, then $\tilde{M}$ admits a concurrent vector field. Concurrent vector fields have also been studied in Finsler geometry since the beginning of 1950s (cf. [17, 18]).

The simplest example of Riemannian manifold with a concurrent vector field is a Euclidean space.

Example 2.1. The position vector field $\mathbf{x}$ of the Euclidean m-space $\mathbb{E}^m$ is a concurrent vector field.

Definition 2.3. Let B and F be two Riemannian manifolds of positive dimensions equipped with metrics g_B and g_F, respectively, and let f be a positive smooth function on B.

The *warped product* $M = B \times_f F$ is the product manifold $B \times F$ equipped with the warped product metric

$$g = g_B + f^2 g_F. \tag{2.9}$$

The function f is called the *warping function* of the warped product (cf. [2, 11]).

For a warped product $B \times_f F$, B is called the *base* and F the *fiber*. The leaves $B \times \{q\} = \eta^{-1}(q)$, $q \in F$, and the fibers $\{b\} \times F = \pi^{-1}(p)$, $b \in B$ are Riemannian submanifolds of $B \times_f F$.

Example 2.2. It is direct to verify that $\mathbb{E}^m_* = \mathbb{E}^m - \{0\} \subset \mathbb{E}^m$ can be regarded as the warped product $\mathbf{R}^+ \times_s S^{m-1}$ equipped with the warped product metric

$$g = ds^2 + s^2 g_S,$$

where g_S is the metric tensor of the unit $(m-1)$-sphere S^{m-1}. In this case, the position vector field $\mathbf{x}$ of $\mathbb{E}^m_*$ is given by $s\frac{\partial}{\partial s}$.

The distance function δ from the origin $o \in \mathbb{E}^m$ to a point of $\mathbb{E}^m$ is given by

$$\delta = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}.$$

Example 2.3. Let F be any Riemannian manifold and let $I = (a, b)$ be an open interval with $0 \notin I$. Consider the warped product $I \times_s F$ equipped with the warped product metric

$$\tilde{g} = ds^2 + s^2 g_F, \tag{2.10}$$

where g_F denotes the Riemannian metric of F. Then the vector field $\tilde{Z} = s\frac{\partial}{\partial s}$ is a concurrent vector field on $I \times_s F$ (cf. Example 3.1 of [5]). Moreover, in this case the vector field $\tilde{Z} = s\frac{\partial}{\partial s}$ can be considered as the radial vector field of $I \times_s F$.

3. Theorems

Now, we define the notion of r-distance function on a submanifold M of a Riemannian manifold $\tilde{M}$ equipped with a concurrent vector field as follows.

Definition 3.4. Let M be a submanifold of a Riemannian manifold $\tilde{M}$ equipped with a concurrent vector field $\tilde{Z}$. Denote by Z the restriction of $\tilde{Z}$ on M. Then the function

$$\delta_Z^r(p) = |Z_p|^r = \langle Z_p, Z_p\rangle^{r/2}$$

is called the *r-distance function* (associated with Z) (or simply the *r-distance function* if there is no confusion arisen).

Lemma 3.1. *Let M be a submanifold of a Riemannian manifold $\tilde{M}$ equipped with a concurrent vector field $\tilde{Z}$ on $\tilde{M}$. Then the corresponding canonical vector field Z^T and the gradient of the 2-distance function δ_Z^2 of M are related by*

$$Z^T = \frac{1}{2}\nabla\delta_Z^2. \tag{3.1}$$

Proof. Let M be a submanifold of a Riemannian manifold $\tilde{M}$ equipped with a concurrent vector field $\tilde{Z}$ on $\tilde{M}$. Then the 2-distance function δ_Z^2 of M is given by

$$\delta_Z^2 = \langle Z, Z\rangle, \tag{3.2}$$

where Z is the restriction of $\tilde{Z}$ on M.

Let $\{e_1,\ldots,e_n\}$ be an orthonormal local frame on M. Then it follows from (2.6), (2.8) and (3.2) that

$$\begin{aligned}\nabla\delta_Z^2 &= \sum_{i=1}^n (e_i\langle Z,Z\rangle)e_i = 2\sum_{i=1}^n \langle\tilde{\nabla}_{e_i}Z, Z\rangle e_i \\ &= 2n\sum_{i=1}^n \langle e_i, Z\rangle e_i = 2Z^T,\end{aligned}$$

which proves (3.1). ☐

The next result provides a simple characterization of an incompressible canonical vector field on a submanifold arisen from a concurrent vector field on its ambient space.

Theorem 3.1. *Let M be a submanifold of a Riemannian manifold $\tilde{M}$ with a concurrent vector field $\tilde{Z}$ on $\tilde{M}$. Then the canonical vector field Z^T on M is incompressible if and only if the mean curvature vector field H of M in $\tilde{M}$ satisfies*

$$\langle H, Z\rangle = -1 \tag{3.3}$$

identically.

Proof. Let M be a submanifold of a Riemannian manifold $\tilde{M}$ equipped with a concurrent vector field $\tilde{Z}$ on $\tilde{M}$. Then, according to Definition 3.4, the canonical vector field Z^T is the tangential component of the restriction Z of the concurrent vector field $\tilde{Z}$ along M.

Now, let us compute the divergence $\operatorname{div}(Z^T)$. It follows from (2.1), (2.4), (2.5) and Lemma 3.1 that

$$\begin{aligned}\operatorname{div}(Z^T) &= \frac{1}{2}\sum_{i=1}^{n}\langle\nabla_{e_i}\nabla\delta_Z^2, e_i\rangle = \sum_{i,j=1}^{n}\langle\nabla_{e_i}(\langle e_j, Z\rangle e_j), e_i\rangle \\ &= \sum_{i,j=1}^{n}\left(\langle\tilde{\nabla}_{e_i}e_j, Z\rangle\langle e_j, e_i\rangle + \langle e_j, e_i\rangle^2 + \langle e_j, Z\rangle\langle\nabla_{e_i}e_j, e_i\rangle\right) \\ &= n(1+\langle H, Z\rangle) + \sum_{i,j=1}^{n}\left(\langle\nabla_{e_i}e_j, Z\rangle\langle e_j, e_i\rangle + \langle e_j, Z\rangle\langle\nabla_{e_i}e_j, e_i\rangle\right). \end{aligned} \tag{3.4}$$

Let us put

$$\nabla_X e_i = \sum_{k=1}^{n}\omega_i^k(X)e_k \tag{3.5}$$

for tangent vectors X of M. Then we find from the fact that ∇ is a metric connection that

$$\omega_i^k = -\omega_k^i \tag{3.6}$$

for $1 \le i, k \le n$.

From (3.5) and (3.6) we obtain

$$\begin{aligned}&\sum_{i,j=1}^{n}\left(\langle\nabla_{e_i}e_j, Z\rangle\langle e_j, e_i\rangle + \langle e_j, Z\rangle\langle\nabla_{e_i}e_j, e_i\rangle\right) \\ &= \sum_{i,k=1}^{n}\omega_i^k(e_i)\langle e_k, \mathbf{x}\rangle + \sum_{i,j=1}^{n}\omega_j^i(e_i)\langle e_j, Z\rangle \\ &= 0. \end{aligned} \tag{3.7}$$

Therefore, after combining (3.4) and (3.7) we have

$$\operatorname{div}(Z^T) = n\{1+\langle H, Z\rangle\}.$$

Consequently, the canonical vector field Z^T is incompressible if and only if $\langle H, Z\rangle = -1$ holds identically. ☐

Remark 3.1. Lemma 3.1 and Theorem (3.1) generalize statement (a) and statement (b) Theorem 3.1 of [11], respectively.

The next result is the **main theorem** of this article. This main theorem provides a very simple link between harmonicity of the 2-distance function δ_Z^2 and the incompressibility of the canonical vector field Z^T.

Theorem 3.2. *Let M be a submanifold of a Riemannian manifold $\tilde{M}$ with a concurrent vector field $\tilde{Z}$. Then the 2-distance function δ_Z^2 is harmonic if and only if the canonical vector field Z^T is incompressible.*

Proof. Let M be a submanifold of a Riemannian manifold $\tilde{M}$. Assume that $\tilde{M}$ admits a concurrent vector field $\tilde{Z}$. Let us compute the Laplacian of the 2-distance function δ_Z^2 of M as follows.

$$\begin{aligned}\Delta\delta_Z^2 &= -\sum_{i=1}^n e_i e_i(\delta_Z^2) + \sum_{i=1}^n \nabla_{e_i} e_i(\delta_Z^2)\\ &= -2\sum_{i=1}^n e_i\langle e_i, Z\rangle + 2\sum_{i=1}^n \langle\nabla_{e_i} e_i Z, Z\rangle\\ &= -2\sum_{i=1}^n \langle\tilde{\nabla}_{e_i} e_i, Z\rangle - 2n + 2\sum_{i=1}^n \langle\nabla_{e_i} e_i Z, Z\rangle\\ &= -2\sum_{i=1}^n \langle h(e_i, e_i), Z\rangle - 2n\\ &= -2n\{\langle H, Z\rangle + 1\}.\end{aligned} \tag{3.8}$$

Now, by combining (3.8) and Theorem 3.1 we obtain the theorem. □

For a Euclidean submanifold M, if we denote the tangential component of the position vector field $\mathbf{x}$ of M by $\mathbf{x}^T$, then $\mathbf{x}^T$ is the canonical vector field of the Euclidean submanifold M.

For Euclidean submanifolds, Theorem 3.2 yields the following.

Theorem 3.3. *Let M be an arbitrary Euclidean submanifold M of $\mathbb{E}^m$. Then the canonical vector field $\mathbf{x}^T$ of M is incompressible if and only if the 2-distance function $\delta^2 = \langle\mathbf{x}, \mathbf{x}\rangle$ of M is a harmonic function.*

Proof. This is an immediate consequence of Theorem 3.2 since the position vector field $\mathbf{x}$ is a concurrent vector field on $\mathbb{E}^m$. □

4. Some applications

In this section we make the following.

Assumption. *Let M be a submanifold of the warped product $\tilde{M} = I \times_s F$. We consider the canonical concurrent vector field $\tilde{Z} = s\frac{\partial}{\partial s}$ on $I \times_s F$.*

Now, we provide the following applications of Theorems 3.1–3.3.

Corollary 4.1. *Let $\tilde{M} = I \times_s F$ be a warped product with warped product metric $\tilde{g} = ds^2 + s^2 g_F$. Then, for every submanifold B of F, the canonical vector field Z^T of $I \times_s B$ is never incompressible.*

Proof. Under the hypothesis, the restriction Z of $\tilde{Z}$ on $I \times_s B$ is always tangent to $I \times_s B$, i.e., $Z^\perp = 0$. Therefore, condition (2.1) never holds at each point. Consequently, the canonical vector field Z^T of $I \times_s B$ is never incompressible according to Theorem 3.1. □

Corollary 4.2. *Let $\tilde{M} = I \times_s F$ be a warped product with warped product metric $\tilde{g} = ds^2 + s^2 g_F$. Then the canonical vector field Z^T of every fiber $\{s_o\} \times_s F$ in $I \times_s F$ is always incompressible.*

Proof. Let M be a submanifold of the warped product $\tilde{M} = I \times_s F$ endowed with a concurrent vector field $\tilde{Z} = s\frac{\partial}{\partial s}$. Then the 2-distance function of M is given by $\delta_Z^2 = s^2$.

Suppose that M is a fiber of $I \times_s F$ defined by $\{s_o\} \times F$. Then the 2-distance function δ_Z^2 is the constant s_o^2. Hence it is a harmonic function trivially. Consequently, Theorem 3.2 implies that the canonical vector field Z^T is always incompressible. □

Corollary 4.3. *Let $\tilde{M} = I \times_s S^{m-1}$ be the warped product of $I = (0, \infty)$ and the unit $(m-1)$-sphere S^{m-1} equipped with the warped product metric $\tilde{g} = ds^2 + s^2 g_S$. Consider the canonical concurrent vector field $\tilde{Z} = s\frac{\partial}{\partial s}$ on $I \times_s S^{m-1}$. Then, for any map $\gamma : I \to S^{m-1}$, the curve defined by*

$$\psi : I \to I \times_s S^{m-1}; I \ni s \mapsto (\sqrt{1+2s}, \gamma(s)) \in I \times_s S^{m-1} \tag{4.1}$$

has incompressible canonical vector field Z^T.

Proof. Under the hypothesis, the 2-distance function δ_Z^2 of the curve ψ given by (4.1) is $\delta_Z^2 = 1 + 2s$, which is a harmonic function. Consequently, the canonical vector field Z^T is incompressible according to Theorem 3.3. □

Example 4.1. Consider the map $\gamma : I \to S^1$, $I = (0, \infty)$, defined by

$$\gamma(s) = \left(\frac{\cos\sqrt{2s} + \sqrt{2s}\sin\sqrt{2s}}{\sqrt{1+2s}}, \frac{\sin\sqrt{2s} - \sqrt{2s}\cos\sqrt{2s}}{\sqrt{1+2s}} \right). \tag{4.2}$$

Then the curve ψ in (4.1) of Corollary 4.3 is given by

$$\psi(s) = \left(\sqrt{1+2s}, \gamma(s) \right) \in I \times_s S^1. \tag{4.3}$$

Therefore, according to Corollary 4.3, the canonical vector field $\mathbf{x}^T = Z^T$ is an incompressible vector field.

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