ON THE BOUNDEDNESS OF A GENERALIZED FRACTIONAL INTEGRAL ON GENERALIZED MORREY SPACES

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Abstract. In this paper we extend Nakai’s result on the boundedness of a generalized fractional integral operator from a generalized Morrey space to another generalized Morrey or Campanato space.

1. Introduction and Main Results

For a given function \( \rho : (0, \infty) \to (0, \infty) \), let \( \mathcal{T}_\rho \) be the generalized fractional integral operator, given by

\[
\mathcal{T}_\rho f(x) = \int_{\mathbb{R}^n} \frac{f(y)\rho(|x - y|)}{|x - y|^n} dy,
\]

and put

\[
\tilde{\mathcal{T}}_\rho f(x) = \int_{\mathbb{R}^n} f(y) \left( \frac{\rho(|x - y|)}{|x - y|^n} - \frac{\rho(|y|)(1 - \chi_{B_0}(y))}{|y|^n} \right) dy,
\]

the modified version of \( \mathcal{T}_\rho \), where \( B_0 \) is the unit ball about the origin, and \( \chi_{B_0} \) is the characteristic function of \( B_0 \).

In [4], Nakai proved the boundedness of the operators \( \tilde{\mathcal{T}}_\rho \) and \( \mathcal{T}_\rho \) from a generalized Morrey space \( \mathcal{M}_{1, \psi} \) to another generalized Morrey space \( \mathcal{M}_{1, \psi} \) or generalized Campanato space \( \mathcal{L}_{1, \psi} \). More precisely, he proved that

\[
\|\mathcal{T}_\rho f\|_{\mathcal{M}_{1, \psi}} \leq C \|f\|_{\mathcal{M}_{1, \phi}} \quad \text{and} \quad \|\tilde{\mathcal{T}}_\rho f\|_{\mathcal{L}_{1, \psi}} \leq C \|f\|_{\mathcal{M}_{1, \phi}},
\]

where \( C > 0 \), with some appropriate conditions on \( \rho, \phi \) and \( \psi \). Using the techniques developed by Kurata et al.[1], we investigate the boundedness of these operators from generalized Morrey spaces \( \mathcal{M}_{p, \phi} \) to generalized Morrey spaces \( \mathcal{M}_{p, \psi} \) or generalized Campanato spaces \( \mathcal{L}_{p, \psi} \) for \( 1 < p < \infty \).

The generalized Morrey and Campanato spaces are defined as follows. For a given function \( \phi : (0, \infty) \to (0, \infty) \), and \( 1 < p < \infty \), let

\[
\|f\|_{\mathcal{M}_{p, \phi}} = \sup_B \frac{1}{\phi(B)} \left( \frac{1}{|B|} \int_B |f(y)|^p dy \right)^{\frac{1}{p}},
\]
and
\[
\|f\|_{L_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left( \frac{1}{|B|} \int_B |f(y) - f_B|^p \, dy \right)^{\frac{1}{p}},
\]
where the supremum is taken over all open balls $B = B(a, r)$ in $\mathbb{R}^n$, $|B|$ is the Lebesgue measure of $B$ in $\mathbb{R}^n$, $\phi(B) = \phi(r)$, and $f_B = \frac{1}{|B|} \int_B f(y) \, dy$. We define the generalized Morrey space $\mathcal{M}_{p, \phi}$ by
\[
\mathcal{M}_{p, \phi} = \{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{M}_{p, \phi}} < \infty \},
\]
and the generalized Campanato space $\mathcal{L}_{p, \phi}$ by
\[
\mathcal{L}_{p, \phi} = \{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{L}_{p, \phi}} < \infty \}.
\]

Our results are the following:

**Theorem 1.** If $\rho, \phi, \psi : (0, \infty) \to (0, \infty)$ satisfying the conditions below:
\[
\frac{1}{2} \leq \frac{t}{r} \leq 2 \Rightarrow \frac{1}{A_1} \leq \frac{\rho(t)}{\phi(r)} \leq A_1, \text{ and } \frac{1}{A_2} \leq \frac{\rho(t)}{\phi(r)} \leq A_2,
\]
\[
\int_0^1 \frac{\rho(t)}{t} \, dt < \infty, \text{ and for all } r > 0, \text{ we have } \int_r^\infty \frac{\phi(t)^p}{t} \, dt \leq A_3 \phi(r)^p,
\]
\[
\phi(r) \int_0^r \frac{\rho(t)}{t} \, dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t} \, dt \leq A_4\psi(r), \text{ for all } r > 0,
\]
where $A_i > 0$ are independent of $t, r > 0$, then for each $1 < p < \infty$ there exists $C_p > 0$ such that
\[
\|T_p f\|_{\mathcal{L}_{p, \phi}} \leq C_p \|f\|_{\mathcal{M}_{p, \phi}}.
\]

**Theorem 2.** If $\rho, \phi, \psi : (0, \infty) \to (0, \infty)$ satisfying the conditions below:
\[
\frac{1}{2} \leq \frac{t}{r} \leq 2 \Rightarrow \frac{1}{A_1} \leq \frac{\rho(t)}{\phi(r)} \leq A_1, \text{ and } \frac{1}{A_2} \leq \frac{\rho(t)}{\phi(r)} \leq A_2,
\]
\[
\int_0^1 \frac{\rho(t)}{t} \, dt < \infty, \text{ and for all } r > 0, \text{ we have } \int_r^\infty \frac{\phi(t)^p}{t} \, dt \leq A_3 \phi(r)^p,
\]
\[
\left| \frac{\rho(r)}{r^n} - \frac{\rho(t)}{t^n} \right| \leq A_4 |r - t|^{\frac{1}{r^n+1}}, \text{ for } \frac{1}{2} \leq \frac{t}{r} \leq 2,
\]
\[
\phi(r) \int_0^r \frac{\rho(t)}{t} \, dt + r \int_r^\infty \frac{\rho(t)\phi(t)}{t^2} \, dt \leq A_5\psi(r), \text{ for all } r > 0,
\]
where $A_i > 0$ are independent of $t, r > 0$, then for each $1 < p < \infty$ there exists $C_p > 0$ such that
\[
\|T_p f\|_{L_{p, \phi}} \leq C_p \|f\|_{\mathcal{M}_{p, \phi}}.
\]
2. Proof of the Theorems

To prove the theorems, we shall use the following result of Nakai [2] (in a slightly modified version) about the boundedness of the standard maximal function \( Mf \) on a generalized Morrey space \( M_{p,\phi} \). The standard maximal function \( Mf \) is defined by

\[
Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \, dy, \quad x \in \mathbb{R}^n,
\]

where the supremum is taken over all open balls \( B \) containing \( x \).

**Theorem (Nakai).** If \( \phi : (0, \infty) \to (0, \infty) \) satisfying the conditions below:

(a) \( \frac{t}{2} \leq \phi(t) \leq 2 \Rightarrow \frac{1}{t} \leq \frac{\phi(t)}{\phi(1)} \leq A_1, \)

(b) \( \int_r^\infty \frac{\phi(t)}{t} \, dt \leq A_2 \phi(r) \), for all \( r > 0, \)

where \( A_i > 0 \) are independent of \( t, r > 0 \), then for each \( 1 < p < \infty \) there exists \( C_p > 0 \) such that

\[
\|Mf\|_{M_{p,\phi}} \leq C_p \|f\|_{M_{p,\phi}}.
\]

From now on, \( C \) and \( C_p \) will denote positive constants, which may vary from line to line. In general, these constants depend on \( n \).

**Proof of Theorem 1.** For \( x \in \mathbb{R}^n \), and \( r > 0 \), write

\[
T_p f(x) = \int_{|x-y|<r} \frac{f(y)\rho(|x-y|)}{|x-y|^p} \, dy + \int_{|x-y|\geq r} \frac{f(y)\rho(|x-y|)}{|x-y|^p} \, dy = I_1(x) + I_2(x).
\]

Note that, for \( t \in [2^kr, 2^{k+1}r] \), there exist constants \( C_t > 0 \) such that

\[
\rho(2^kr) \leq C_t \int_{2^kr}^{2^{k+1}r} \frac{\rho(t)}{t} \, dt
\]

and

\[
\rho(2^kr)\phi(2^kr) \leq C_t \int_{2^kr}^{2^{k+1}r} \frac{\rho(t)\phi(t)}{t} \, dt.
\]

So, we have

\[
|I_1(x)| \leq \int_{|x-y|<r} \frac{|f(y)|\rho(|x-y|)}{|x-y|^p} \, dy
\]

\[
\leq \sum_{k=-\infty}^{-1} \int_{2^{k}r \leq |x-y| < 2^{k+1}r} \frac{|f(y)|\rho(|x-y|)}{|x-y|^p} \, dy
\]

\[
\leq C \sum_{k=-\infty}^{-1} \frac{\rho(2^kr)}{(2^kr)^n} \int_{x-y \leq 2^{k+1}r} |f(y)| \, dy
\]
\[ \leq C \sum_{k=-\infty}^{-1} \rho(2^k r) MF(x) \]
\[ \leq CMf(x) \sum_{k=-\infty}^{-1} \int_{2^k r}^{2^{k+1} r} \frac{\rho(t)}{t} \, dy \]
\[ \leq CMf(x) \int_{0}^{r} \frac{\rho(t)}{t} \, dy \]
\[ \leq C \frac{\psi(r)}{\phi(r)} MF(x). \]

Meanwhile,
\[ |I_2(x)| \leq \int_{|x-y| \geq r} \frac{|f(y)| |\rho(|x-y|)|}{|x-y|^n} \, dy \]
\[ \leq \sum_{k=0}^{\infty} \int_{2^k r \leq |x-y| < 2^{k+1} r} \frac{|f(y)| |\rho(|x-y|)|}{|x-y|^n} \, dy \]
\[ \leq C \sum_{k=0}^{\infty} \frac{\rho(2^{k+1} r)}{(2^k r)^n} \int_{|x-y| < 2^{k+1} r} |f(y)| \, dy \]
\[ \leq C \sum_{k=0}^{\infty} \frac{\rho(2^{k+1} r)}{\phi(2^{k+1} r)} \|f\|_{\Lambda^\infty} \]
\[ \leq C \|f\|_{\Lambda^\infty} \sum_{k=0}^{\infty} \int_{2^k r}^{2^{k+2} r} \phi(t) \rho(t) \, dt \]
\[ \leq C \|f\|_{\Lambda^\infty} \int_{r}^{\infty} \frac{\phi(t) \rho(t)}{t} \, dt \]
\[ \leq C \psi(r) \|f\|_{\Lambda^\infty}. \]

Now, for \( 1 \leq p < \infty \), we have
\[ |I_\partial f(x)|^p \leq 2^{p-1} (|I_1(x)|^p + |I_2(x)|^p), \]
and by Nakai’s Theorem, we have for all balls \( B = B(a,r) \)
\[ \frac{1}{\psi(p) |B|} \int_B |I_1(x)|^p \, dx \leq \frac{C}{\phi(p) r |B|} \int_B MF(x)^p \, dx \leq C \|MF\|_{\Lambda^p}^p \leq C_p \|f\|_{\Lambda^p}^p, \]
and
\[ \frac{1}{\psi(p) |B|} \int_B |I_2(x)|^p \, dx \leq C \|f\|_{\Lambda^p}^p. \]

Combining the two estimates, we obtain
\[ \frac{1}{\psi(p) |B|} \int_B |I_\partial f(x)|^p \, dx \leq C_p \|f\|_{\Lambda^p}^p. \]
and the result follows.

**Proof of Theorem 2.** Let $\tilde{T}_p = B(a, 2r)$. For $x \in B = B(a, r)$, we have

$$\tilde{T}_p f(x) - C_B = E^1_B(x) + E^2_B(x),$$

where

$$C_B = \int_{\mathbb{R}^n} f(y) \left( \frac{\rho(|a - y|)}{|a - y|^n} - \frac{\rho(|y|)}{|y|^n} \right) dy,$$

$$E^1_B(x) = \int_{\tilde{B}} f(y) \frac{\rho(|x - y|)}{|x - y|^n} dy,$$

and

$$E^2_B(x) = \int_{\tilde{B}^c} f(y) \left( \frac{\rho(|x - y|)}{|x - y|^n} - \frac{\rho(|a - y|)}{|a - y|^n} \right) dy.$$

From (2.3), we have

$$|C_B| \leq C \left( \int_{|a - y| < k} |f(y)| dy + |a| \int_{|a - y| \geq k} |f(y)| \frac{\rho(|a - y|)}{|a - y|^{n+1}} dy \right),$$

where $k = \max(2|a|, 2r)$, and so we know that $C_B$ is finite for every ball $B = B(a, r)$.

With the same technique as in the proof of the previous theorem, we have

$$|E^1_B(x)| \leq \int_{|a - y| < 2r} |f(y)| \frac{\rho(|x - y|)}{|x - y|^n} dy$$

$$\leq \int_{|x - y| < 3r} |f(y)| \frac{\rho(|x - y|)}{|x - y|^n} dy$$

$$\leq CM f(x) \int_0^{3r} \frac{\rho(t)}{t} dt$$

$$\leq CM f(x) \int_0^t \frac{\rho(t)}{t} dt,$$

and by (2.3)

$$|E^2_B(x)| \leq \int_{|a - y| \geq 2r} |f(y)| \left( \frac{\rho(|x - y|)}{|x - y|^n} - \frac{\rho(|a - y|)}{|a - y|^n} \right) dy$$

$$\leq C|x - a| \int_{|a - y| \geq 2r} |f(y)| \frac{\rho(|a - y|)}{|a - y|^{n+1}} dy$$

$$\leq C\|f\|_{\mathcal{M}_{p, \phi}} \int_0^{\infty} \frac{\rho(t)\phi(t)}{t^2} dt,$$

and the result follows as before.

**Remark.** We also suspect that $\tilde{T}_p$, the modified version of $T_p$, is bounded from $\mathcal{L}_{p, \psi}$ to $\mathcal{L}_{p, \psi}$ under the same hypothesis on $\rho$, $\phi$ and $\psi$ as in Theorem 2. However, we have not obtained the proof and the research in this direction is still ongoing.
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References


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