TAMKANG JOURNAL OF MATHEMATICS
Volume 33, Number 4, Winter 2002

ON THE BOUNDEDNESS OF A GENERALIZED FRACTIONAL INTEGRAL ON GENERALIZED MORREY SPACES

ERIDANI

Abstract. In this paper we extend Nakai's result on the boundedness of a generalized fractional integral operator from a generalized Morrey space to another generalized Morrey or Campanato space.

1. Introduction and Main Results

For a given function $\rho : (0,\infty) \to (0,\infty)$, let $\mathcal{T}_\rho$ be the generalized fractional integral operator, given by

$$\mathcal{T}_\rho f(x) = \int_{\boldsymbol{R}^n} \frac{f(y)\rho(|x-y|)}{|x-y|^n} dy,$$

and put

$$\tilde{\mathcal{T}}_\rho f(x) = \int_{\boldsymbol{R}^n} f(y) \left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n} \right) dy,$$

the modified version of $\mathcal{T}_\rho$, where B_0 is the unit ball about the origin, and χ_{B_0} is the characteristic function of B_0.

In [4], Nakai proved the boundedness of the operators $\tilde{\mathcal{T}}_\rho$ and $\mathcal{T}_\rho$ from a generalized Morrey space $\mathcal{M}_{1,\phi}$ to another generalized Morrey space $\mathcal{M}_{1,\psi}$ or generalized Campanato space $\mathcal{L}_{1,\psi}$. More precisely, he proved that

$$\|\mathcal{T}_\rho f\|_{\mathcal{M}_{1,\psi}} \le C\|f\|_{\mathcal{M}_{1,\phi}} \quad \text{and} \quad \|\tilde{\mathcal{T}}_\rho f\|_{\mathcal{L}_{1,\psi}} \le C\|f\|_{\mathcal{M}_{1,\phi}},$$

where $C > 0$, with some appropriate conditions on ρ, ϕ and ψ. Using the techniques developed by Kurata *et.al.*[1], we investigate the boundedness of these operators from generalized Morrey spaces $\mathcal{M}_{p,\phi}$ to generalized Morrey spaces $\mathcal{M}_{p,\psi}$ or generalized Campanato spaces $\mathcal{L}_{p,\psi}$ for $1 < p < \infty$.

The generalized Morrey and Campanato spaces are defined as follows. For a given function $\phi : (0,\infty) \to (0,\infty)$, and $1 < p < \infty$, let

$$\|f\|_{\mathcal{M}_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left(\frac{1}{|B|} \int_B |f(y)|^p dy \right)^{\frac{1}{p}},$$

Received September 19, 2001; revised January 29, 2002.
2000 *Mathematics Subject Classification*. Primary 42B20, 46E30, 43A15, 26A33.
Key words and phrases. Generalized fractional integrals, generalized Morrey spaces, generalized Campanato spaces.

and

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left(\frac{1}{|B|} \int_B |f(y) - f_B|^p dy \right)^{\frac{1}{p}},$$

where the supremum is taken over all open balls $B = B(a,r)$ in $\boldsymbol{R}^n$, $|B|$ is the Lebesgue measure of B in $\boldsymbol{R}^n$, $\phi(B) = \phi(r)$, and $f_B = \frac{1}{|B|}\int_B f(y)dy$. We define the generalized Morrey space $\mathcal{M}_{p,\phi}$ by

$$\mathcal{M}_{p,\phi} = \{f \in L^p_{loc}(\boldsymbol{R}^n) : \|f\|_{\mathcal{M}_{p,\phi}} < \infty\},$$

and the generalized Campanato space $\mathcal{L}_{p,\psi}$ by

$$\mathcal{L}_{p,\phi} = \{f \in L^p_{loc}(\boldsymbol{R}^n) : \|f\|_{\mathcal{L}_{p,\phi}} < \infty\}.$$

Our results are the following:

Theorem 1. *If $\rho, \phi, \psi : (0,\infty) \longrightarrow (0,\infty)$ satisfying the conditions below:*

$$\frac{1}{2} \le \frac{t}{r} \le 2 \Rightarrow \frac{1}{A_1} \le \frac{\phi(t)}{\phi(r)} \le A_1, \text{ and } \frac{1}{A_2} \le \frac{\rho(t)}{\rho(r)} \le A_2, \tag{1.1}$$

$$\int_0^1 \frac{\rho(t)}{t} dt < \infty, \text{ and for all } r > 0, \text{ we have } \int_r^\infty \frac{\phi(t)^p}{t} dt \le A_3 \phi(r)^p, \tag{1.2}$$

$$\phi(r) \int_0^r \frac{\rho(t)}{t} dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t} dt \le A_4 \psi(r), \text{ for all } r > 0, \tag{1.3}$$

where $A_i > 0$ are independent of $t, r > 0$, then for each $1 < p < \infty$ there exists $C_p > 0$ such that

$$\|\mathcal{T}_\rho f\|_{\mathcal{M}_{p,\psi}} \le C_p \|f\|_{\mathcal{M}_{p,\phi}}.$$

Theorem 2. *If $\rho, \phi, \psi : (0,\infty) \to (0,\infty)$ satisfying the conditions below:*

$$\frac{1}{2} \le \frac{t}{r} \le 2 \Rightarrow \frac{1}{A_1} \le \frac{\phi(t)}{\phi(r)} \le A_1, \text{ and } \frac{1}{A_2} \le \frac{\rho(t)}{\rho(r)} \le A_2, \tag{2.1}$$

$$\int_0^1 \frac{\rho(t)}{t} dt < \infty, \text{ and for all } r > 0, \text{ we have } \int_r^\infty \frac{\phi(t)^p}{t} dt \le A_3 \phi(r)^p, \tag{2.2}$$

$$\left| \frac{\rho(r)}{r^n} - \frac{\rho(t)}{t^n} \right| \le A_4 |r - t| \frac{\rho(r)}{r^{n+1}}, \text{ for } \frac{1}{2} \le \frac{t}{r} \le 2, \tag{2.3}$$

$$\phi(r) \int_0^r \frac{\rho(t)}{t} dt + r \int_r^\infty \frac{\rho(t)\phi(t)}{t^2} dt \le A_5 \psi(r), \text{ for all } r > 0, \tag{2.4}$$

where $A_i > 0$ are independent of $t, r > 0$, then for each $1 < p < \infty$ there exists $C_p > 0$ such that

$$\|\tilde{\mathcal{T}}_\rho f\|_{\mathcal{L}_{p,\psi}} \le C_p \|f\|_{\mathcal{M}_{p,\phi}}.$$

2. Proof of the Theorems

To prove the theorems, we shall use the following result of Nakai [2] (in a slightly modified version) about the boundedness of the standard maximal function Mf on a generalized Morrey space $\mathcal{M}_{p,\phi}$. The standard maximal function Mf is defined by

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy,\ x \in \boldsymbol{R}^n,$$

where the supremum is taken over all open balls B containing x.

Theorem(Nakai). *If* $\phi : (0, \infty) \longrightarrow (0, \infty)$ *satisfying the conditions below:*

(a) $\frac{1}{2} \le \frac{t}{r} \le 2 \Rightarrow \frac{1}{A_1} \le \frac{\phi(t)}{\phi(r)} \le A_1$,
(b) $\int_r^\infty \frac{\phi(t)^p}{t} dt \le A_2 \phi(r)^p$, *for all* $r > 0$,

where $A_i > 0$ *are independent of* $t, r > 0$, *then for each* $1 < p < \infty$ *there exists* $C_p > 0$ *such that*

$$\|Mf\|_{\mathcal{M}_{p,\phi}} \le C_p \|f\|_{\mathcal{M}_{p,\phi}}.$$

From now on, C and C_p will denote positive constants, which may vary from line to line. In general, these constants depend on n.

Proof of Theorem 1. For $x \in \boldsymbol{R}^n$, and $r > 0$, write

$$\mathcal{T}_\rho f(x) = \int_{|x-y|<r} \frac{f(y)\rho(|x-y|)}{|x-y|^n} dy + \int_{|x-y|\ge r} \frac{f(y)\rho(|x-y|)}{|x-y|^n} dy = I_1(x) + I_2(x).$$

Note that, for $t \in [2^k r, 2^{k+1} r]$, there exist constants $C_i > 0$ such that

$$\rho(2^k r) \le C_1 \int_{2^k r}^{2^{k+1} r} \frac{\rho(t)}{t} dt$$

and

$$\rho(2^k r)\phi(2^k r) \le C_2 \int_{2^k r}^{2^{k+1} r} \frac{\rho(t)\phi(t)}{t} dt.$$

So, we have

$$\begin{aligned}
|I_1(x)| &\le \int_{|x-y|<r} \frac{|f(y)|\rho(|x-y|)}{|x-y|^n} dy \\
&\le \sum_{k=-\infty}^{-1} \int_{2^k r \le |x-y| < 2^{k+1} r} \frac{|f(y)|\rho(|x-y|)}{|x-y|^n} dy \\
&\le C \sum_{k=-\infty}^{-1} \frac{\rho(2^k r)}{(2^k r)^n} \int_{|x-y|<2^{k+1} r} |f(y)| dy
\end{aligned}$$

$$\begin{aligned}
&\le C\sum_{k=-\infty}^{-1}\rho(2^k r)Mf(x)\\
&\le CMf(x)\sum_{k=-\infty}^{-1}\int_{2^k r}^{2^{k+1}r}\frac{\rho(t)}{t}dy\\
&\le CMf(x)\int_0^r\frac{\rho(t)}{t}dy\\
&\le C\frac{\psi(r)}{\phi(r)}Mf(x).
\end{aligned}$$

Meanwhile,

$$\begin{aligned}
|I_2(x)| &\le \int_{|x-y|\ge r}\frac{|f(y)|\rho(|x-y|)}{|x-y|^n}dy\\
&\le \sum_{k=0}^{\infty}\int_{2^k r\le|x-y|<2^{k+1}r}\frac{|f(y)|\rho(|x-y|)}{|x-y|^n}dy\\
&\le C\sum_{k=0}^{\infty}\frac{\rho(2^{k+1}r)}{(2^k r)^n}\int_{|x-y|<2^{k+1}r}|f(y)|dy\\
&\le C\sum_{k=0}^{\infty}\rho(2^{k+1}r)\phi(2^{k+1}r)\|f\|_{\mathcal{M}_{p,\phi}}\\
&\le C\|f\|_{\mathcal{M}_{p,\phi}}\sum_{k=0}^{\infty}\int_{2^{k+1}r}^{2^{k+2}r}\frac{\phi(t)\rho(t)}{t}dt\\
&\le C\|f\|_{\mathcal{M}_{p,\phi}}\int_r^{\infty}\frac{\phi(t)\rho(t)}{t}dt\\
&\le C\psi(r)\|f\|_{\mathcal{M}_{p,\phi}}.
\end{aligned}$$

Now, for $1\le p<\infty$, we have

$$|\mathcal{T}_\rho f(x)|^p \le 2^{p-1}(|I_1(x)|^p + |I_2(x)|^p),$$

and by Nakai's Theorem, we have for all balls $B=B(a,r)$

$$\frac{1}{\psi(r)^p|B|}\int_B |I_1(x)|^p dx \le \frac{C}{\phi(r)^p|B|}\int_B Mf(x)^p dx \le C\|Mf\|_{\mathcal{M}_{p,\phi}}^p \le C_p\|f\|_{\mathcal{M}_{p,\phi}}^p,$$

and

$$\frac{1}{\psi(r)^p|B|}\int_B |I_2(x)|^p dx \le C\|f\|_{\mathcal{M}_{p,\phi}}^p.$$

Combining the two estimates, we obtain

$$\frac{1}{\psi(r)^p|B|}\int_B |\mathcal{T}_\rho f(x)|^p dx \le C_p\|f\|_{\mathcal{M}_{p,\phi}}^p,$$

and the result follows.

Proof of Theorem 2. Let $\tilde{B} = B(a, 2r)$. For $x \in B = B(a, r)$, we have

$$\tilde{\mathcal{T}}_\rho f(x) - C_B = E_B^1(x) + E_B^2(x),$$

where

$$C_B = \int_{\boldsymbol{R}^n} f(y) \left(\frac{\rho(|a-y|)(1-\chi_{\tilde{B}}(y))}{|a-y|^n} - \frac{\rho(|y|)(1-\chi_{B_0}(y))}{|y|^n} \right) dy,$$

$$E_B^1(x) = \int_{\tilde{B}} f(y) \frac{\rho(|x-y|)}{|x-y|^n} dy,$$

and

$$E_B^2(x) = \int_{\tilde{B}^c} f(y) \left(\frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n} \right) dy.$$

From (2.3), we have

$$|C_B| \le C \left(\int_{|a-y|<k} |f(y)| dy + |a| \int_{|a-y|\ge k} |f(y)| \frac{\rho(|a-y|)}{|a-y|^{n+1}} dy \right),$$

where $k = \max(2|a|, 2r)$, and so we know that C_B is finite for every ball $B = B(a, r)$.

With the same technique as in the proof of the previous theorem, we have

$$\begin{aligned} |E_B^1(x)| &\le \int_{|a-y|<2r} \frac{|f(y)|\rho(|x-y|)}{|x-y|^n} dy \\ &\le \int_{|x-y|<3r} \frac{|f(y)|\rho(|x-y|)}{|x-y|^n} dy \\ &\le CMf(x) \int_0^{3r} \frac{\rho(t)}{t} dt \\ &\le CMf(x) \int_0^{r} \frac{\rho(t)}{t} dt, \end{aligned}$$

and by (2.3)

$$\begin{aligned} |E_B^2(x)| &\le \int_{|a-y|\ge 2r} |f(y)| \left| \frac{\rho(|x-y|)}{|x-y|^n} - \frac{\rho(|a-y|)}{|a-y|^n} \right| dy \\ &\le C|x-a| \int_{|a-y|\ge 2r} |f(y)| \frac{\rho(|a-y|)}{|a-y|^{n+1}} dy \\ &\le C\|f\|_{\mathcal{M}_{p,\phi}} r \int_r^\infty \frac{\rho(t)\phi(t)}{t^2} dt, \end{aligned}$$

and the result follows as before.

Remark. We also suspect that $\tilde{\mathcal{T}}_\rho$, the modified version of $\mathcal{T}_\rho$, is bounded from $\mathcal{L}_{p,\phi}$ to $\mathcal{L}_{p,\psi}$ under the same hypothesis on ρ, ϕ and ψ as in Theorem 2. However, we have not obtained the proof and the research in this direction is still ongoing.

Acknowledgements

The author would like to thank Dr. Hendra Gunawan and Dr. Eiichi Nakai for their useful comments and suggestions during the preparation of this note. He also thanks the referee(s) for their remarks on the earlier version of this note.

References

[1] K. Kurata, S. Nishigaki and S. Sugano, *Boundedness of integral operators on generalized Morrey spaces and its application to Schrödinger operators,* Proc. Amer. Math. Soc. **128**(1999), 1125-1134.

[2] E. Nakai, *Hardy-Littlewood maximal operator, singular integral operators, and the Riesz potentials on generalized Morrey spaces,* Math. Nachr. **166**(1994), 95-103.

[3] E. Nakai, *On generalized fractional integrals,* Proceedings of the International Conference on Mathematical Analysis and its Applications 2000 (Kaohsiung, Taiwan), Taiwanese J. Math. **5**(2001), 587-602.

[4] E. Nakai, *On generalized fractional integrals on the weak Orlicz spaces, BMO_ϕ, the Morrey sapces and the Campanato spaces,* Proceedings of the Conference on Function Spaces, Interpolation Theory and related topics in honour of Jaak Peetre on his 65th birthday, Lund University Sweden, to appear.

Department of Mathematics, Bandung Institute of Technology, Bandung 40132, Indonesia.

Permanent address: Department of Mathematics, University of Airlangga, Surabaya 60115, Indonesia.