

**MONOTONE ITERATIVE TECHNIQUE
FOR CARATHEODORY THEORY OF NONLINEAR
FUNCTIONAL RANDOM INTEGRAL EQUATIONS**

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Abstract. In this paper some random fixed point theorems for the mappings on an ordered Banach space are proved. As applications, the existence of the extremal solutions of some nonlinear functional random integral equations is obtained under certain monotonicity conditions.

1. Introduction

Let (Ω, \mathcal{A}) be a measurable space and set X be a real separable Banach space. Let β_X denote the σ -algebra of all Borel subsets of X . A mapping $x : \Omega \rightarrow X$ is called measurable if for any $B \in \beta_X$, $x^{-1}(B) = \{\omega \in \Omega : x(\omega) \in B\} \in \mathcal{A}$. A mapping $T : \Omega' \times X \rightarrow X$ is called a random operator if $T(\cdot, x)$ is measurable for all $x \in X$. A random operator T is also denoted by $T(\omega)x := T(\omega, x)$ or by simply $T(\omega)$. A random operator $T(\omega)$ is called continuous if $T(\omega)(\cdot)$ is continuous for each $\omega \in \Omega$, and completely continuous, if it is continuous and for any bounded set B in X , $T(\omega)(B)$ is precompact subset of X for each $\omega \in \Omega$. Similarly a random operator $T : \Omega \times X \rightarrow X$ is called contraction if $\|T(\omega)x - T(\omega)y\| \leq \alpha(\omega)\|x - y\|$ ($\alpha(\omega) < 1$) for all $x, y \in X$ and for each $\omega \in \Omega$. A measurable mapping $\xi : \Omega \rightarrow X$ is called a random fixed point of the random operator $T(\omega)$ if $T(\omega)\xi(\omega) = \xi(\omega)$ for each $\omega \in \Omega$.

A non-empty closed subset K of X is called a cone in X if (i) $K + K \subseteq K$ (ii) $\lambda K \subseteq K$ for $\lambda \in \mathbf{R}$, $\lambda \geq 0$ and (iii) $\{-K\} \cap K = 0$, where 0 is a zero element of X . Now we introduce an order relation \leq in X as follows : Let $x, y \in X$. Then $x \leq y$ if and only if $y - x \in K$. The Banach space X together with this order relation \leq becomes a ordered Banach space and it is denoted by $(X, \|\cdot\|, \leq)$ or simply by (X, K) . We need the following properties of the cone K in the sequel. A cone K in X is called normal if every order bounded set in X is bounded in norm. Similarly a cone K in X is called regular if every increasing sequence which is bounded above is convergent. Let $u, v \in X$ be such that $u \leq v$. Then an order interval $[u, v]$ is a set in X defined by $[u, v] = \{x \in X | u \leq x \leq v\}$. A random operator $T(\omega)$ is called nondecreasing if for any $x, y \in X$, $x \leq y$ implies $T(\omega)x \leq T(\omega)y$ for each $\omega \in \Omega$. A measurable mapping

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$\xi^* : \Omega \rightarrow X$ is called a maximal random fixed point of the random operator $T(\omega)$ if for any random fixed point ξ of $T(\omega)$, $\xi(\omega) \leq \xi^*(\omega)$ for each $\omega \in \Omega$. Similarly a minimal random fixed point ξ_* of the random operator $T(\omega)$ is defined.

Several classical or deterministic fixed point theorems have been extended to random operators in the literature by different authors and some of the random fixed point theorems have nice applications to nonlinear random differential and integral equations for proving the existence and uniqueness results. See for details, Bharucha Reid [3] and the references given therein. In this paper we prove some fixed point theorems involving a single and a pair of random operators on an ordered Banach space which extend the classical or deterministic well-known fixed point theorems of Amann [1] and Dhage [4] to random operators.

2. Random Fixed Point Theory

Before going to the main results of this paper, we recall some useful definitions:

The measure of noncompactness of a bounded set S in a Banach space X is a non-negative real number $\alpha(S)$ to be defined by

$$\alpha(S) = \inf \{ r > 0; S = \bigcup_{i=1}^n S_i \text{ diam } (S_i) \leq r, \forall_i \}. \quad (2.1)$$

The function $\alpha(\cdot)$ is called a Kuratowski measure of noncompactness and the details may be found in Banas and Goebel [2]. A random operator $T : \Omega \times X \rightarrow X$ is called α -condensing if for any bounded set S in X , $T(\omega)(S)$ is bounded and $\alpha(T(\omega)S) < \alpha(S)$ if $\alpha(S) > 0$ for each $\omega \in \Omega$. It is known that every completely continuous and contraction random operators are α -condensing.

Let $a, b : \Omega \rightarrow X$. Then by $a \leq b$ we mean $a(\omega) \leq b(\omega) \forall \omega \in \Omega$. In this case we also write $a \leq b$ on Ω . By the order interval $[a, b]$ we denote a set in X defined by

$$\begin{aligned} [a, b] &= \{ x \in X \mid a(\omega) \leq x \leq b(\omega) \quad \forall \omega \in \Omega \} \\ &= \bigcap_{\omega \in \Omega} [a(\omega), b(\omega)]. \end{aligned}$$

Our first result is

Theorem 2.1. *Let $a, b : \Omega \rightarrow X$ be two measurable functions such that $a \leq b$ and let $Q : \Omega \times [a, b] \rightarrow [a, b]$ be a continuous and nondecreasing random operator. Then Q has a minimal random fixed point ξ_* and a maximal random fixed point ξ^* if any one of following conditions holds.*

- (a) $Q(\omega)$ is α -condensing and the cone K in X is normal.
- (b) The cone K is regular in X .

Moreover,

$$\xi_*(\omega) = \lim_n Q^n(\omega)a \quad \text{and} \quad \xi^*(\omega) = \lim_n Q^n(\omega)b. \quad (2.2)$$

Proof. Let $\omega \in \Omega$ be a fixed element. Then the sequences $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ in $[a, b]$ defined by

$$x_n(\omega) = Q^n(\omega)a \text{ and } y_n(\omega) = Q^n(\omega)b \tag{2.3}$$

satisfy the inequality

$$a(\omega) = x_0(\omega) \leq x_1(\omega) \leq \dots \leq x_n(\omega) \leq \dots \leq y_n(\omega) \leq y_0(\omega) = b(\omega). \tag{2.4}$$

for each $\omega \in \Omega$.

Since a and b are measurable function on Ω , and $Q(\omega)$ is a continuous random operator, each $x_n(\omega)$ and $y_n(\omega)$ are measurable and consequently are random variables for each $n \in \mathbb{N}$. We discuss the conclusion of the theorem in the following two cases:

Case I. Suppose that the cone K in X is normal. Then the order interval $[a, b]$ is a bounded subset of X . Consequently $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are bounded sets in X . We show that $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are precompact sets in X . Suppose that $\{x_n(\omega)\}$ is not precompact.

Now

$$\{x_n(\omega)\} = \{a(\omega)\} \cup Q(\{x_n(\omega)\}).$$

Hence

$$\alpha(\{x_n(\omega)\}) \leq \alpha(Q(\omega)(\{x_n\})) < \alpha(x_n(\omega)) \text{ which is a contradiction.}$$

Hence $\{x_n(\omega)\}$ is precompact and $\overline{\{x_n(\omega)\}}$ is compact for each $\omega \in \Omega$. Therefore there is a point ξ_* in $[a, b]$ such that $\lim_n x_n(\omega) = \xi_*(\omega)$. Similarly it is proved that there is a point ξ^* in $[a, b]$ such that $\lim_n y_n(\omega) = \xi^*(\omega)$. The measurability of $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ implies that ξ_* and ξ^* are random variables. From the continuity of $Q(\omega)$ it follows that $Q(\omega)\xi_*(\omega) = \xi_*(\omega)$ and $Q(\omega)\xi^*(\omega) = \xi^*(\omega)$ for each $\omega \in \Omega$. Now suppose that ξ is any other random fixed point of $Q(\omega)$ in $[a, b]$. Then $a(\omega) \leq \xi(\omega) \leq b(\omega)$ for each $\omega \in \Omega$ and from the nondecreasing nature of $Q(\omega)$ it follows that

$$x_n(\omega) = Q^n(\omega)a \leq \xi(\omega) \leq Q^n(\omega)b = y_n(\omega) \text{ for each } \omega \in \Omega.$$

Hence $\xi_*(\omega) \leq \xi(\omega) \leq \xi^*(\omega)$ for each $\omega \in \Omega$. Thus ξ_* and ξ^* are respectively the minimal and maximal random fixed points of $Q(\omega)$ in $[a, b]$.

Case II. Next suppose that the condition (b) holds. By the regularity of cone K in X , there are points ξ_* and ξ^* in $[a, b]$ such that $\lim_n x_n(\omega) = \xi_*(\omega)$ and $\lim_n y_n(\omega) = \xi^*(\omega)$, $\omega \in \Omega$, where $x_n(\omega)$ and $y_n(\omega)$ are defined by (2.3). The rest of the proof is similar to Case I and we omit the details. This completes the proof.

Corollary 2.1. *Let $a, b : \Omega \rightarrow X$ be two measurable functions such that $a \leq b$ and let $Q : \Omega \times [a, b] \rightarrow [a, b]$ be a completely continuous and nondecreasing random operator.*

Further if the cone K in X is normal, then $Q(\omega)$ has a minimal random fixed point ξ_* and a maximal random fixed point ξ^* in $[a, b]$. Moreover,

$$\xi_*(\omega) = \lim_n Q^n(\omega)a \quad \text{and} \quad \xi^*(\omega) = \lim_n Q^n(\omega)b.$$

Theorem 2.2. Let $Q : \Omega \times X \rightarrow X$ be a continuous and nondecreasing random operator. Suppose that there exist two measurable functions $a, b : \Omega \rightarrow X$ such that $a \leq b$ satisfying $a(\omega) \leq Q(\omega)a$ and $Q(\omega)b \leq b(\omega)$ for each $\omega \in \Omega$. Then $Q(\omega)$ has a minimal random fixed point ξ_* and a maximal random fixed point ξ^* in $[a, b]$ if any one of the following conditions holds.

- (a) $Q(\omega)$ is α -condensing and the cone K in X is normal.
- (b) The cone K in X is regular.

Moreover

$$\xi_*(\omega) = \lim_n Q^n(\omega)a \quad \text{and} \quad \xi^*(\omega) = \lim_n Q^n(\omega)b.$$

Proof. Consider the order interval $[a, b]$ in X . We shall show that Q defines a mapping $Q(\omega) : \Omega \times [a, b] \rightarrow [a, b]$. Let $x \in [a, b]$ be any element. Then $a \leq x \leq b$. Since Q is nondecreasing and $a(\omega) \leq Q^n(\omega)a$ and $Q^n(\omega)b \leq b(\omega)$ for each $\omega \in \Omega$, one has

$$a(\omega) \leq Q(\omega)a \leq Q(\omega)x \leq Q(\omega)b \leq b(\omega)$$

for each $\omega \in \Omega$. Hence $Q(\omega)x \in [a, b]$ for all $x \in [a, b]$. Now the desired conclusion follows by an application of Theorem 2.1.

Corollary 2.2. Let $Q : \Omega \times X \rightarrow X$ be a completely continuous and nondecreasing random operator. Suppose that there exist two measurable functions $a, b : \Omega \rightarrow X$ such that $a \leq b$ satisfying $a(\omega) \leq Q(\omega)a$ and $Q(\omega)b \leq b(\omega)$ for each $\omega \in \Omega$. Further if the cone K in X is normal, then the operator $Q(\omega)$ has a minimal random fixed point ξ_* and a maximal random fixed point ξ^* in $[a, b]$. Moreover,

$$\xi_*(\omega) = \lim_n Q^n(\omega)a \quad \text{and} \quad \xi^*(\omega) = \lim_n Q^n(\omega)b \quad \text{for each } \omega \in \Omega.$$

Theorem 3.3. Let $a, b : \Omega \rightarrow X$ be two measurable functions such that $a \leq b$ and let $A, B : \Omega \times [a, b] \rightarrow [a, b]$ be two non-decreasing random operators such that for each $\omega \in \Omega$,

- (a) $A(\omega)$ is contraction,
- (b) $B(\omega)$ is completely continuous, and
- (c) $A(\omega)x + B(\omega)x \in [a, b]$ for each $x \in [a, b]$.

Further if the cone K in X is normal, then the operator equation

$$A(\omega)x + B(\omega)x = x \tag{2.4}$$

has a minimal random solution ξ_* and a maximal random solution ξ^* in $[a, b]$. Moreover,

$$\xi_*(\omega) = \lim_n x_n(\omega) \text{ and } \xi^*(\omega) = \lim_n y_n(\omega),$$

$$\left. \begin{aligned} \text{where } x_{n+1}(\omega) &= A(\omega)x_n + B(\omega)x_n, \quad n \geq 0, \quad \text{with } x_0 = a \\ \text{and } y_{n+1}(\omega) &= A(\omega)y_n + B(\omega)y_n, \quad n \geq 0, \quad \text{with } y_0 = b \end{aligned} \right\} \quad (2.5)$$

Proof. Define a mapping $Q : \Omega \times X \rightarrow X$ by

$$Q(\omega)x = A(\omega)x + B(\omega)x \quad (2.6)$$

for each $\omega \in \Omega$. Clearly by hypothesis (c), Q defines a mapping $Q : \Omega \times [a, b] \rightarrow [a, b]$. Also Q is nondecreasing random operator in view of the fact that the sum of two measurable functions is again measurable. As $A(\omega)$ is contraction, it is continuous and since sum of two continuous mappings is again continuous, the random operator $Q(\omega)$ is continuous. It is easily shown that Q is a α -condensing random operator on $\Omega \times [a, b]$. Now $Q(\omega)$ satisfies all the conditions of Theorem 2.1 and hence a direct application of it yields the desired conclusion. The proof is complete.

Corollary 3.3. Let $A, B : \Omega \times X \rightarrow X$ be two non-decreasing random operators such that

- (a) $A(\omega)$ is a contraction,
- (b) $B(\omega)$ is completely continuous, and
- (c) there exist two measurable functions $a, b : \Omega \rightarrow X$ such that $a \leq b$ and satisfy for each $\omega \in \Omega$,

$$a(\omega) \leq A(\omega)a + B(\omega)a \text{ and } A(\omega)b + B(\omega)b \leq b(\omega).$$

Further if the cone K in X is normal, then the operator equation (2.4) has a minimal random solution ξ_* and a maximal random solution ξ^* in $[a, b]$. Moreover,

$$\xi_*(\omega) = \lim_n x_n(\omega) \text{ and } \xi^*(\omega) = \lim_n y_n(\omega), \text{ for each } \omega \in \Omega,$$

where $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are defined by (2.5).

To state our final fixed point theorem we need the following property of the cone K in X .

A cone K in a separable Banach algebra X is called a positive if it satisfies (d) $K_0K \subseteq K$ where "o" is a multiplicative composition in X .

A random operator $T : \Omega \times X \rightarrow X$ is called positive if $\text{range}(T) \subset K \setminus \{0\}$.

Lemma 2.1. Let K be a positive cone K in a Banach algebra X and let $u_1, u_2, v_1, v_2 \in K$. If $u_1 \leq u_2$ and $v_1 \leq v_2$, then $u_1v_1 \leq u_2v_2$.

Proof.

$$\begin{aligned} \text{Now } u_2v_2 - u_1v_1 &= u_2v_2 - u_1v_2 + u_1v_2 - u_1v_1 \\ &= (u_2 - u_1)v_2 + u_1(v_2 - v_1) \\ &= k_1v_2 + u_1k_2 \\ &= k \end{aligned}$$

for some $k = k_1v_2 + u_1k_2 \in K$. By the definition of the order relation \leq which implies that $u_1v_1 \leq u_2v_2$.

Theorem 3.4. *Let $a, b : \Omega \rightarrow K$ be two measurable functions such that $a \leq b$ and let $A, B : \Omega \times [a, b] \rightarrow X$ be two positive and non-decreasing random operators satisfying for each $\omega \in \Omega$.*

- (a) $A(\omega)$ is Lipschitzian with Lipschitz constant $\alpha(\omega)$,
- (b) $B(\omega)$ is completely continuous, and
- (c) $A(\omega)x B(\omega)x \in [a, b]$ for each $x \in [a, b]$.

Further if the cone K in X is positive and normal, then the random equation

$$A(\omega)x B(\omega)x = x \tag{2.7}$$

has a minimal random solution ξ_* and a maximal positive random solution ξ^* in $[a, b]$ whenever $\alpha(\omega)M(\omega) < l$ for each $\omega \in \Omega$, where $M(\omega) = \|B(\omega)([a, b])\| = \sup\{\|B(\omega)x\| : x \in [a, b]\}$.

Moreover,

$$\xi_*(\omega) = \lim_n x_n(\omega) \quad \text{and} \quad \xi^*(\omega) = \lim_n y_n(\omega),$$

$$\left. \begin{array}{l} \text{where} \quad x_{n+1}(\omega) = A(\omega)x_n B(\omega)x_n, \quad n \geq 0, \quad \text{with } x_0 = a \\ \text{and} \quad y_{n+1}(\omega) = A(\omega)y_n B(\omega)y_n, \quad n \geq 0, \quad \text{with } y_0 = b \end{array} \right\} \tag{2.8}$$

Proof. Define a mapping $Q : \Omega \times X \rightarrow X$ by

$$Q(\omega)x = A(\omega)x B(\omega)x. \tag{2.9}$$

Now $A(\omega)$ and $B(\omega)$ are continuous and so from the continuity of multiplication in Banach algebra, it follows that Q is a random operator. By hypothesis (c), Q defines a mapping $Q : \Omega \times [a, b] \rightarrow [a, b]$. It is shown as in Dhage [4] that $Q(\omega)$ is a continuous and α -condensing random operator. Also $Q(\omega)$ is nondecreasing in view of Lemma 2.1. Now the desired result follows by an application of Theorem 2.1.

Corollary 3.4. *Let $A, B : \Omega \times X \rightarrow X$ be two non decreasing and positive random operators satisfying for each $\omega \in \Omega$.*

- (a) $A(\omega)$ is Lipschitzian with Lipschitz constant $\alpha(\omega)$,
- (b) $B(\omega)$ is completely continuous, and
- (c) there exist measurable functions $a, b : \Omega \rightarrow K \setminus \{0\}$ such that $a(\omega) \leq A(\omega)a B(\omega)a$ and $A(\omega)b B(\omega)b \leq b(\omega)$.

Further if the cone K in X is positive and normal, then the operator equation (2.7) has a minimal random fixed point ξ_* and a maximal fixed point ξ^* in $[a, b]$ whenever $a(\omega)M(\omega) < l$ for each $\omega \in \Omega$, where $M(\omega) = \|B(\omega)([a, b])\|$ Moreover,

$$\begin{aligned} \xi_*(\omega) &= \lim_n x_n(\omega), \quad \text{and} \\ \xi^*(\omega) &= \lim_n y_n(\omega) \end{aligned}$$

where $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are defined by (2.8).

In the following section we shall obtain the existence of extremal solutions of a certain nonlinear functional random integral equations.

3. Functional Random Integral Equations

Given a closed and bounded interval $J = [0, 1]$ in \mathbf{R} , the set of all real numbers, consider the nonlinear functional random integral equation (in short RIE).

$$x(t, \omega) = q(t, \omega) + \int_0^{\sigma(t)} f(s, x(\eta(s), \omega), \omega) ds \tag{3.1}$$

for all $t \in J$, here $q : J \times \Omega \rightarrow \mathbf{R}$, $f : J \times \mathbf{R} \times \Omega \rightarrow \mathbf{R}$ and $\sigma, \eta : J \rightarrow J$. The special cases of the RIE(3.1) have been studied extensively in the literature. See Bharucha - Reid [3].

We shall obtain the existence of the extremal random solutions of the RIE(3.1) in the space $BM(J, \mathbf{R})$ of all measurable and bounded real-valued functions defined on J . We equip the space $BM(J, \mathbf{R})$ with a norm $\| \cdot \|$ defined by

$$\|x\| = \sup_{t \in J} \|x(t)\|.$$

Clearly $BM(J, \mathbf{R})$ is a separable Banach space with this supremum norm. We define an order relation \leq in the space $BM(J, \mathbf{R})$ by the cone K in $BM(J, \mathbf{R})$ define by

$$K = \{x \in BM(J, \mathbf{R}) | x(t) \geq 0, \quad t \in J\} \tag{3.2}$$

Clearly the cone K is normal $BM(J, \mathbf{R})$.

A function $u : \Omega \rightarrow BM(J, \mathbf{R})$ is called a lower solution of the RIE(3.1) if for each $\omega \in \Omega$,

$$u(t, \omega) \leq q(t, \omega) + \int_0^{\sigma(t)} f(s, x(\eta(s), \omega), \omega) ds$$

for all $t \in J$, Similarly u is called an upper solution of the RIE (3.1) if the reverse inequality is satisfied. Finally a function $u : \Omega \rightarrow BM(J, \mathbf{R})$ is called a solution of the RIE (3.1) if it is a lower as well as an upper solution of the RIE (3.1) on J . In this case u is called a wide sense solution of RIE (3.1). Moreover if $u(t, \omega)$ is measurable in ω , then it is called a random solution of RIE (3.1).

We need the following definition in the sequel.

Definition 3.1. A function $\beta : J \times \mathbf{R} \times \Omega \rightarrow \mathbf{R}$ is said to be L^1_ω -Caratheodory, if for each $\omega \in \Omega$,

- (i) $t \rightarrow \beta(t, x, \omega)$ is measurable for all $x \in \mathbf{R}$,
- (ii) $x \rightarrow \beta(t, \cdot, \omega)$ is continuous for almost all $t \in J$, and
- (iii) for given $k > 0$, there exists a function $h_k : \Omega \rightarrow L^1(J, \mathbf{R})$ such that

$$|\beta(t, x, \omega)| \leq h_k(t, \omega), \quad \text{a.e } t \in J.$$

for all $x \in \mathbf{R}$ with $|x| \leq k$.

We consider the following set of assumptions

(**H**₀) The functions $\sigma, \eta : J \rightarrow J$, are continuous.

(**H**₁) $t \rightarrow q(t, \omega)$ is continuous for each $\omega \in \Omega$ and $\omega \rightarrow q(t, \omega)$ is measurable for all $t \in J$,

(**H**₂) The function $\omega \rightarrow f(t, x, \omega)$ is measurable for all $(t, x) \in J \times \mathbf{R}$.

(**H**₃) $f(t, x, \omega)$ is L^1_ω -Caratheodory.

(**H**₄) For each $\omega \in \Omega$, $f(t, x, \omega)$ is nondecreasing in x for almost all $t \in J$.

(**H**₅) There exists two measurable functions $a, b : \Omega \rightarrow BM(J, \mathbf{R})$ such that $a \leq b$ and a and b are respectively the lower and upper solutions of the RIE (3.1) on J .

Theorem 3.1. *Assume that the hypotheses (**H**₀) – (**H**₅) hold. Then the RIE (3.1) admits a maximal random solution ξ^* and a minimal random solution ξ_* in $[a, b]$.*

Moreover,

$$\xi_*(\omega) = \lim_n x_n(\omega) \quad \text{and} \quad \xi^*(\omega) = \lim_n y_n(\omega)$$

where

$$x_n(\omega) = q(t, \omega) + \int_0^{\sigma(t)} f(s, x_{n-1}(\eta(s), \omega), \omega) ds, \quad n \geq 1, \quad x_0(\omega) = a(\omega),$$

and

$$y_n(\omega) = q(t, \omega) + \int_0^{\sigma(t)} f(s, y_{n-1}(\eta(s), \omega), \omega) ds, \quad n \geq 1, \quad y_0(\omega) = b(\omega),$$

Proof. Define a mapping Q on $\Omega \times BM(J, \mathbf{R})$ by

$$Q(\omega)x(t) = q(t, \omega) + \int_0^{\sigma(t)} f(s, x(\eta(s), \omega), \omega) ds, \quad t \in J. \quad (3.3)$$

Take

$$y(t, \omega) = \int_0^{\sigma(t)} f(s, x(\eta(s), \omega), \omega) ds, \quad t \in J \text{ and } \omega \in \Omega.$$

Since σ is continuous, $t \rightarrow y(t, \omega)$ is continuous and hence is bounded on J for each $\omega \in \Omega$

Further $\omega \rightarrow y(t, \omega) = \int_0^{\sigma(t)} f(s, x(\eta(s), \omega), \omega) ds$ exists for each $\omega \in \Omega$, and hence is a limit of the finite sum of the measurable functions. So $y : \Omega \rightarrow BM(J, \mathbf{R})$ is a measurable function. This further in view of the hypothesis (**H**₁) implies that Q defines a measurable mapping $Q : \Omega \times BM(J, \mathbf{R}) \rightarrow BM(J, \mathbf{R})$ and hence Q is a random operator. By hypothesis (**H**₅), $a(\omega) \leq Q(\omega)a$ and $Q(\omega)b \leq b(\omega)$ for each $\omega \in \Omega$.

Let $x, y \in BM(J, \mathbf{R})$ be such that $x \leq y$. Then by (\mathbf{H}_4) for each $\omega \in \Omega$,

$$\begin{aligned} Q(\omega)x(t) &= q(t, \omega) + \int_0^{\sigma(t)} f(s, x(\eta(s), \omega), \omega) ds \\ &\leq q(t, \omega) + \int_0^{\sigma(t)} f(s, x(\eta(s), \omega), \omega) ds \\ &= Q(\omega)y(t) \end{aligned}$$

for all $t \in J$, and so $Q(\omega)x \leq Q(\omega)y$ for each $\omega \in \Omega$. Hence $Q(\omega)$ is a nondecreasing random operator on $BM(J, \mathbf{R})$. Next we show that $Q(\omega)$ is a completely continuous random operator on $BM(J, \mathbf{R})$. Obviously by using a standard arguments (see Granas et.at.[5]) and using dominated convergence theorem, it is shown that $Q(\omega)$ is a continuous random operator. Let $k > 0$ be a fixed real number and consider a sequence $\{x_n\}$ in $BM(J, \mathbf{R})$ such that $|x_n(t)| \leq k$ for all $t \in J$. Then by (\mathbf{H}_8) ,

$$\begin{aligned} |Q(\omega)x_n(t)| &\leq \left| \int_0^{\sigma(t)} f(s, x(\eta(s), \omega), \omega) ds \right| + |q(t, \omega)| \\ &\leq \|q(\omega)\| + \int_0^{\sigma(t)} h_k(s, \omega) ds \\ &\leq \|q(\omega)\| + \|h_k(\omega)\|_{L^1} \end{aligned} \tag{3.4}$$

which shows that $\|Q(\omega)x_n\|$ is uniformly bounded for each $\omega \in \Omega$. Now let $t, \tau \in J$. Then

$$|Q(\omega)x_n(t) - Q(\omega)x_n(\tau)| \leq |q(t, \omega) - q(\tau, \omega)| + |p(t, \omega) - p(\tau, \omega)| \tag{3.5}$$

where $p(t, \omega) = \int_0^{\sigma(t)} h_k(s, \omega) ds$.

Since $q(\cdot, \omega)$ and $p(\cdot, \omega)$ are uniformly continuous on J , it follows that $\{Q(\omega)x_n(t)\}$ is a equi-continuous set in $BM(J, \mathbf{R})$. Hence $Q(\omega)$ is a completely continuous random operator on $M(J, \mathbf{R})$ by Arzela-Ascoli theorem. Now the desired conclusion follows by application of Corollary 2.2. This completes the proof.

As an application of the main result of this section, we consider the functional first order random differential equation (in short RDE)

$$\left. \begin{aligned} \frac{dx(t, \omega)}{dt} &= f(t, x(\eta(t), \omega), \omega) \quad \text{a.e. } t \in J, \\ x(0, \omega) &= q(\omega) \in \mathbf{R} \end{aligned} \right\} \tag{3.6}$$

for each $\omega \in \Omega$, here $f : J \times \mathbf{R} \times \Omega \rightarrow \mathbf{R}$ and $\eta : J \rightarrow J$ is continuous.

By the random solution of the RDE (3.6) we mean a measurable function $\xi : \Omega \rightarrow AC(J, \mathbf{R})$ which satisfies (3.6) on J .

We equip the space $AC(J, \mathbf{R})$ with a norm $\|\cdot\|$ and order relation \leq by

$$\|x\| = \sup_{t \in J} |x(t)|,$$

and $x \leq y \Leftrightarrow x(t) \leq y(t)$ for all $t \in J$.

Clearly $AC(J, \mathbf{R})$ is an ordered separable Banach space, where in the cone K defined by

$$K = \{x \in AC(J, \mathbf{R}) | x \geq 0\}$$

is normal.

We need the following set of hypotheses:

(A₁) $q : \Omega \rightarrow \mathbf{R}$ is measurable.

(A₂) There exists two measurable functions $u, v : \Omega \rightarrow AC(J, \mathbf{R})$ such that $u \leq v$, and for each $\omega \in \Omega$,

$$\begin{aligned} \frac{du(t, \omega)}{dt} &\leq f(t, u(\eta(t), \omega), \omega) \quad \text{a.e. } t \in J. \\ u(0, \omega) &\leq q(\omega) \\ \text{and} \quad \frac{dv(t, \omega)}{dt} &\geq f(t, v(\eta(t), \omega), \omega) \quad \text{a.e. } t \in J. \\ v(0, \omega) &\leq q(\omega). \end{aligned}$$

Theorem 3.2. *Assume that the hypotheses (A₁)–(A₂) and (H₂)–(H₄) hold. Then the RDE (3.6) admits a minimal random solution ξ_* and a maximal random solution ξ^* in $[u, v]$. Moreover*

$$\xi_*(\omega) = \lim_n x_n(\omega) \quad \text{and} \quad \xi^*(\omega) = \lim_n y_n(\omega)$$

for each $\omega \in \Omega$, where

$$x_n(\omega) = q(\omega) + \int_0^t f(s, x_{n-1}(\eta(s), \omega), \omega) ds, \quad n \geq 1, \quad \text{with } x_0 = u.$$

with

$$y_n(\omega) = q(\omega) + \int_0^t f(s, y_{n-1}(\eta(s), \omega), \omega) ds, \quad n \geq 1, \quad \text{with } y_0 = v.$$

Proof. The problem (2.6) is equivalent to the RIE

$$x(t, \omega) = q(\omega) + \int_0^t f(s, x(\eta(s), \omega), \omega) ds, \quad t \in J.$$

Now the desired conclusion follows by an application of Theorem 3.1 since $AC(J, \mathbf{R}) \subset BM(J, \mathbf{R})$.

4. Conclusion

Finally while concluding this paper, we mention that we have not been able to relax the continuity of the random mappings in any result of this paper for proving the fixed point theorem or for proving the existence results for random differential and integral

equations. This is because of the reason that the image of a random variable under a discontinuous random operator may not be a random variable. Again the fixed point theorems of this paper have some nice applications to a variety of nonlinear random differential and integral equations and some of the results in this direction will be reported else where.

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