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A NOTE ON CERTAIN INTEGRAL INEQUALITY

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Abstract. In this note we establish an integral inequality which can be used in the study of certain integral equations. The discrete analogue and some applications of the main result are also given.

1. Introduction

The inequalities which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of differential, integral and finite difference equations, see [2,5,6] and the references given therein. In [2,p.11] Bainov and Simeonov have given the following useful integral inequaltiy.

Lemma BS. *Let $u(t)$, $a(t)$, $b(t)$ be continuous functions in $I=[\alpha,\beta]$, let $a(t)$ and $b(t)$ be nonnegative in I, and suppose*

$$u(t)\leq c+\int_{\alpha}^{t}a(s)u(s)ds+\int_{\alpha}^{\beta}b(s)u(s)ds,$$

for $t\in I$, where c is a constant. If

$$q=\int_{\alpha}^{\beta}b(s)\exp\left(\int_{\alpha}^{s}a(\sigma)d\sigma\right)ds<1,$$

then

$$u(t)\leq\frac{c}{1-q}\exp\left(\int_{\alpha}^{t}a(s)ds\right),$$

for $t\in I$

The main purpose of the present note is to establish a useful version of the above inequality which can be used as ready tool to study the qualitative behavior of solutions of a certain Volterra-Fredholm integral equation. The discrete analogue of the main result and some applications are also given to illustrate the usefulness.

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2. Statement of Results

In whtat follows, R denotes the set of real numbers, $R_+ = [0, \infty)$, $I = [\alpha, \beta]$ are the given subsets of R and Z be the set of integers. For $\alpha, \beta \in Z$, $\alpha \leq \beta$, let $N_{\alpha,\beta} = \{n \in Z : \alpha \leq n \leq \beta\}$. We denote by

$$\Delta = \{(t, s) \in I^2 : \alpha \leq s \leq t \leq \beta\}$$

and

$$E = \{(n, s) \in N_{\alpha,\beta}^2 : \alpha \leq s \leq n \leq \beta\}.$$

Let R^n denote the n dimensional Euclidean Space with norm $|\cdot|$ and $C(A, B)$ denotes the class of continuous functions from A to B. We use the usual conventions that the empty sums and products are taken to be 0 and 1 respectively. We shall also assume that all the integrals, sums and products involved throughout the discussion exist in the respective domains of their definitions.

Our main result is established in the following theorem.

Theorem 1. *Let $u(t) \in C(I, R_+)$, $a(t, s)$, $b(t, s) \in C(\Delta, R_+)$ and $a(t, s)$, $b(t, s)$ be nondecreasing in t, for each $s \in I$ and suppose that*

$$u(t) \leq c + \int_\alpha^t a(t, s)u(s)ds + \int_\alpha^\beta b(t, s)u(s)ds, \tag{2.1}$$

for $t \in I$, *where $c \geq 0$ is a constant. If*

$$p(t) = \int_\alpha^\beta b(t, s) \exp\left(\int_\alpha^s a(s, \sigma)d\sigma\right) ds < 1, \tag{2.2}$$

then

$$u(t) \leq \frac{c}{1 - p(t)} \exp\left(\int_\alpha^t a(t, s)ds\right), \tag{2.3}$$

for $t \in I$

Remark 1. In the special case when $a(t, s) = a(s)$, $b(t, s) = b(s)$, the inequality given in Theorem 1 reduces to the inequality given in Lemma BS, in case $u(t)$ and c therein are nonnegative.

The discrete analogue of Theorem 1 is given in the following theorem.

Theorem 2. *Let $u(n)$ be a real-valued nonnegative function defined on $N_{\alpha,\beta}$. Let $a(n, s)$, $b(n, s)$ be real-valued nonnegative functions defined on E and nondecreasing in n for each $s \in N_{\alpha,\beta}$ and suppose that*

$$u(n) \leq c + \sum_{\sigma=\alpha}^{n-1} a(n, \sigma)u(\sigma) + \sum_{\sigma=\alpha}^{\beta} b(n, \sigma)u(\sigma), \tag{2.4}$$

for $n \in N_{\alpha,\beta}$, *where* $c \geq 0$ *is a constant. If*

$$r(n) = \sum_{s=\alpha}^{\beta} b(n,s) \prod_{\sigma=\alpha}^{s-1} [1 + a(s,\sigma)] < 1, \tag{2.5}$$

then

$$u(n) \leq \frac{c}{1 - r(n)} \prod_{\sigma=\alpha}^{n-1} [1 + a(n,\sigma)], \tag{2.6}$$

for $n \in N_{\alpha,\beta}$

3. Proof of Theorem 1

Fix any T, $\alpha \leq T \leq \beta$, then for $\alpha \leq t \leq T$ we have

$$u(t) \leq c + \int_{\alpha}^{t} a(T,s)u(s)ds + \int_{\alpha}^{\beta} b(T,s)u(s)ds. \tag{3.1}$$

Define a function $z(t)$, $\alpha \leq t \leq T$ by the right side of (3.1). Then $u(t) \leq z(t)$, $\alpha \leq t \leq T$,

$$z(\alpha) = c + \int_{\alpha}^{\beta} b(T,s)u(s)ds, \tag{3.2}$$

and

$$\begin{aligned} z'(t) &= a(T,t)u(t) \\ &\leq a(T,t)z(t) \end{aligned} \tag{3.3}$$

for $\alpha \leq t \leq T$. By setting $t = \sigma$ in (3.3) and integrating it with respect to σ form α to T we get

$$z(T) \leq z(\alpha) \exp\left(\int_{\alpha}^{T} a(T,\sigma)d\sigma \right). \tag{3.4}$$

Since T is arbitrary, from (3.4) and (3.2) with T replaced by t and $u(t) \leq z(t)$ we have

$$u(t) \leq z(\alpha) \exp\left(\int_{\alpha}^{t} a(t,\sigma)d\sigma \right), \tag{3.5}$$

where

$$z(\alpha) = c + \int_{\alpha}^{\beta} b(t,s)u(s)ds. \tag{3.6}$$

Using (3.5) on the right side of (3.6) and (2.2) it is easy to observe that

$$z(\alpha) \leq \frac{c}{1 - p(t)}. \tag{3.7}$$

Using (3.7) in (3.5) we get the desired inequality in (2.3). The proof is complete.

4. Proof of Theorem 2

Fix any $m \in N_{\alpha,\beta}$, $\alpha \leq m \leq \beta$, Then for $\alpha \leq n \leq m$, we have

$$u(n) \leq c + \sum_{s=\alpha}^{n-1} a(m,s)u(s) + \sum_{s=\alpha}^{\beta} b(m,s)u(s). \tag{4.1}$$

Define a function $z(n)$, $\alpha \leq n \leq m$ by the right side of (4.1). Then $u(n) \leq z(n)$, $\alpha \leq n \leq m$,

$$z(\alpha) = c + \sum_{s=\alpha}^{\beta} b(m,s)u(s), \tag{4.2}$$

and

$$\begin{aligned} z(n+1) - z(n) &= a(m,n)u(n) \\ &\leq a(m,n)z(n), \end{aligned}$$

i.e.

$$z(n+1) \leq [1 + a(m,n)]z(n) \tag{4.3}$$

for $\alpha \leq n \leq m$. By setting $n = \sigma$ in (4.3) and substitiuting $\sigma = \alpha,\ \alpha+1, \ldots m-1$ successively, we obtain

$$z(m) \leq z(\alpha) \prod_{\sigma=\alpha}^{m-1} [1 + a(m,\sigma)]. \tag{4.4}$$

Since m is arbitrary, from (4.4) and (4.2) with m replaced by n and using $u(n) \leq z(n)$ we have

$$u(n) \leq z(\alpha) \prod_{\sigma=\alpha}^{n-1} [1 + a(n,\sigma)], \tag{4.5}$$

where

$$z(\alpha) = c + \sum_{s=\alpha}^{\beta} b(n,s)u(s). \tag{4.6}$$

Using (4.5) on the right side of (4.6) and (2.5) we observe that

$$z(\alpha) \leq \frac{c}{1 - r(n)}. \tag{4.7}$$

Using (4.7) in (4.5) we get (2.6) and the proof is complete

5. Applications

In this section we present some applications of Theorem 1 to study certatin properties of the solutions of the nonlinear Volterra-Fredholm integral equation of the form

$$x(t) = f(t) + \int_{\alpha}^{t} g(t,s,x(s))ds + \int_{\alpha}^{\beta} h(t,s,x(s))ds \tag{5.1}$$

for $t \in I$, where $x(t)$ is an unknown function, $f \in C(I, R^n), g, h \in C(\Delta \times \Delta \times R^n, R^n)$. For the study of Volterra-Fredholm integral equations of the type (5.1), we refer the interested readers to [1,3,4] and the references cited therein.

The following theorem deals with the estimate on the solution of equation (5.1).

Theorem 3. *Suppose that the functions f, g, h in equation* (5.1) *satisfy the conditions*

$$|f(t)| \leq c \tag{5.2}$$

$$|g(t,s,x)| \leq a(t,s)|x| \tag{5.3}$$

$$|h(t,s,x)| \leq b(t,s)|x| \tag{5.4}$$

where $a(t,s)$, $b(t,s)$ and c are defined as in Theorem 1. *Let $p(t)$ be as in* (2.2). *If $x(t)$ is a solution of* (5.1) *on I, then*

$$|x(t)| \leq \frac{c}{1-p(t)} \exp\left(\int_\alpha^t a(t,s)ds\right) \tag{5.5}$$

for $t \in I$

Proof. Since $x(t)$ is a solution of (5.1), from (5.1)-(5.4) we have

$$|x(t)| \leq c + \int_\alpha^t a(t,s)|x(s)|ds + \int_\alpha^\beta b(t,s)|x(s)|ds. \tag{5.6}$$

Now an application of Theorem 1 to (5.6) yields the required estimate in (5.5).

The next result deals with the uniqueness of solutions of equation (5.1).

Theorem 4. *Suppose that the functions g, h in equation* (5.1) *satisfy the conditions*

$$|g(t,s,x) - g(t,s,y)| \leq a(t,s)|x-y|, \tag{5.7}$$

$$|h(t,s,x) - h(t,s,y)| \leq b(t,s)|x-y|, \tag{5.8}$$

where $a(t,s)$, $b(t,s)$ be as defined in Theorem 1. *Let $p(t)$ be as in Theorem* 1. *Then the equation* (5.1) *has at most one solution on I.*

Proof. Let $u(t)$ and $v(t)$ be two solutions of (5.1) on I. From (5.1), (5.7) and (5.8) we have

$$|u(t) - v(t)| \leq \int_\alpha^t a(t,s)|u(s) - v(s)|ds + \int_\alpha^\beta b(t,s)|u(s) - v(s)|ds. \tag{5.9}$$

Now an application of Theorem 1 to (5.9) yields $u(t) = v(t)$, *i.e* there is at most one solution of the equation (5.1).

Remark 2. It is easy to observe that the inequality given in Theorem 2 can be used to study the similar properties as in Theorems 3 and 4 for the following sum-difference equation

$$y(n) = f(n) + \sum_{s=\alpha}^{n-1} g(n,s,y(s)) + \sum_{s=\alpha}^{\beta} h(n,s,y(s)), \tag{5.10}$$

under some suitable conditions on the functions involved in (5.10). For similar applications, see [6].

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