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A VISCOSITY ITERATIVE ALGORITHM TECHNIQUE FOR SOLVING A GENERAL EQUILIBRIUM PROBLEM SYSTEM

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Abstract. In the recent decade, a considerable number of Equilibrium problems have been solved successfully based on the iteration methods. In this paper, we suggest a viscosity iterative algorithm for nonexpansive semigroup in the framework of Hilbert space. We prove that, the sequence generated by this algorithm under the certain conditions imposed on parameters strongly convergence to a common solution of general equilibrium problem system. Results presented in this paper extend and unify the previously known results announced by many other authors. Further, we give some numerical examples to justify our main results.

1. Introduction

The viscosity iterative algorithms for finding a common element of the set of fixed points for nonlinear operators and the set of solutions of variational inequality problems have been investigated by many authors [11, 21, 24, 26, 27] and references therein. The viscosity technique for nonexpansive mappings in Hilbert space was proposed by Moudafi[9, 10]. This technique allow us to apply this method to convex optimization, linear programming and monoton inclusions [15, 17, 20, 22, 23, 25]. It is well known that the generalized equilibrium problems include variational inequality problems, optimization problems, problems of Nash equilibria, saddle point problems, fixed point problems and complementarity problems as special cases [1, 9, 22, 23].

Let C be a nonempty closed convex subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. A mapping $T : C \to C$ is said to be contraction if there exists a constant $\alpha \in (0,1)$ such that $\|T(x) - T(y)\| \leq \alpha \|x - y\|$, $\forall x, y \in C$. If $\alpha = 1$ T is called nonexpansive on C.

The generalized equilibrium problem (GEP) is defined as follows:

$$\text{Find} \quad \bar{x} \in C : \quad F(\bar{x}, y) + \langle A\bar{x}, y - \bar{x} \rangle \geq 0 \qquad \forall y \in C, \tag{1.1}$$

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where $A : C \to H$ is a nonlinear mapping, and $F : C \times C \to \mathbb{R}$ is a bifunction. The set of solutions this problem is denoted by GEP(F, A)., i.e.,

$$\mathrm{GEP}(F, A) = \{\bar{x} \in C : \quad F(\bar{x}, y) + \langle A\bar{x}, y - \bar{x}\rangle \geq 0, \quad \forall y \in C\},$$

which was studied by Takahashi [23].

To study the generalized equilibrium problem (1.1), we may assume that F satisfies the following conditions:

(A1) $F(x, x) \geq 0, \ \forall x \in C$,

(A2) F is monotone, i.e. $F(x, y) + F(y, x) \leq 0, \ \forall x \in C$,

(A3) F is upper hemicontinuouse, i.e. for each $x, y, z \in C$, $\limsup_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y)$,

(A4) For each $x \in C$ fixed, the function $x \to F(x, y)$ is convex and lower semi-continuous;

A family $S := \{T(s) : 0 \leq s < \infty\}$ of mapping from C into itself is called a nonexpansive semi-group on C if it satisfies the following conditions:

(1) $T(0)x = x$ for all $x \in C$,

(2) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$,

(3) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$,

(4) For all $x \in C$, $s \to T(s)x$ is continuous.

Plubtieng and Punpaeng introduced the following iterative method for nonexpansive semigroup[13]:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds.$$

In 2010 Kang et.al, introduced and inspired by results in [6], prove a strong convergence of the iterative scheme in a real Hilbert space by

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds,$$

where A is a strong positive bounded linear operator on C.

Cianciaruso et al. [3] considered the following iterative method:

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0;$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + (1 - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds.$$

Recently, Sahebi et al. [14, 15, 16, 17] considered a general viscosity iterative algorithm for finding a common element of the set general equilibrium problem system and the set of fixed points of a nonexpansive semigroup in a Hilbert space. They proved, under the certain appropriate conditions, the iterative algorithm converges strongly to the unique solution of a variational inequality. In this paper, by intuition from [3, 6, 13, 14, 15, 16, 17] a new iterative algorithm scheme is introduced. The results presented in this paper generalize, improve and unify many previously known results in this research area.

2. Preliminaries

For each point $x \in H$, there exists a unique nearest point of C, denote by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$ for all $y \in C$. P_C is called the metric projection of H onto C. It is well known that P_C is nonexpansive mapping and is characterized by the following property:

$$\langle x - P_C x, y - P_C y\rangle \le 0 \tag{2.1}$$

Definition 2.1. A mapping $T : H \to H$ is said to be firmly nonexpansive, if

$$\langle Tx - Ty, x - y\rangle \ge \|Tx - Ty\|^2, \ \ \forall x, y \in H.$$

Definition 2.2. A mapping $M : C \to H$ is said to be monotone, if

$$\langle Mx - My, x - y\rangle \ge 0, \quad \forall x, y \in C.$$

M is called α-inverse-strongly-monotone if there exist a positive real number α such that

$$\langle Mx - My, x - y\rangle \ge \alpha\|Mx - My\|^2, \quad \forall x, y \in C.$$

Definition 2.3. A mapping $B : H \to H$ is said to be strongly positive linear bounded operator, if there exists a constant $\bar{\gamma} > 0$ such that $\langle Bx, x\rangle \ge \bar{\gamma}\|x\|^2$, $\forall x \in H$.

Notation. Let $\{x_n\}$ be a sequence in H, then $x_n \to x$ (respectively, $x_n \rightharpoonup x$) denote strong (respectively, weak) convergence of the sequence $\{x_n\}$ to a point $x \in H$.

It is known that H satisfies Opial's condition [12], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.2}$$

holds for every $y \in H$ with $y \ne x$.

Lemma 2.4 ([5])**.** *Let C be a nonempty, closed convex subset of H and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Then For $r > 0$ and $x \in H$, there exists $z \in C$ such that $F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0$, $\forall y \in C$.*

Further define

$$T_r^F x = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C\},$$

for all $r > 0$ and $x \in H$. Then, the following hold:

(i) T_r^F *is single-valued.*

(ii) T_r^F *is firmly nonexpansive, i. e.,*

$$\|T_r^F(x)-T_r^F(y)\|^2 \leq \langle T_r^F(x)-T_r^F(y), x-y\rangle, \quad \forall x,y\in H.$$

(iii) $Fix(T_r^F)=EP(F)$.

(iv) $EP(F)$ *is compact and convex.*

Lemma 2.5 ([4]). *Let* $F: C\times C\to\mathbb{R}$ *be a bifunction satisfying* (A1)−(A4) *and let* T_r^F *be defined as in Lemma 2.4, for* $r>0$. *Let* $x,y\in H$ *and* $r_1,r_2>0$. *Then,*

$$\|T_{r_2}^F y - T_{r_1}^F x\| \leq \|x-y\| + |\frac{r_2-r_1}{r_2}|\|T_{r_2}^F y - y\|.$$

Lemma 2.6 ([8]). *Assume that* B *is a strong positive linear bounded self adjoint operator on a Hilbert space* H *with coefficient* $\bar{\gamma}>0$ *and* $0<\rho\leq\|B\|^{-1}$. *Then* $\|I-\rho B\|\leq 1-\rho\bar{\gamma}$.

Lemma 2.7 ([18]). *Let* C *be a nonempty bounded closed convex subset of a Hilbert space* H *and let* $S:=\{T(s): 0\leq s<\infty\}$ *be a nonexpansive semigroup on* C, *for each* $x\in C$ *and* $t>0$. *Then, for any* $0\leq h<\infty$,

$$\lim_{t\to\infty}\sup_{x\in C}\|\frac{1}{t}\int_0^t T(s)x\,ds - T(h)(\frac{1}{t}\int_0^t T(s)x\,ds)\|=0.$$

Lemma 2.8 ([19]). *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space* X *and* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *with* $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$. *Suppose* $x_{n+1}=(1-\beta_n)y_n+\beta_n x_n$, *for all integers* $n\geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\leq 0$. *Then* $\lim_{n\to\infty}\|y_n-x_n\|=0$.

Lemma 2.9 ([23]). *Let* $F: C\times C\to\mathbb{R}$ *be a bifunction satisfying* (A1)−(A4) *and let* T_r^F *be defined as in Lemma 2.4, for* $r>0$. *Let* $x\in H$ *and* $s,t>0$. *Then,*

$$\|T_s^F x - T_t^F x\|^2 \leq \frac{s-t}{s}\langle T_s^F(x)-T_t^F(x), T_s^F(x)-x\rangle.$$

Lemma 2.10 ([25]). *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers such that* $a_{n+1}\leq(1-\alpha_n)a_n+\delta_n$, $n\geq 0$ *where* α_n *is a sequence in* $(0,1)$ *and* δ_n *is a sequence in* $\mathbb{R}$ *such that*

(i) $\sum_{n=1}^{\infty}\alpha_n=\infty$;

(ii) $\limsup_{n\to\infty}\frac{\delta_n}{\alpha_n}\geq 0$ *or* $\sum_{n=1}^{\infty}\delta_n<\infty$.
Then $\lim_{n\to\infty}a_n=0$.

Lemma 2.11 ([2]). *The following inequality holds in real space* H:

$$\|x+y\|^2\leq\|x\|^2+2\langle y,x+y\rangle, \quad \forall x,y\in H.$$

3. Viscosity iterative algorithm

Let C be a nonempty closed convex subset of real Hilbert space H. For each $i \in \{1,\dots,k\}$, let $F_i : C\times C \to \mathbb{R}$ be bifunctions satisfying $(A1)-(A4)$ and ψ_i be $\bar{\alpha}_i$-inverse strongly monotone mappings from C into H. Let $S = \{T(s) : s \in [0,+\infty)\}$ be a nonexpansive semigroup on C such that $\Gamma = \bigcap_{i=1}^{k} \mathrm{Fix}(S) \cap \mathrm{GEP}(F_i,\psi_i) \neq \emptyset$. Also $f : C \to C$ be an α-contraction mapping and A, B be strongly positive bounded linear self adjoint operators on H with coefficients $\bar{\delta} > 0$ and $\bar{\beta} > 0$ respectively such that $0 < \gamma < \frac{\bar{\delta}}{\alpha} < \gamma + \frac{1}{\alpha}$, $\bar{\delta} \le \|A\| \le 1$ and $\|B\| = \bar{\beta}$.

Algorithm 3.1. *For given $x_0 \in C$ arbitrary, let the sequence $\{x_n\}$ be generated by the manner:*

$$\begin{cases} u_{n,i} = T^{F_i}_{r_{n,i}}(x_n - r_{n,i}\psi_i x_n) \\ w_n = \frac{1}{k}\sum\limits_{i=1}^{k} u_{n,i} \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n B x_n + ((1-\epsilon_n)I - \beta_n B - \alpha_n A)\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T(s) w_n ds, \end{cases} \tag{3.1}$$

where $\{r_{n,i}\} \subseteq (0, 2\bar{\alpha}_i)$, $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\}$, $\{\epsilon_n\} \subset [0,1)$ and $\{s_n\} \subset (0,\infty)$ satisfying the following control conditions:

(C1) $\epsilon_n \le \alpha_n$, $\lim\limits_{n\to\infty} \alpha_n = \lim\limits_{n\to\infty} \beta_n = \lim\limits_{n\to\infty} \epsilon_n = 0$, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $\lim\limits_{n\to\infty} s_n = \infty$, $\sup\limits_{n\in\mathbb{N}} |s_{n+1} - s_n|$ *is bounded;*

(C3) $\lim\limits_{n\to\infty} |r_{n+1,i} - r_{n,i}| = 0$, $0 < b < r_{n,i} < a < 2\bar{\alpha}_i$.

Lemma 3.2. *For any $0 < \gamma < \frac{\bar{\delta}}{\alpha} < \gamma + \frac{1}{\alpha}$, there exist a unique fixed point for sequence $\{x_n\}$.*

Proof. We define the sequence of mappings $\{P_n : H \to H\}$ as follows:

$$P_n x := \alpha_n\gamma f(x) + \beta_n B x + ((1-\epsilon_n)I - \beta_n B - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s) x ds, \quad \forall x \in H.$$

We may assume without loss of generality that $\alpha_n \le (1-\epsilon_n - \beta_n\|B\|)\|A\|^{-1}$. Since A and B are linear bounded self adjoint operators, we have

$$\|A\| = \sup\{|\langle Ax, x\rangle| : x \in H, \|x\| = 1\},$$
$$\|B\| = \sup\{|\langle Bx, x\rangle| : x \in H, \|x\| = 1\}$$

observe that

$$\begin{aligned} \langle((1-\epsilon_n)I - \beta_n B - \alpha_n A)x, x\rangle &= (1-\epsilon_n)\langle x, x\rangle - \beta_n\langle Bx, x\rangle - \alpha_n\langle Ax, x\rangle \\ &\ge 1 - \epsilon_n - \beta_n\|B\| - \alpha_n\|A\| \\ &\ge 0. \end{aligned}$$

Therefore, $(1-\epsilon_n)I-\beta_nB-\alpha_nA$ is positive. Then, by strong positivity of A and B, we get

$$\begin{aligned}\|(1-\epsilon_n)I-\beta_nB-\alpha_nA\| &= \sup\{\langle((1-\epsilon_n)I-\beta_nB-\alpha_nA)x,x\rangle\ x\in H,\|x\|=1\}\\ &= \sup\{(1-\epsilon_n)\langle x,x\rangle-\beta_n\langle Bx,x\rangle-\alpha_n\langle Ax,x\rangle : x\in H,\|x\|=1\}\\ &\leq 1-\epsilon_n-\beta_n\bar{\beta}-\alpha_n\bar{\delta}\\ &\leq 1-\beta_n\bar{\beta}-\alpha_n\bar{\delta}. \end{aligned}\tag{3.2}$$

For any $x,y\in C$

$$\begin{aligned}\|P_nx-P_ny\| &\leq \alpha_n\gamma\|f(x)-f(y)\|+\beta_n\|B\|\|x-y\|\\ &\quad+\|(1-\epsilon_n)I-\beta_nB-\alpha_nA\|\frac{1}{s_n}\int_0^{s_n}\|T(s)x-T(s)y\|ds\\ &\leq \alpha_n\gamma\alpha\|x-y\|+\beta_n\bar{\beta}\|x-y\|+(1-\beta_n\bar{\beta}-\alpha_n\bar{\delta})\|x-y\|\\ &= (1-(\bar{\delta}-\gamma\alpha)\alpha_n)\|x-y\|.\end{aligned}$$

Therefore, Banach contraction principle guarantees that P_n has a unique fixed point in H, and so the iteration (3.1) is well defined. □

Lemma 3.3. *The sequence $\{x_n\}$ generated by Algorithm 3.1 is bounded.*

Proof. Let $p\in\Gamma=\bigcap_{i=1}^k \mathrm{Fix}(S)\cap \mathrm{GEP}(F_i,\psi_i)$. By intuition from [14], we have

$$\|u_{n,i}-p\|^2\leq\|x_n-p\|^2+r_{n,i}(r_{n,i}-2\bar{\alpha}_i)\|\psi_ix_n-\psi_ip\|^2.$$

Then

$$\begin{aligned}\|w_n-p\|^2 &\leq \frac{1}{k}\sum_{i=1}^k\|u_{n,i}-p\|^2\\ &\leq \|x_n-p\|^2+\frac{1}{k}\sum_{i=1}^k r_{n,i}(r_{n,i}-2\bar{\alpha}_i)\|\psi_ix_n-\psi_ip\|^2,\end{aligned}\tag{3.3}$$

and

$$\|w_n-p\|\leq\|x_n-p\|.$$

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n\gamma f(x_n)+\beta_nBx_n+((1-\epsilon_n)I-\beta_nB-\alpha_nA)\frac{1}{s_n}\int_0^{s_n}T(s)w_nds-p\|\\ &\leq \alpha_n\|\gamma f(x_n)-Ap\|+\beta_n\|Bx_n-Bp\|+\epsilon_n\|p\|\\ &\quad+\|((1-\epsilon_n)I-\beta_nB-\alpha_nA)\|\frac{1}{s_n}\int_0^{s_n}\|T(s)w_n-T(s)p\|ds\\ &\leq \alpha_n(\|\gamma f(x_n)-\gamma f(p)\|+\|\gamma f(p)-Ap\|)+\beta_n\|Bx_n-Bp\|+\epsilon_n\|p\|\\ &\quad+(1-\beta_n\bar{\beta}-\alpha_n\bar{\delta})\|w_n-p\|\\ &\leq \alpha_n\gamma\alpha\|x_n-p\|+\alpha_n\|\gamma f(p)-Ap\|+\beta_n\bar{\beta}\|x_n-p\|+\alpha_n\|p\|\\ &\quad+(1-\beta_n\bar{\beta}-\alpha_n\bar{\delta})\|x_n-p\|\\ &= (1-(\bar{\delta}-\gamma\alpha)\alpha_n)\|x_n-p\|+\alpha_n(\|p\|+\|\gamma f(p)-Ap\|)\end{aligned}$$

$$\leq \max\left\{\|x_n - p\|, \frac{\|\gamma f(p) - Ap\| + \|p\|}{\bar{\delta} - \gamma\alpha}\right\}$$

$$\vdots$$

$$\leq \max\left\{\|x_0 - p\|, \frac{\|\gamma f(p) - Ap\| + \|p\|}{\bar{\delta} - \gamma\alpha}\right\}. \tag{3.4}$$

Hence $\{x_n\}$ is bounded. □

Now, set $t_n := \frac{1}{s_n}\int_0^{s_n} T(s)w_n ds$. Then $\{w_n\}$, $\{t_n\}$ and $\{f(x_n)\}$ are bounded.

Lemma 3.4. *The following properties are satisfied for the Algorithm 3.1:*

P1. $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$

P2. $\lim_{n\to\infty} \|x_n - t_n\| = 0.$

P3. $\lim_{n\to\infty} \|\psi_i x_n - \psi_i p\| = 0,\ for\ i \in \{1, 2, \ldots, k\}.$

P4. $\lim_{n\to\infty} \|t_n - w_n\| = 0.$

P5. $\lim_{n\to\infty} \|T(s)t_n - t_n\| = 0.$

Proof.

P1: From Theorem 3.1(ii)[14], we have

$$\|t_{n+1} - t_n\| \leq \|x_{n+1} - x_n\| + M|r_{n+1,i} - r_{n,i}| + \frac{2|s_{n+1} - s_n|}{s_{n+1}}\|w_n - p\|, \tag{3.5}$$

where $M_i = \max\left\{\sup\{\frac{\|T^{F_i}_{r_{n+1,i}}(x_n - r_{n+1,i}\psi_i x_n) - (x_n - r_{n+1,i}\psi_i x_n)\|}{r_{n+1,i}}\}, \sup\{\|\psi_i x_n\|\}\right\}$ and $M = \frac{1}{k}\sum_{i=1}^{k} 2M_i$.

Setting $x_{n+1} = \epsilon_n x_n + (1-\epsilon_n)z_n$, then we have

$$\begin{aligned} z_{n+1} - z_n &= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + \beta_{n+1}Bx_{n+1} + ((1-\epsilon_{n+1})I - \beta_{n+1}B - \alpha_{n+1}A)t_{n+1} - \epsilon_{n+1}x_{n+1}}{1-\epsilon_{n+1}} \\ &\quad - \frac{\alpha_n\gamma f(x_n) + \beta_n Bx_n + ((1-\epsilon_n)I - \beta_n B - \alpha_n A)t_n - \epsilon_n x_n}{1-\epsilon_n} \\ &= \frac{\alpha_{n+1}}{1-\epsilon_{n+1}}(\gamma f(x_{n+1}) - At_{n+1}) + \frac{\alpha_n}{1-\epsilon_n}(At_n - \gamma f(x_n)) + \frac{\beta_{n+1}}{1-\epsilon_{n+1}}B(x_{n+1} - t_{n+1}) \\ &\quad + \frac{\beta_n}{1-\epsilon_n}B(t_n - x_n) + (t_{n+1} - t_n) + \frac{\epsilon_n}{1-\epsilon_n}x_n - \frac{\epsilon_{n+1}}{1-\epsilon_{n+1}}x_{n+1}. \end{aligned}$$

Using (3.5), we have

$$\begin{aligned} \|z_{n+1} - z_n\| \leq\ & \frac{\alpha_{n+1}}{1-\epsilon_{n+1}}\|\gamma f(x_{n+1}) - At_{n+1}\| + \frac{\alpha_n}{1-\epsilon_n}\|\gamma f(x_n) - At_n\| \\ & + \frac{\beta_{n+1}}{1-\epsilon_{n+1}}\|B\|\|x_{n+1} - t_{n+1}\| + \frac{\beta_n}{1-\epsilon_n}\|B\|\|t_n - x_n\| + \|t_{n+1} - t_n\| \end{aligned}$$

$$\begin{aligned}
&+\frac{\epsilon_n}{1-\epsilon_n}\|x_n\|+\frac{\epsilon_{n+1}}{1-\epsilon_{n+1}}\|x_{n+1}\| \\
&\le \frac{\alpha_{n+1}}{1-\epsilon_{n+1}}\|\gamma f(x_{n+1})-At_{n+1}\|+\frac{\alpha_n}{1-\epsilon_n}\|\gamma f(x_n)-At_n\| \\
&+\frac{\beta_{n+1}}{1-\epsilon_{n+1}}\bar{\beta}(\|x_{n+1}\|+\|t_{n+1}\|)+\frac{\beta_n}{1-\epsilon_n}\bar{\beta}(\|t_n\|+\|x_n\|)+\|x_{n+1}-x_n\| \\
&+M|r_{n+1,i}-r_{n,i}|+\frac{2|s_{n+1}-s_n|}{s_{n+1}}\|w_n-p\|+\frac{\epsilon_n}{1-\epsilon_n}\|x_n\|+\frac{\epsilon_{n+1}}{1-\epsilon_{n+1}}\|x_{n+1}\|,
\end{aligned}$$

which implies

$$\begin{aligned}
\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| \le\ & \frac{\alpha_{n+1}}{1-\epsilon_{n+1}}\|\gamma f(x_{n+1})-At_{n+1}\|+\frac{\alpha_n}{1-\epsilon_n}\|\gamma f(x_n)-At_n\| \\
&+\frac{\beta_{n+1}}{1-\epsilon_{n+1}}\bar{\beta}(\|x_{n+1}\|+\|t_{n+1}\|)+\frac{\beta_n}{1-\epsilon_n}\bar{\beta}(\|t_n\|+\|x_n\|) \\
&+M|r_{n+1,i}-r_{n,i}|+\frac{2|s_{n+1}-s_n|}{s_{n+1}}\|w_n-p\| \\
&+\frac{\epsilon_n}{1-\epsilon_n}\|x_n\|+\frac{\epsilon_{n+1}}{1-\epsilon_{n+1}}\|x_{n+1}\|.
\end{aligned}$$

Hence, it follows by conditions $(C1)-(C3)$ that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0. \tag{3.6}$$

From (3.6) and Lemma 2.8, we get $\lim_{n\to\infty}\|z_n-x_n\|=0$ and

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\epsilon_n)\|z_n-x_n\|=0. \tag{3.7}$$

Then we have $\lim_{n\to\infty}\|t_{n+1}-t_n\|=0$.

P2: We can write

$$\begin{aligned}
\|x_n-t_n\| &\le \|x_{n+1}-x_n\|+\|\alpha_n\gamma f(x_n)+\beta_nBx_n+((1-\epsilon_n)I-\beta_nB-\alpha_nA)t_n-t_n\| \\
&\le \|x_{n+1}-x_n\|+\alpha_n\|\gamma f(x_n)-At_n\|+\beta_n\|Bx_n-Bt_n\|+\epsilon_n\|t_n\| \\
&= \|x_{n+1}-x_n\|+\alpha_n\|\gamma f(x_n)-At_n\|+\beta_n\bar{\beta}\|x_n-t_n\|+\epsilon_n\|t_n\|.
\end{aligned}$$

Then

$$(1-\beta_n\bar{\beta})\|x_n-t_n\|\le\|x_{n+1}-x_n\|+\alpha_n\|\gamma f(x_n)-At_n\|+\epsilon_n\|t_n\|.$$

Therefore

$$\begin{aligned}
\|x_n-t_n\| &\le \frac{1}{1-\beta_n\bar{\beta}}\|x_{n+1}-x_n\|+\frac{\alpha_n}{1-\beta_n\bar{\beta}}\|\gamma f(x_n)-At_n\|+\frac{\epsilon_n}{1-\beta_n\bar{\beta}}\|t_n\| \\
&\le \frac{1}{1-\beta_n\bar{\beta}}\|x_{n+1}-x_n\|+\frac{\alpha_n}{1-\beta_n\bar{\beta}}(\|\gamma f(x_n)-At_n\|+\|t_n\|).
\end{aligned}$$

Using (C1) together (P1), we obtain

$$\lim_{n\to\infty}\|x_n - t_n\| = 0. \tag{3.8}$$

P3: We have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n\gamma f(x_n)+\beta_n Bx_n+((1-\epsilon_n)I-\beta_n B-\alpha_n A)t_n-p\|^2\\
&= \|\alpha_n(\gamma f(x_n)-Ap)+\beta_n(Bx_n-Bp)+((1-\epsilon_n)I-\beta_n B-\alpha_n A)(t_n-p)-\epsilon_n p\|^2\\
&\le \|((1-\epsilon_n)I-\alpha_n A)(t_n-p)+\beta_n(Bx_n-Bt_n)-\epsilon_n p\|^2\\
&\quad +2\langle\alpha_n(\gamma f(x_n)-Ap),x_{n+1}-p\rangle\\
&\le ((1-\alpha_n\bar{\delta})\|w_n-p\|+\beta_n\bar{\beta}\|x_n-t_n\|+\epsilon_n\|p\|)^2+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\
&= (1-\alpha_n\bar{\delta})^2\|w_n-p\|^2+(\beta_n\bar{\beta})^2\|x_n-t_n\|^2+(\epsilon_n)^2\|p\|^2\\
&\quad +2(1-\alpha_n\bar{\delta})\beta_n\bar{\beta}\|w_n-p\|\|x_n-t_n\|+2(1-\alpha_n\bar{\delta})\epsilon_n\|p\|\|w_n-p\|\\
&\quad +2\beta_n\epsilon_n\bar{\beta}\|p\|\|x_n-t_n\|+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle.
\end{aligned}\tag{3.9}$$

From (3.3), we have

$$\begin{aligned}
&\le (1-\alpha_n\bar{\delta})^2(\|x_n-p\|^2+\frac{1}{k}\sum_{i=1}^{k} r_{n,i}(r_{n,i}-2\bar{\alpha}_i)\|\psi_i x_n-\psi_i p\|^2)+(\beta_n\bar{\beta})^2\|x_n-t_n\|^2\\
&\quad +(\epsilon_n)^2\|p\|^2+2(1-\alpha_n\bar{\delta})\beta_n\bar{\beta}\|w_n-p\|\|x_n-t_n\|+2(1-\alpha_n\bar{\delta})\epsilon_n\|p\|\|w_n-p\|\\
&\quad +2\beta_n\epsilon_n\bar{\beta}\|p\|\|x_n-t_n\|+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\
&\le \|x_n-p\|^2+(\alpha_n\bar{\delta})^2\|x_n-p\|^2+(1-\alpha_n\bar{\delta})^2\frac{1}{k}\sum_{i=1}^{k} r_{n,i}(r_{n,i}-2\bar{\alpha}_i)\|\psi_i x_n-\psi_i p\|^2\\
&\quad +(\beta_n\bar{\beta})^2\|x_n-t_n\|^2+(\alpha_n)^2\|p\|^2+2(1-\alpha_n\bar{\delta})\beta_n\bar{\beta}\|w_n-p\|\|x_n-t_n\|\\
&\quad +2(1-\alpha_n\bar{\delta})\alpha_n\|p\|\|w_n-p\|+2\beta_n\epsilon_n\bar{\beta}\|p\|\|x_n-t_n\|+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle.
\end{aligned}$$

Using $(C3)$, we obtain

$$\begin{aligned}
&(1-\alpha_n\bar{\delta})^2\frac{1}{k}\sum_{i=1}^{k} b(2\bar{\alpha}_i-a)\|\psi_i x_n-\psi_i p\|^2\\
&\quad\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+(\alpha_n\bar{\delta})^2\|x_n-p\|^2+(\beta_n\bar{\beta})^2\|x_n-t_n\|^2+(\alpha_n)^2\|p\|^2\\
&\qquad +2(1-\alpha_n\bar{\delta})\beta_n\bar{\beta}\|w_n-p\|\|x_n-t_n\|+2(1-\alpha_n\bar{\delta})\alpha_n\|p\|\|w_n-p\|\\
&\qquad +2\beta_n\epsilon_n\bar{\beta}\|p\|\|x_n-t_n\|+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\
&\quad\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+(\alpha_n\bar{\delta})^2\|x_n-p\|^2+(\beta_n\bar{\beta})^2\|x_n-t_n\|^2\\
&\qquad +(\alpha_n)^2\|p\|^2+2(1-\alpha_n\bar{\delta})\beta_n\bar{\beta}\|w_n-p\|\|x_n-t_n\|+2(1-\alpha_n\bar{\delta})\alpha_n\|p\|\|w_n-p\|\\
&\qquad +2\beta_n\epsilon_n\bar{\beta}\|p\|\|x_n-t_n\|+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle.
\end{aligned}$$

By (P1)-(P2) and Lemma 2.4(i),we have $\lim_{n\to\infty}\|\psi_i x_n-\psi_i p\|^2 = 0$.

P4: Theorem 3.1 [14] implies that

$$\begin{aligned}\|w_n-p\|^2 &\leq \frac{1}{k}\sum_{i=1}^{k}\|u_{n,i}-p\|^2\\ &\leq \|x_n-p\|^2-\frac{1}{k}\sum_{i=1}^{k}\|x_n-u_{n,i}\|^2\\ &\quad+\frac{2}{k}\sum_{i=1}^{k}r_{n,i}(\|x_n-u_{n,i}\|\|\psi_i x_n-\psi_i p\|-\bar{\alpha}_i\|\psi_i x_n-\psi_i p\|^2).\end{aligned}\tag{3.10}$$

It follows from (3.9) and (3.10) that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= (1-\alpha_n\bar{\delta})^2\|w_n-p\|^2+(\beta_n\bar{\beta})^2\|x_n-t_n\|^2+(\epsilon_n)^2\|p\|^2\\ &\quad+2(1-\alpha_n\bar{\delta})\beta_n\bar{\beta}\|w_n-p\|\|x_n-t_n\|+2(1-\alpha_n\bar{\delta})\epsilon_n\|p\|\|w_n-p\|\\ &\quad+2\beta_n\epsilon_n\bar{\beta}\|p\|\|x_n-t_n\|+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\ &\leq (1-\alpha_n\bar{\delta})^2(\|x_n-p\|^2-\frac{1}{k}\sum_{i=1}^{k}\|x_n-u_{n,i}\|^2\\ &\quad+\frac{2}{k}\sum_{i=1}^{k}r_{n,i}(\|x_n-u_{n,i}\|\|\psi_i x_n-\psi_i p\|-\bar{\alpha}_i\|\psi_i x_n-\psi_i p\|^2)+(\beta_n\bar{\beta})^2\|x_n-t_n\|^2\\ &\quad+(\epsilon_n)^2\|p\|^2+2(1-\alpha_n\bar{\delta})\beta_n\bar{\beta}\|w_n-p\|\|x_n-t_n\|+2(1-\alpha_n\bar{\delta})\epsilon_n\|p\|\|w_n-p\|\\ &\quad+2\beta_n\epsilon_n\bar{\beta}\|p\|\|x_n-t_n\|+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\ &\leq \|x_n-p\|^2+(\alpha_n\bar{\delta})^2\|x_n-p\|^2-(1-\alpha_n\bar{\delta})^2\frac{1}{k}\sum_{i=1}^{k}\|x_n-u_{n,i}\|^2\\ &\quad+(1-\alpha_n\bar{\delta})^2\frac{2}{k}\sum_{i=1}^{k}r_{n,i}(\|x_n-u_{n,i}\|\|\psi_i x_n-\psi_i p\|-\bar{\alpha}_i\|\psi_i x_n-\psi_i p\|^2)\\ &\quad+(\beta_n\bar{\beta})^2\|x_n-t_n\|^2+(\alpha_n)^2\|p\|^2+2(1-\alpha_n\bar{\delta})\beta_n\bar{\beta}\|w_n-p\|\|x_n-t_n\|\\ &\quad+2(1-\alpha_n\bar{\delta})\epsilon_n\|p\|\|w_n-p\|+2\beta_n\epsilon_n\bar{\beta}\|p\|\|x_n-t_n\|+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle.\end{aligned}$$

Therefore

$$\begin{aligned}&(1-\alpha_n\bar{\delta})^2\frac{1}{k}\sum_{i=1}^{k}\|x_n-u_{n,i}\|^2\\ &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+(\alpha_n\bar{\delta})^2\|x_n-p\|^2\\ &\quad+(1-\alpha_n\bar{\delta})^2\frac{2}{k}\sum_{i=1}^{k}r_{n,i}(\|x_n-u_{n,i}\|\|\psi_i x_n-\psi_i p\|-\bar{\alpha}_i\|\psi_i x_n-\psi_i p\|^2)\\ &\quad+(\beta_n\bar{\beta})^2\|x_n-t_n\|^2+(\alpha_n)^2\|p\|^2+2(1-\alpha_n\bar{\delta})\beta_n\bar{\beta}\|w_n-p\|\|x_n-t_n\|\\ &\quad+2(1-\alpha_n\bar{\delta})\epsilon_n\|p\|\|w_n-p\|+2\beta_n\epsilon_n\bar{\beta}\|p\|\|x_n-t_n\|+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle\\ &\leq (\|x_n-p\|+\|x_{n+1}-p\|)\|x_{n+1}-x_n\|+(\alpha_n\bar{\delta})^2\|x_n-p\|^2\\ &\quad+(1-\alpha_n\bar{\delta})^2\frac{2}{k}\sum_{i=1}^{k}r_{n,i}(\|x_n-u_{n,i}\|\|\psi_i x_n-\psi_i p\|-\bar{\alpha}_i\|\psi_i x_n-\psi_i p\|^2)\end{aligned}$$

$$+(\beta_n\bar{\beta})^2\|x_n-t_n\|^2+(\alpha_n)^2\|p\|^2+2(1-\alpha_n\bar{\delta})\beta_n\bar{\beta}\|w_n-p\|\|x_n-t_n\|$$
$$+2(1-\alpha_n\bar{\delta})\epsilon_n\|p\|\|w_n-p\|+2\beta_n\epsilon_n\bar{\beta}\|p\|\|x_n-t_n\|+2\alpha_n\langle\gamma f(x_n)-Ap,x_{n+1}-p\rangle.$$

From (C1) together (P1)-(P3), we obtain $\lim_{n\to\infty}\|x_n-u_{n,i}\|=0$.

It is easy to prove

$$\lim_{n\to\infty}\|w_n-x_n\|=0. \tag{3.11}$$

Using (3.8) and (3.11), we estimate $\|t_n-w_n\|\le\|t_n-x_n\|+\|x_n-w_n\|$. Then $\lim_{n\to\infty}\|t_n-w_n\|=0$.

P5: Let $E:=\{w\in C:\|w-p\|\le\|x_0-p\|,\frac{1}{\bar{\delta}-\gamma\alpha}\|\gamma f(p)-Ap\|+\|p\|\}$, E is a nonempty bounded closed convex subset of C which is $T(s)$-invariant for each $s\in[0,+\infty)$ and contains $\{x_n\}$. Without loss of generality, we may assume that $S:=\{T(s):s\in[0,+\infty)\}$ is a nonexpansive semigroup on E. From (27)[7], we have

$$\|T(s)x_n-x_n\|\le 2\left\|\frac{1}{s_n}\int_0^{s_n}T(s)w_n ds-x_n\right\|$$
$$+\left\|T(s)\frac{1}{s_n}\int_0^{s_n}T(s)w_n ds-\frac{1}{s_n}\int_0^{s_n}T(s)w_n ds\right\|.$$

Using Lemma 2.7 and (3.8), we obtain $\lim_{n\to\infty}\|T(s)x_n-x_n\|=0$.
Therefore

$$\|T(s)t_n-t_n\|\le\|T(s)t_n-T(s)x_n\|+\|T(s)x_n-x_n\|+\|x_n-t_n\|$$
$$\le\|t_n-x_n\|+\|T(s)x_n-x_n\|+\|x_n-t_n\|.$$

Then we have $\lim_{n\to\infty}\|T(s)t_n-t_n\|=0$. □

4. Main result

Theorem 4.1. *The Algorithm defined by (3.1) is convergence strongly to $z\in\Gamma=\bigcap_{i=1}^k Fix(S)\cap GEP(F_i,\psi_i)$, which is a unique solution in of the variational inequality*

$$\langle(\gamma f-A)z,y-z\rangle\le 0,\quad\forall y\in\Gamma.$$

Proof. For all $x,y\in H$, we have

$$\|P_\Gamma(I-A+\gamma f)(x)-P_\Gamma(I-A+\gamma f)(y)\|\le\|(I-A+\gamma f)(x)-(I-A+\gamma f)(y)\|$$
$$\le\|I-A\|\|x-y\|+\gamma\|f(x)-f(y)\|$$
$$\le(1-\bar{\delta})\|x-y\|+\gamma\alpha\|x-y\|$$
$$=(1-(\bar{\delta}-\gamma\alpha))\|x-y\|.$$

Then $P_\Gamma(I-A+\gamma f)$ is a contraction mapping from H into itself. Therefore by the Banach contraction principle, there exists $z\in H$ such that $z=P_\Gamma(I-A+\gamma f)z$.

The proof of Theorem 3.2 [14] show that

$$\langle(\gamma f-A)z, x_n-z\rangle\le 0. \tag{4.1}$$

Finally, we prove x_n is strongly convergent to z.

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \alpha_n\langle\gamma f(x_n)-Az, x_{n+1}-z\rangle+\beta_n\langle Bx_n-Bz, x_{n+1}-z\rangle-\epsilon_n\langle z, x_{n+1}-z\rangle\\ &\quad+\langle((1-\epsilon_n)I-\beta_nB-\alpha_nA)(t_n-z), x_{n+1}-z\rangle\\ &\le \alpha_n(\gamma\langle f(x_n)-f(z), x_{n+1}-z\rangle+\langle\gamma f(z)-Az, x_{n+1}-z\rangle)+\beta_n\|B\|\|x_n-z\|\|x_{n+1}-z\|\\ &\quad-\epsilon_n\|z\|\|x_{n+1}-z\|+\|(1-\epsilon_n)I-\beta_nB-\alpha_nA\|\|t_n-z\|\|x_{n+1}-z\|\\ &\le \alpha_n\alpha\gamma\|x_n-z\|\|x_{n+1}-z\|+\alpha_n\langle\gamma f(z)-Az, x_{n+1}-z\rangle+\beta_n\bar\beta\|x_n-z\|\|x_{n+1}-z\|\\ &\quad-\epsilon_n\|z\|\|x_{n+1}-z\|+(1-\beta_n\bar\beta-\alpha_n\bar\delta)\|x_n-z\|\|x_{n+1}-z\|\\ &= (1-\alpha_n(\bar\delta-\alpha\gamma))\|x_n-z\|\|x_{n+1}-z\|-\epsilon_n\|z\|\|x_{n+1}-z\|+\alpha_n\langle\gamma f(z)-Az, x_{n+1}-z\rangle\\ &\le \frac{1-\alpha_n(\bar\delta-\alpha\gamma)}{2}(\|x_n-z\|^2+\|x_{n+1}-z\|^2)-\epsilon_n\|z\|\|x_{n+1}-z\|\\ &\quad+\alpha_n\langle\gamma f(z)-Az, x_{n+1}-z\rangle\\ &\le \frac{1-\alpha_n(\bar\delta-\alpha\gamma)}{2}\|x_n-z\|^2+\frac{1}{2}\|x_{n+1}-z\|^2-\epsilon_n\|z\|\|x_{n+1}-z\|\\ &\quad+\alpha_n\langle\gamma f(z)-Az, x_{n+1}-z\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}2\|x_{n+1}-z\|^2 \le{}& (1-\alpha_n(\bar\delta-\alpha\gamma))\|x_n-z\|^2+\|x_{n+1}-z\|^2-2\alpha_n\|z\|\|x_{n+1}-z\|\\ &+2\alpha_n\langle\gamma f(z)-Az, x_{n+1}-z\rangle.\end{aligned}$$

Then

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le (1-\alpha_n(\bar\delta-\alpha\gamma))\|x_n-z\|^2-2\alpha_n\|z\|\|x_{n+1}-z\|+2\alpha_n\langle\gamma f(z)-Az, x_{n+1}-z\rangle\\ &= (1-k_n)\|x_n-z\|^2+2\alpha_nl_n,\end{aligned} \tag{4.2}$$

where $k_n=\alpha_n(\bar\delta-\alpha\gamma)$ and $l_n=\langle\gamma f(z)-Az, x_{n+1}-z\rangle-\|z\|\|x_{n+1}-z\|$.

Since $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=0}^{\infty}\alpha_n=\infty$, it is easy to see that $\lim_{n\to\infty}k_n=0$, $\sum_{n=0}^{\infty}k_n=\infty$ and $\limsup_{n\to\infty}l_n\le 0$. Hence, from (4.1), (4.2) and Lemma 2.10, we deduce that $x_n\to z$, where $z=P_\Gamma(I-A+\gamma f)z$. □

Remark 4.2. Putting $\psi_i = 0$ and $\{\epsilon_n\}, \{\beta_n\} = 0$ we obtain method introduced in Theorem 4.1 [3]. Taking $\{\epsilon_n\} = 0, F_i = \psi_i = 0, w_n = x_n$ and $A = B = I$, then the conclusion Theorem 3.3 [13] is obtained. Taking $\{\epsilon_n\} = 0, F_i = \psi_i = 0, w_n = x_n$ and $B = I$, then the conclusion Theorem 3.1 [6] is obtained. Putting $\{\epsilon_n\} = 0$ and $B = I$, then the main Theorems [14, 15, 16, 17] are obtained.

5. Numerical examples

In this section, we give some examples and numerical results for supporting our main theorem.

All the numerical results have been produced in Matlab 2017 on a Linux workstation with a 3.8 GHZ Intel annex processor and 8 Gb of memory.

Example 5.1. Let $H = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy, \ \forall x, y \in \mathbb{R}$, and induced usual norm $|\,.\,|$. Let $C = [-4, 2]$; let $F_1, F_2 : C \times C \to \mathbb{R}$ be defined by $F_1(x, y) = (3 - x^2)(x - y)$, $F_2(x, y) = (x + 6)(y - x)$, $\ \forall x, y \in C$; let $\psi_1, \psi_2 : C \to H$ be defined by $\psi_1(x) = 2x, \psi_2(x) = x$, $\forall x \in C$ and let for each $x \in \mathbb{R}$, we define $f(x) = \frac{1}{6}x$, $A(x) = \frac{1}{3}x$, $B(x) = \frac{1}{10}x$, and let, for each $x \in C$, $T(s)x = x$. Then there exist unique sequences $\{x_n\} \subset \mathbb{R}$, $\{u_{n,i}\} \subset C$, and $\{w_n\} \subset C$ generated by the iterative schemes

$$u_{n,i} = T^{F_i}_{r_{n,i}}(x_n - r_{n,i}\psi_i x_n), \qquad w_n = \frac{1}{2}(u_{n,1} + u_{n,2}) \tag{5.1}$$

$$x_{n+1} = \frac{1}{n}x_n + \frac{1}{10(n+1)^2}x_n + \Big((1-\frac{2}{n^2})I - \frac{1}{(n+1)^2}B - \frac{3}{n}A\Big)\frac{1}{s_n}\int_0^{s_n} w_n ds \tag{5.2}$$

where $\alpha_n = \frac{3}{n}$, $\beta_n = \frac{1}{(n+1)^2}$, $\epsilon_n = \frac{2}{n^2}$ and $s_n = n$, $r_{n,1} = r_{n,2} = 1 + \frac{1}{n}$. Then $\{x_n\}$ converges to $\{-3\} \in \bigcap_{i=1}^k \mathrm{Fix}(S) \cap \mathrm{GEP}(F_i, \psi_i)$.

Proof. The bifunctions F_1 and F_2 satisfy the (A1)−(A4). Further, f is contraction mapping with constant $\alpha = \frac{1}{3}$ and A and B are strongly positive bounded linear operator with constant $\bar{\delta} = 1$ on $\mathbb{R}$. Therefore, we can choose $\gamma = 2$ which satisfies $0 < \gamma < \frac{\bar{\delta}}{\alpha} < \gamma + \frac{1}{\alpha}$. Furthermore, it is easy to observe that $\bigcap_{i=1}^k \mathrm{Fix}(S) \cap \mathrm{GEP}(F_i, \psi_i) = \{-3\} \neq \emptyset$. We have computed $u_{n,i}$ for each example i = 1,2 as follow

$$u_{n,1} = -\frac{1 + \sqrt{1 - 4(1+\frac{1}{n})((1+\frac{2}{n})x_n - 3(1+\frac{1}{n}))}}{2 + \frac{2}{n}},$$

$$u_{n,2} = -\frac{\frac{1}{n}x_n + 6(1+\frac{1}{n})}{2+\frac{1}{n}}, \qquad w_n = \frac{1}{2}(u_{n,1} + u_{n,2})$$

$$x_{n+1} = \Big(\frac{10n^2 + 21n + 10}{10n(n+1)^2}\Big)x_n + \Big(\frac{10n^4 + 10n^3 - 31n^2 - 50n - 20}{10n^2(n+1)^2}\Big)w_n.$$

We obtain the following figure of the result , with initial point $x_1 = 1$. □

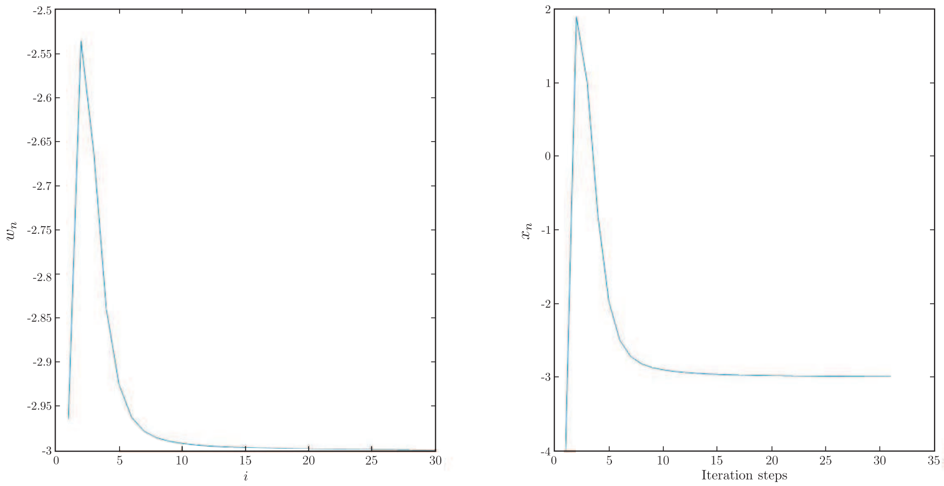

Figure 1: The graph of $\{x_n\}$ with initial value $x_1 = 1$.

Example 5.2. *Let $H = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy$, $\forall x, y \in \mathbb{R}$, and induced usual norm $|\,.\,|$. Let $C = [0,2]$; let $F_1, F_2, F_3 : C \times C \to \mathbb{R}$ be defined by $F_1(x, y) = -2x^2(x-y)$, $F_2(x, y) = -x^2(x-y)^2$, $F_3(x, y) = -3x^2 + xy + 2y^2$, $\forall x, y \in C$; let $\psi_1, \psi_2, \psi_3 : C \to H$ be defined by $\psi_1(x) = \psi_2(x) = \psi_3(x) = 0$, $\forall x \in C$ and let for each $x \in \mathbb{R}$, we define $f(x) = \frac{1}{8}x$, $A(x) = B(x) = I$, and let, for each $x \in C$, $T(s)x = \frac{1}{1+2s}x$. Then there exist unique sequences $\{x_n\} \subset \mathbb{R}$, $\{u_{n,i}\} \subset C$, and $\{w_n\} \subset C$ generated by the iterative schemes*

$$u_{n,i} = T^{F_i}_{r_{n,i}}(x_n - r_{n,i}\psi_i x_n), \qquad w_n = \frac{1}{3}(u_{n,1} + u_{n,2} + u_{n,3}) \tag{5.3}$$

$$x_{n+1} = \frac{2}{8\sqrt{n}}x_n + \frac{1}{n^2}x_n + \Big((1-\frac{1}{n})I - \frac{1}{n^2}B - \frac{1}{\sqrt{n}}A\Big)\frac{1}{s_n}\int_0^{s_n} \frac{1}{1+2s}w_n ds \tag{5.4}$$

where $\alpha_n = \frac{1}{\sqrt{n}}$, $\beta_n = \frac{1}{n^2}$, $\epsilon_n = \frac{1}{n}$ and $s_n = n$, $r_{n,1} = r_{n,2} = 1 + \frac{8}{n}$. Then $\{x_n\}$ converges to $\{0\} \in \bigcap_{i=1}^k Fix(S) \cap GEP(F_i, \psi_i)$.

Proof. It is easy to prove that the bifunctions F_1, F_2 and F_3 satisfy the $(A1)-(A4)$. Further, f is contraction mapping with constant $\alpha = \frac{1}{5}$ and $A = B = I$ are strongly positive bounded linear operator with constant $\bar{\delta} = 1$ on $\mathbb{R}$. Therefore, we can choose $\gamma = 2$ which satisfies $0 < \gamma < \frac{\bar{\delta}}{\alpha} < \gamma + \frac{1}{\alpha}$. Furthermore, it is easy to observe that $\bigcap_{i=1}^k \text{Fix}(S) \cap \text{GEP}(F_i, \psi_i) = \{0\} \neq \emptyset$. We have computed $u_{n,i}$ for i = 1,2 as follow

$$u_{n,1} = \frac{-1 + \sqrt{1 + (8 + \frac{64}{n})x_n}}{4 + \frac{32}{n}}, \qquad u_{n,2} = x_n, \qquad u_{n,3} = \frac{n}{6n+40}x_n$$

$$w_n = \frac{1}{3}(u_{n,1} + u_{n,2} + u_{n,3})$$

$$x_{n+1} = \Big(\frac{1}{4\sqrt{n}} + \frac{1}{n^2}\Big)x_n + (\frac{w_n}{2n})\ln(1+2n)\Big(1 - \frac{1}{n} - \frac{1}{n^2} - \frac{1}{\sqrt{n}}\Big)$$

Choose $x_1 = 1$. we obtain the following figure. □

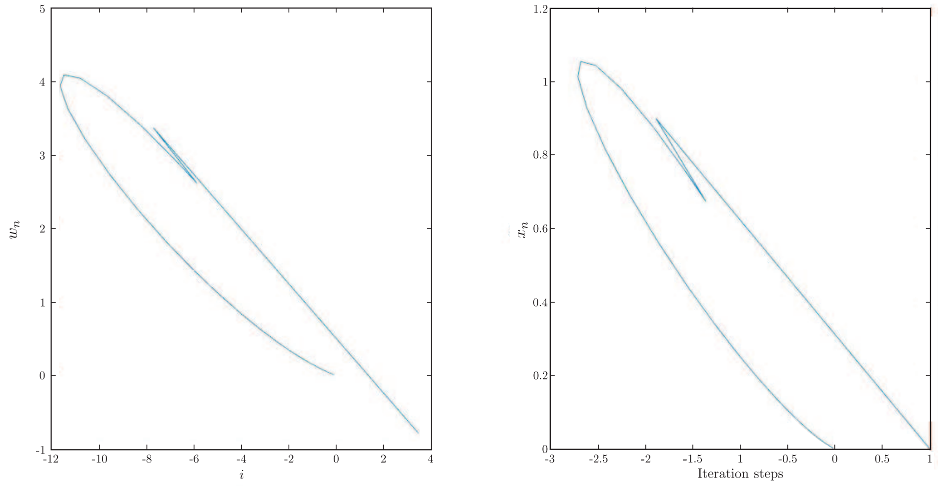

Figure 2: The graph of $\{x_n\}$ with initial value $x_1 = 1$.

Example 5.3. *Let $H = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy,\ \ \forall x, y \in \mathbb{R}$, and induced usual norm $|\ .\ |$. Let $C = [0,3]$; let $F_1, F_2 : C \times C \to \mathbb{R}$ be defined by $F_1(x, y) = 5x(x-y)$, $F_2(x, y) = -2x(y-x),\ \ \forall x, y \in C$; let $\psi_1, \psi_2 : C \to H$ be defined by $\psi_1(x) = 3x, \psi_2(x) = 4x,\ \forall x \in C$ and let for each $x \in \mathbb{R}$, we define $f(x) = \frac{1}{5}(x+2),\ A(x) = x,\ B(x) = \frac{1}{3}x$, and let, for each $x \in C$, $T(s)x = \frac{1}{1+3s}x$. Then there exist unique sequences $\{x_n\} \subset \mathbb{R}$, $\{u_{n,i}\} \subset C$, and $\{w_n\} \subset C$ generated by the iterative schemes*

$$u_{n,i} = T^{F_i}_{r_{n,i}}(x_n - r_{n,i}\psi_i x_n), \qquad w_n = \frac{1}{2}(u_{n,1} + u_{n,2}) \tag{5.5}$$

$$x_{n+1} = \frac{1}{5\sqrt{n}}(x_n+2) + \frac{1}{3(2n^2-3)}x_n + \Big((1-\frac{1}{n^2})I - \frac{1}{2n^2-3}B - \frac{1}{2\sqrt{n}}A\Big)\frac{1}{s_n}\int_0^{s_n}\frac{1}{1+3s}w_n ds \tag{5.6}$$

where $\alpha_n = \frac{1}{2\sqrt{n}}$, $\beta_n = \frac{1}{2n^2-3}$, $\epsilon_n = \frac{1}{n^2}$ and $s_n = 2n,\ r_{n,1} = r_{n,2} = 1 + \frac{1}{5n^2}$. Then $\{x_n\}$ converges to $\{0\} \in \bigcap_{i=1}^{k} Fix(S) \cap GEP(F_i, \psi_i)$.

Proof. It is easy to prove that the f is contraction mapping with constant $\alpha = \frac{1}{3}$ and A and B are strongly positive bounded linear operator with constant $\bar{\delta} = 1$ on $\mathbb{R}$. Therefore, we can choose $\gamma = 2$ which satisfies $0 < \gamma < \frac{\bar{\delta}}{\alpha} < \gamma + \frac{1}{\alpha}$. Furthermore, it is easy to observe that $\bigcap_{i=1}^{k} \mathrm{Fix}(S) \cap \mathrm{GEP}(F_i, \psi_i) = \{0\} \neq \emptyset$. As mention

$$u_{n,1} = \frac{10n^2+3}{50n^2+5}x_n, \qquad u_{n,2} = \frac{15n^2+4}{5n^2+2}x_n,$$

$$w_n = \frac{1}{2}(u_{n,1} + u_{n,2}),$$

$$x_{n+1} = \left(\frac{6n^2 + 5\sqrt{n} - 9}{15\sqrt{n}(2n^2 - 3)}\right)x_n + \frac{2}{5\sqrt{n}} + \frac{1}{6n}\ln(1+6n)\left(1 - \frac{1}{n^2} - \frac{1}{3(2n^2-3)} - \frac{1}{2\sqrt{n}}\right)w_n$$

Choose $x_1 = 3$. we obtain the following figure. □

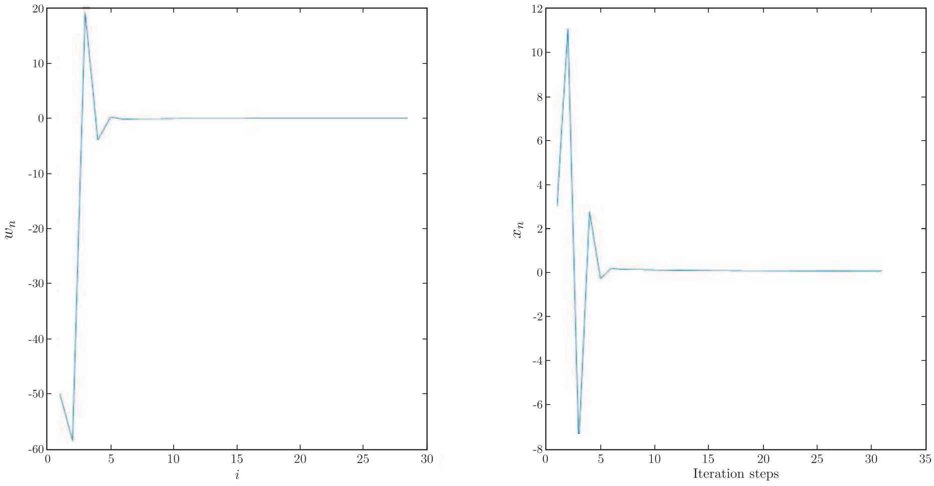

Figure 3: The graph of $\{x_n\}$ with initial value $x_1 = 3$.

Example 5.4. *Let $H = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy$, $\forall x, y \in \mathbb{R}$, and induced usual norm $|\,.\,|$. Let $C = [0,3]$; let $F_1, F_2 : C \times C \to \mathbb{R}$ be defined by $F_1(x, y) = 5x(x - y)$, $F_2(x, y) = -2x(y - x)$, $\forall x, y \in C$; let $\psi_1, \psi_2 : C \to H$ be defined by $\psi_1(x) = 3x, \psi_2(x) = 4x$, $\forall x \in C$ and let for each $x \in \mathbb{R}$, we define $f(x) = \frac{1}{5}(x+2)$, $A(x) = x$, $B(x) = \frac{1}{3}x$, and let, for each $x \in C$, $T(s)x = e^{-3s}x$. Then there exist unique sequences $\{x_n\} \subset \mathbb{R}$, $\{u_{n,i}\} \subset C$, and $\{w_n\} \subset C$ generated by the iterative schemes*

$$u_{n,i} = T^{F_i}_{r_{n,i}}(x_n - r_{n,i}\psi_i x_n), \qquad w_n = \frac{1}{2}(u_{n,1} + u_{n,2}) \tag{5.7}$$

$$x_{n+1} = \frac{1}{5\sqrt{n}}(x_n + 2) + \frac{1}{3(2n^2-3)}x_n + \left((1 - \frac{1}{n^2})I - \frac{1}{2n^2-3}B - \frac{1}{2\sqrt{n}}A\right)\frac{1}{s_n}\int_0^{s_n} e^{-3s} w_n ds \tag{5.8}$$

where $\alpha_n = \frac{1}{2\sqrt{n}}$, $\beta_n = \frac{1}{2n^2-3}$, $\epsilon_n = \frac{1}{n^2}$ and $s_n = 2n$, $r_{n,1} = r_{n,2} = 1 + \frac{1}{5n^2}$. Then $\{x_n\}$ converges to $\{0\} \in \bigcap_{i=1}^k Fix(S) \cap GEP(F_i, \psi_i)$.

Proof. By the same arguments example (5.3), we have

$$u_{n,1} = \frac{10n^2 + 3}{50n^2 + 5}x_n, \qquad u_{n,2} = \frac{15n^2 + 4}{5n^2 + 2}x_n,$$

$$w_n = \frac{1}{2}(u_{n,1} + u_{n,2}),$$

$$x_{n+1} = (\frac{6n^2 + 5\sqrt{n} - 9}{15\sqrt{n}(2n^2 - 3)})x_n + \frac{2}{5\sqrt{n}} - \frac{1}{6n}(1 - \frac{1}{n^2} - \frac{1}{3(2n^2 - 3)} - \frac{1}{2\sqrt{n}})e^{-6n}w_n$$

Choose $x_1 = 3$. we obtain the following figure. □

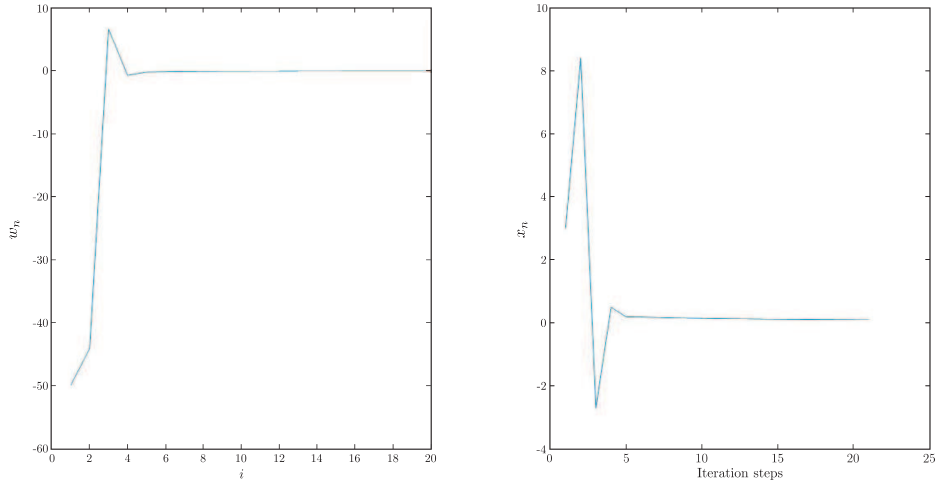

Figure 4: The graph of $\{x_n\}$ with initial value $x_1 = 3$.

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